

Journées ALEA 2011

CIRM

Processus d'exclusion  
par approche cellulaire

exercices

8 Mars 2011

Luminy

Xavier Viennot

LaBRI, Bordeaux

# 1. Tableaux alternatifs et tableaux de permutations

## **1. Tableaux alternatifs et tableaux de permutations**

Construire une bijection entre les tableaux alternatifs de taille  $n$  et les tableaux de permutations de taille  $(n+1)$ . (conseil: penser au passage Q-tableau complet vers Q-tableau). En déduire l'interprétation (Corteel-Williams) des probabilités stationnaires du PASEP  $(q, \alpha, \beta)$  en termes de tableaux de permutations.



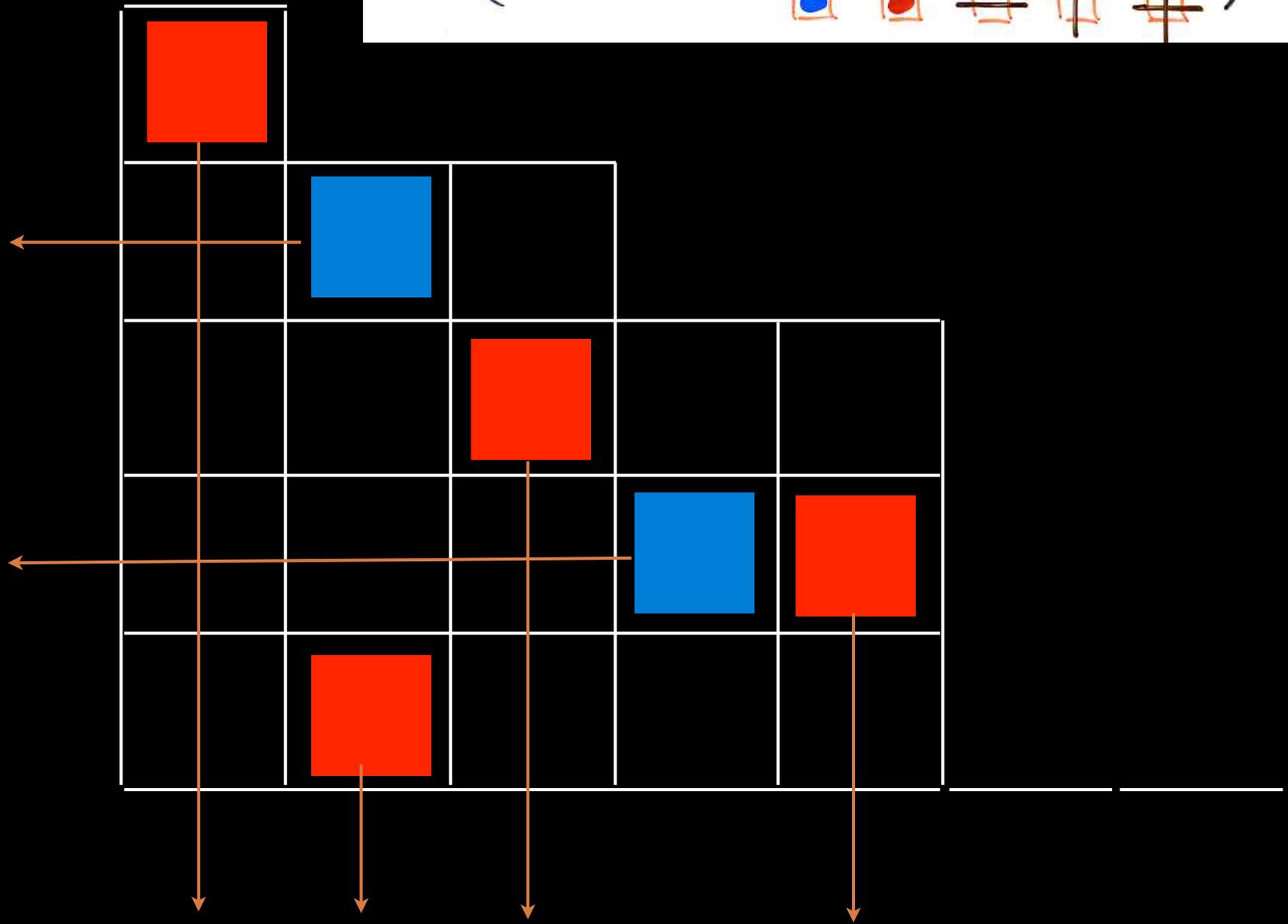
(i) mark the cells

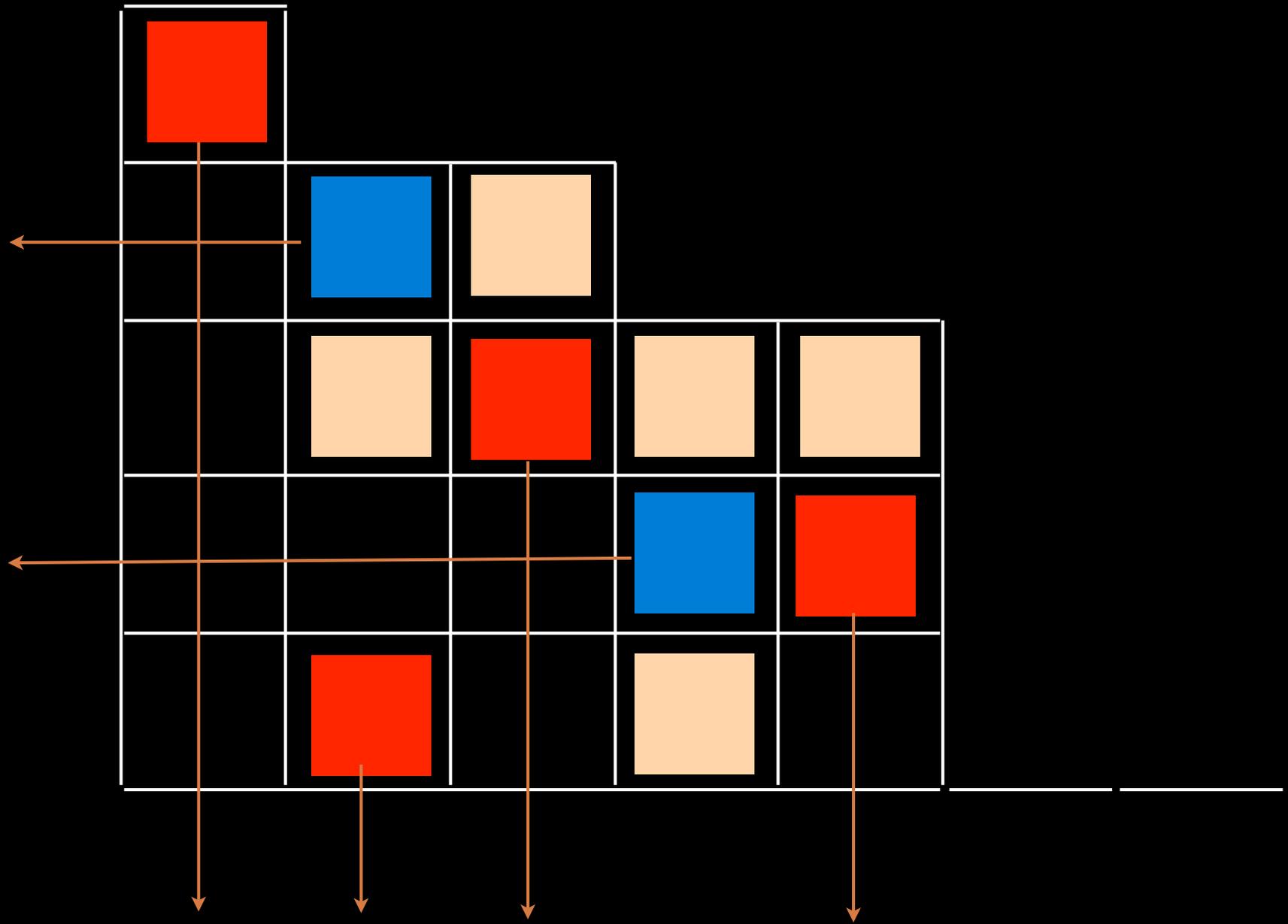


Red				
	Blue			
		Red		
			Blue	Red
	Red			

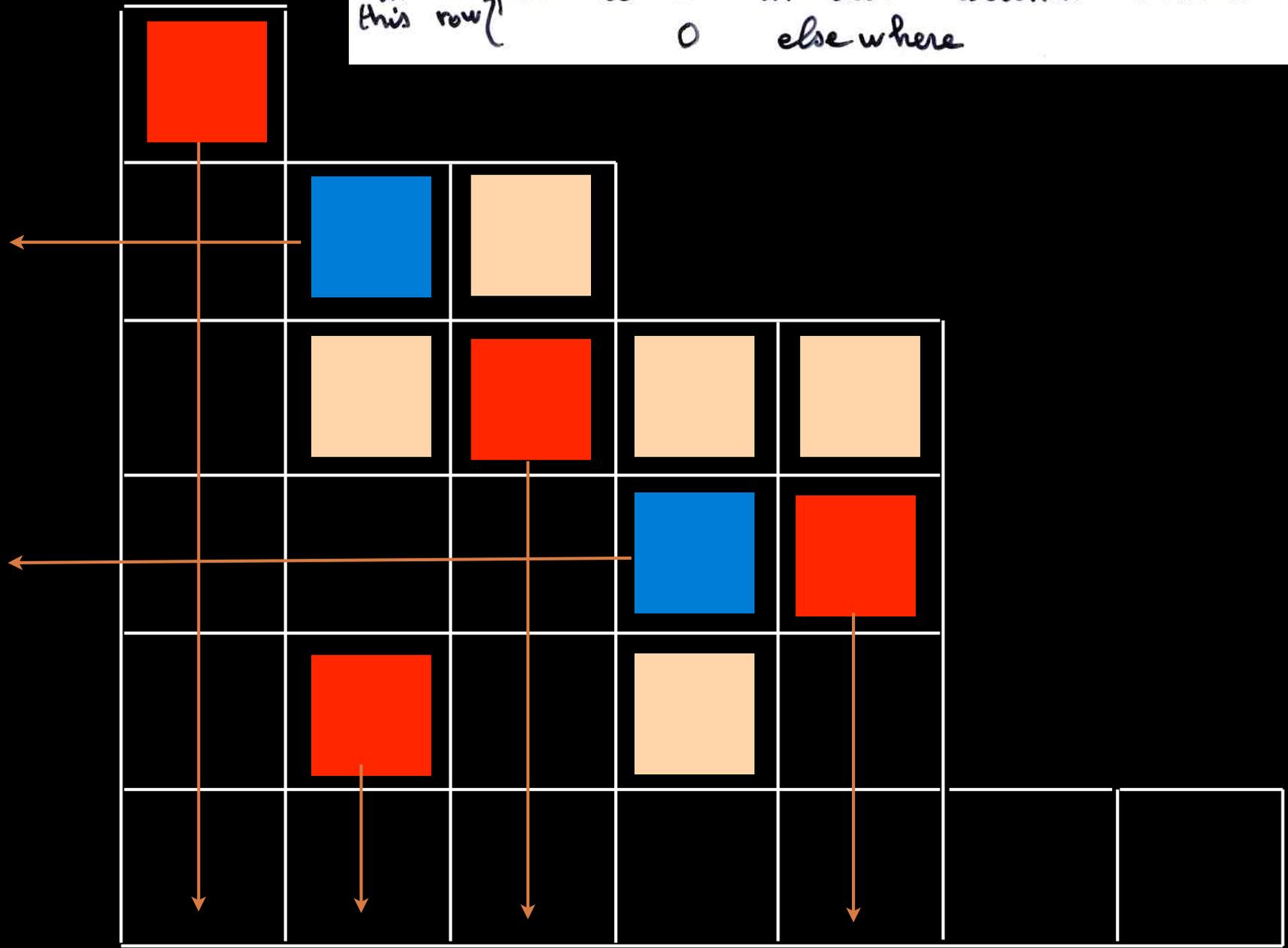
alternative tableau

(ii) mark the empty cells by   
(other than      )

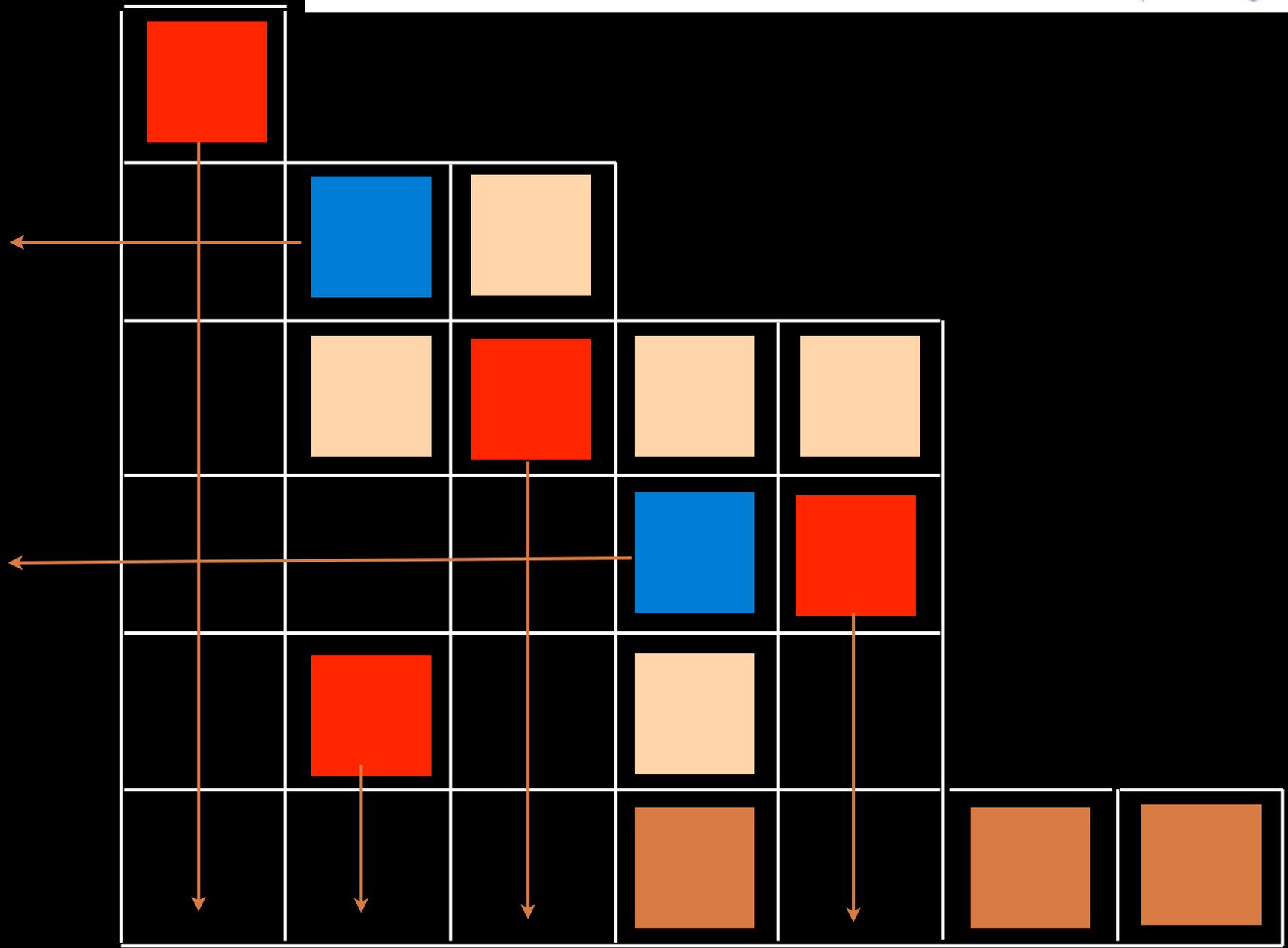




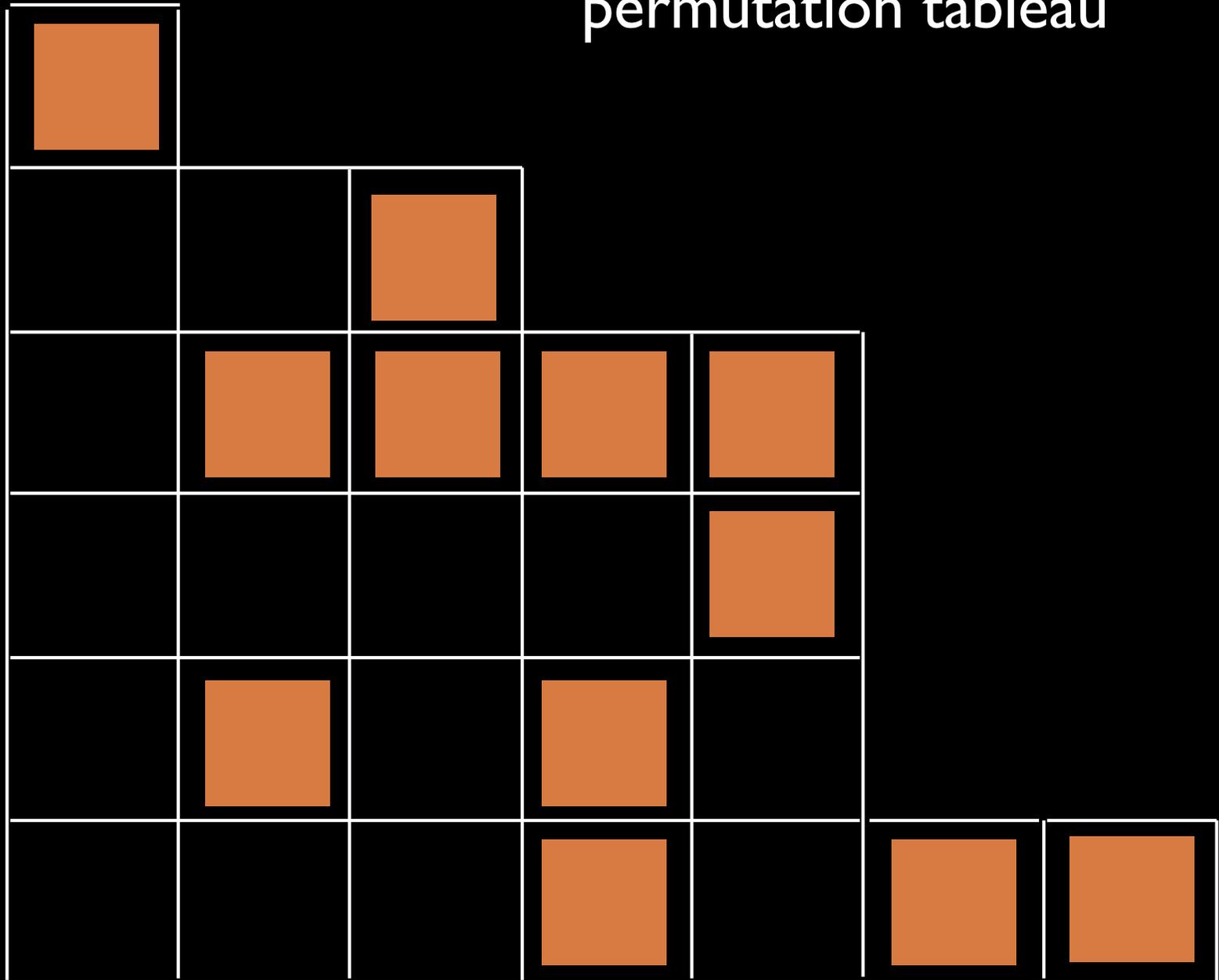
(iv) add a new row below  $F$   
in this row { put a 1 in each column without   
0 elsewhere

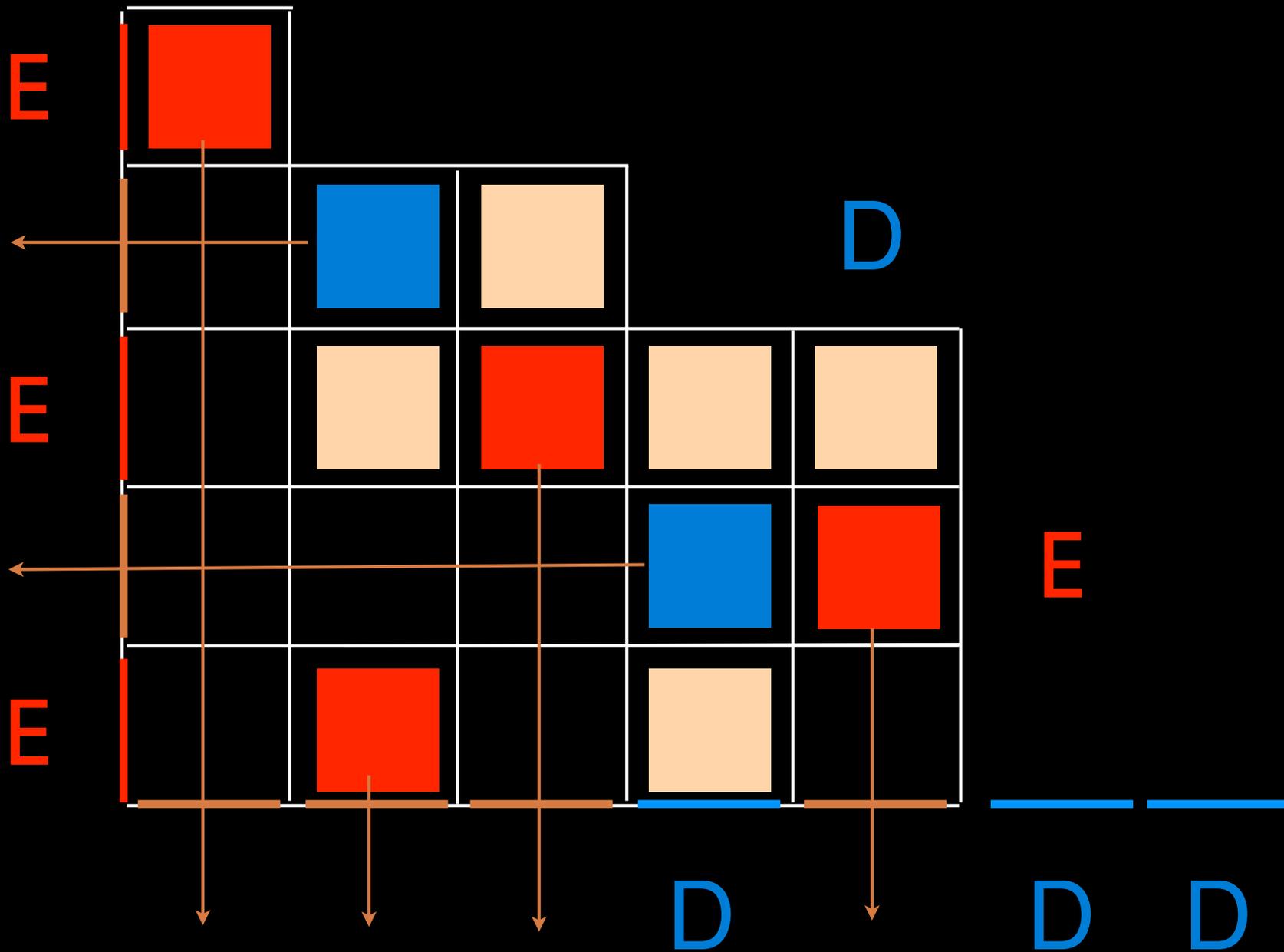


(iii) • replace the cells  or  by **1**  
• replace the cells     by **0**



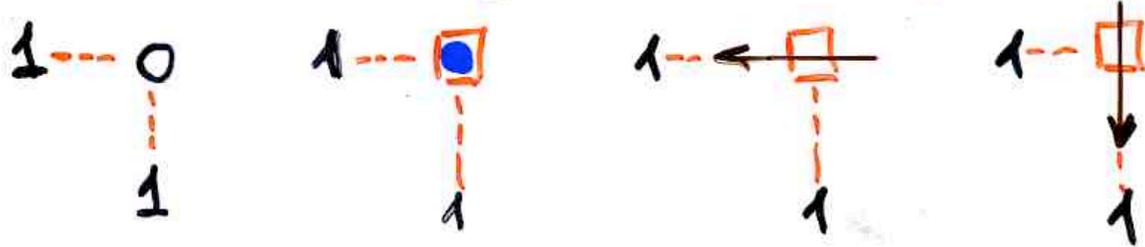
# permutation tableau





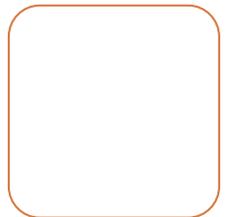
check:  $AT \xrightarrow{\varphi} PT$  size  $(n+1)$

- there exist at least a 1 in each column of  $PT = \varphi(AT)$



impossible

inverse bijection  $\psi = \varphi^{-1}$

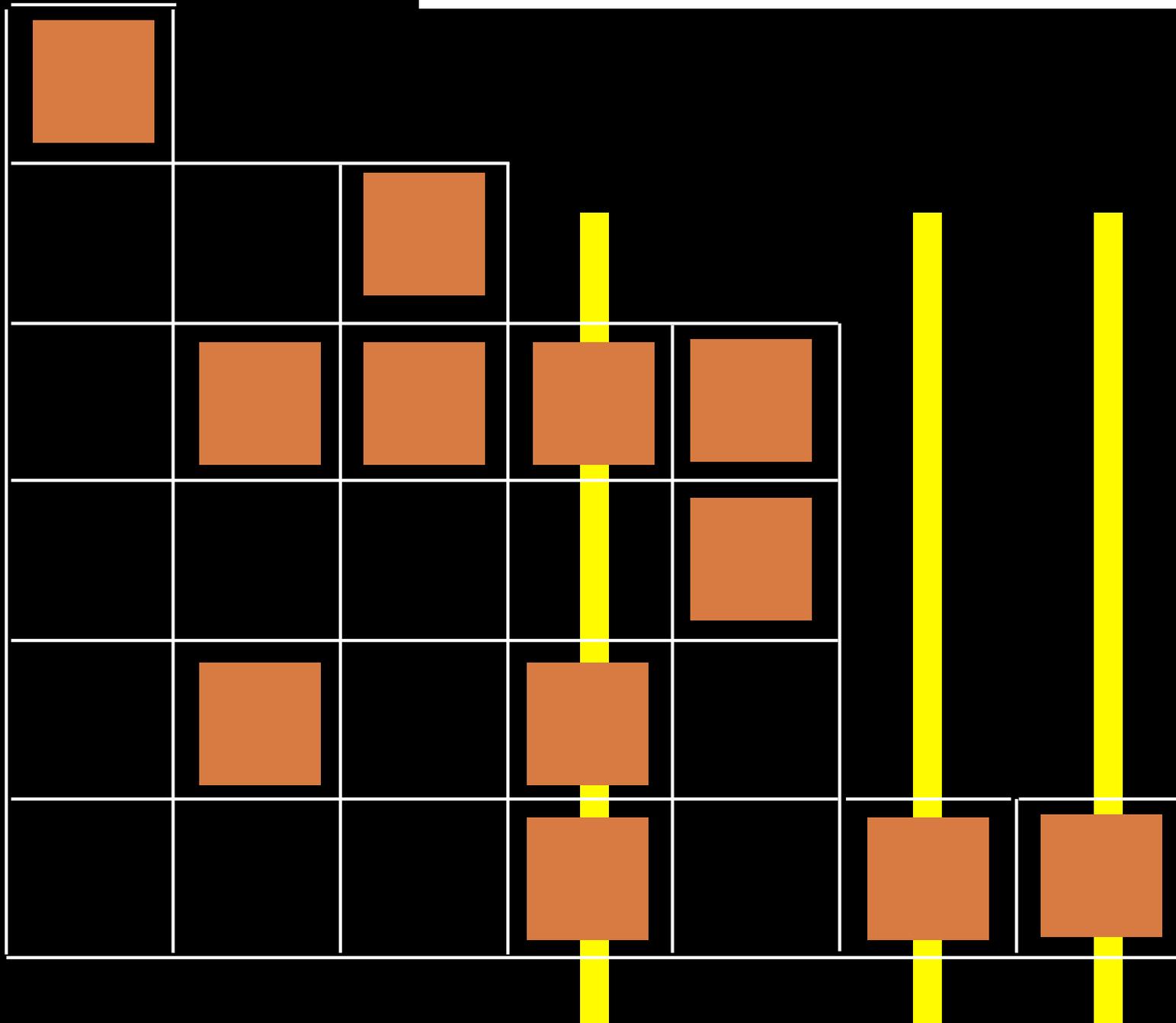


(i) mark the columns with  
a 1 in the first row

1						
		1				
	1	1	1	1		
					1	
	1		1			
			1		1	1

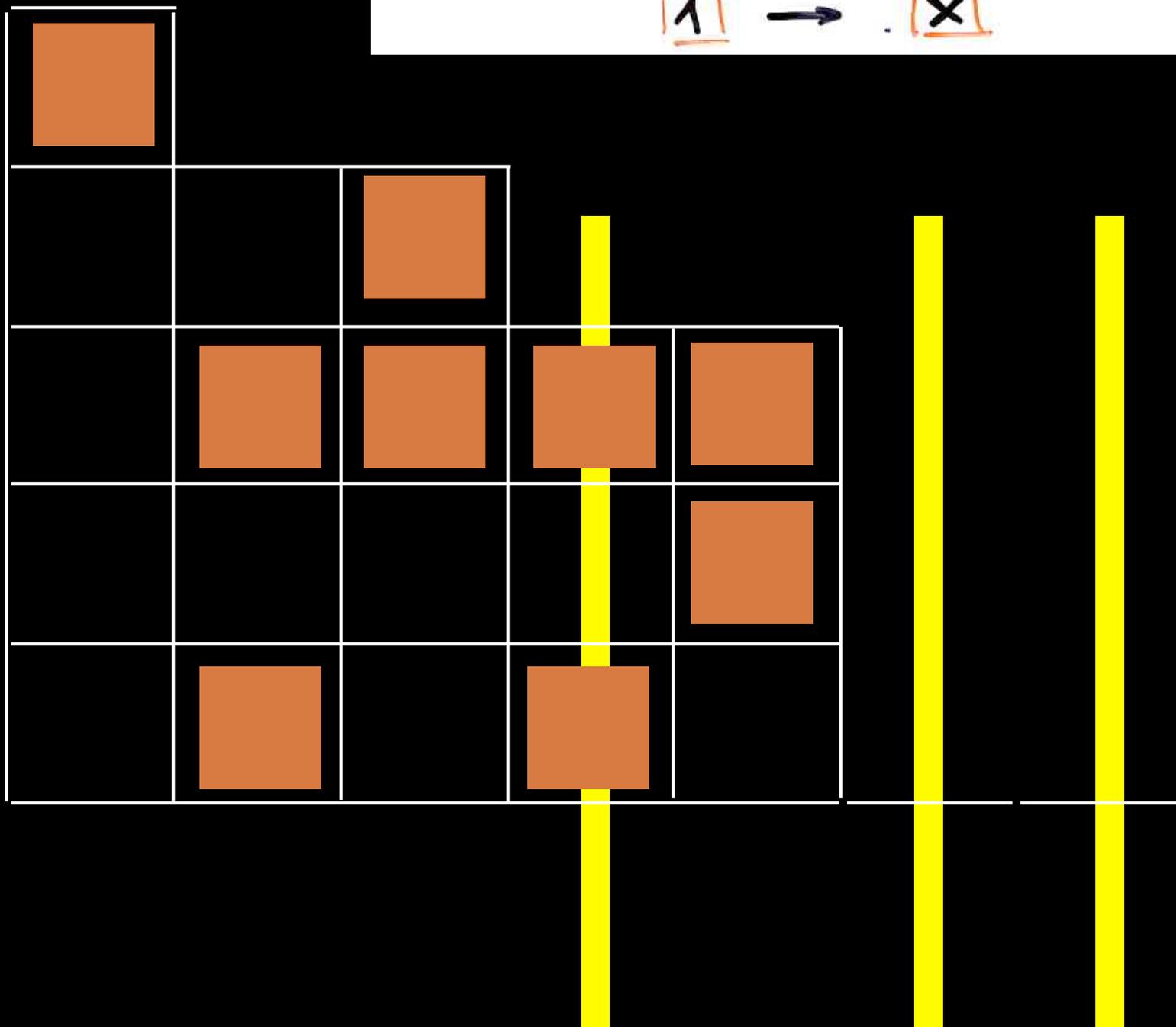
permutation tableau

(ii) delete the first row

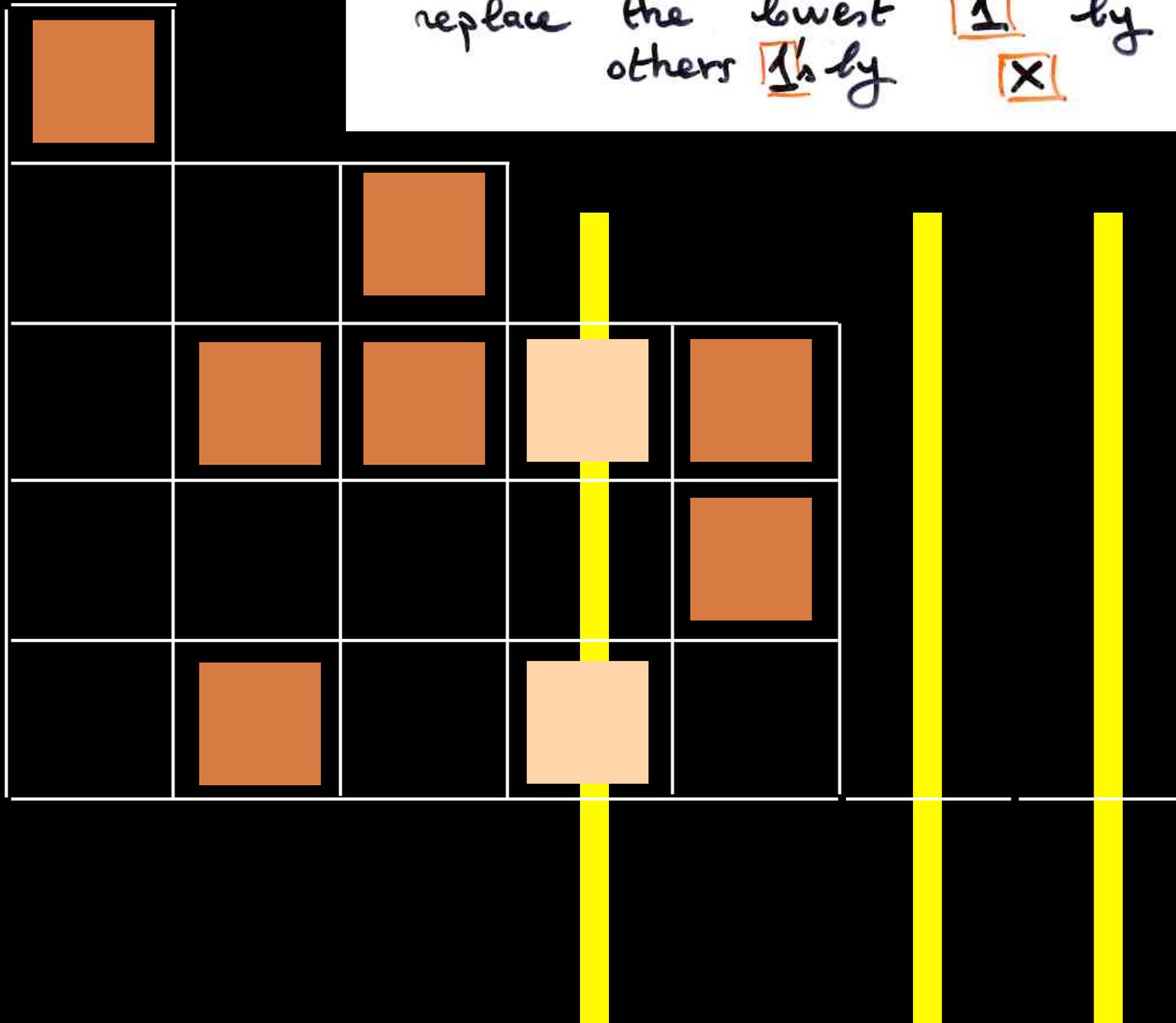


(iii) in each marked column

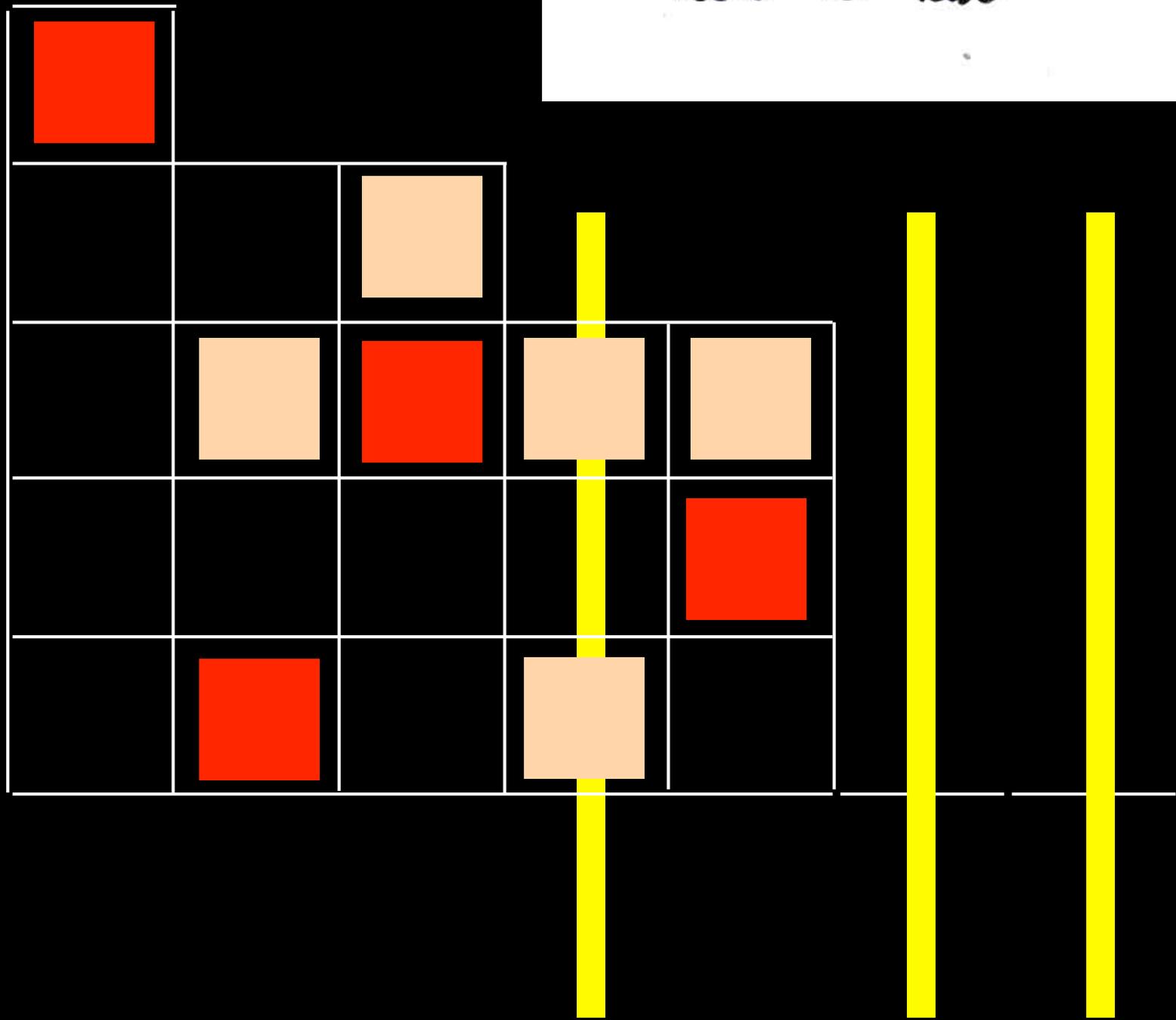
$\boxed{1} \rightarrow \boxed{X}$



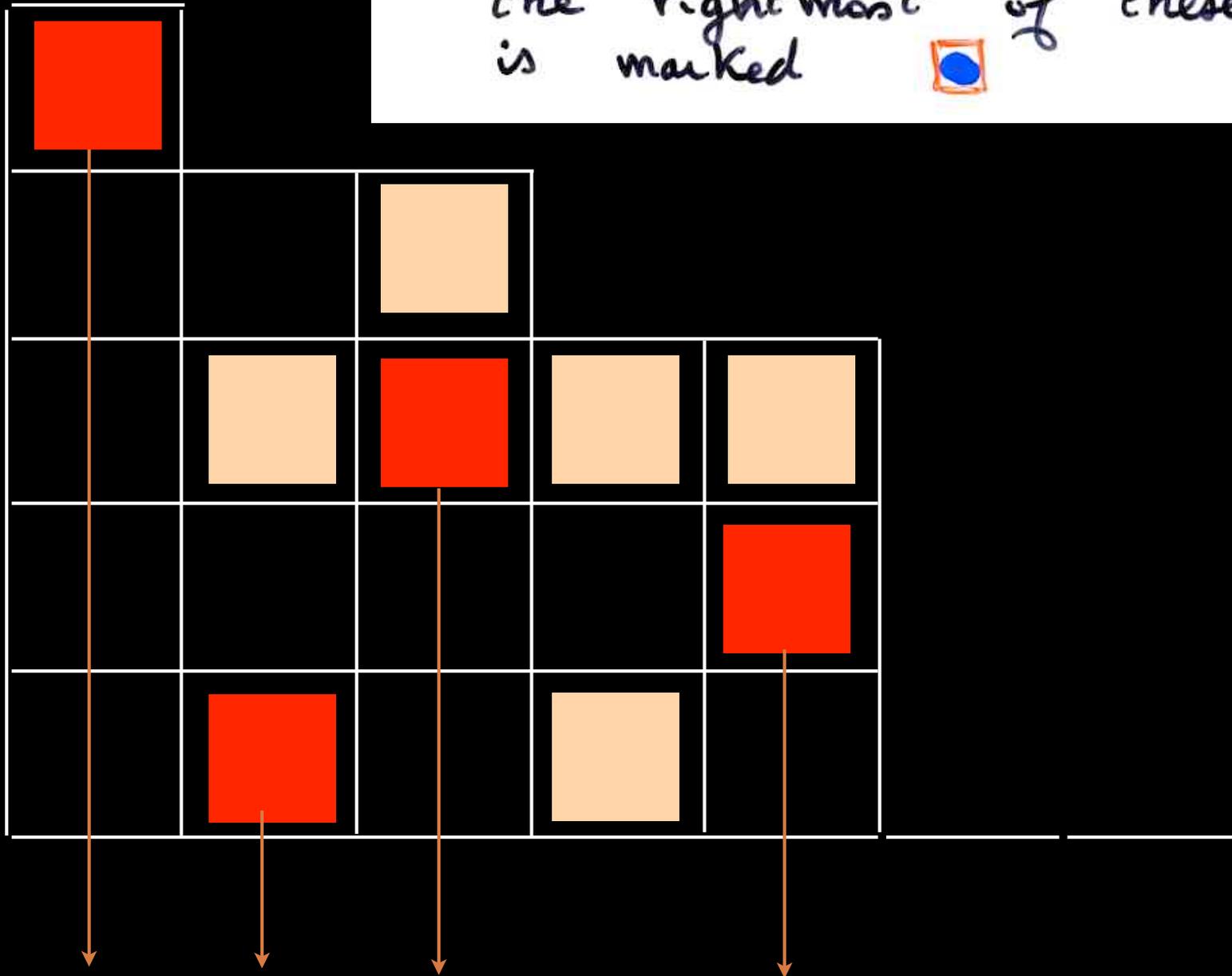
(iv) in each non marked column  
( $\exists$  some cells with 1)  
replace the lowest 1 by 0  
others 1s by X



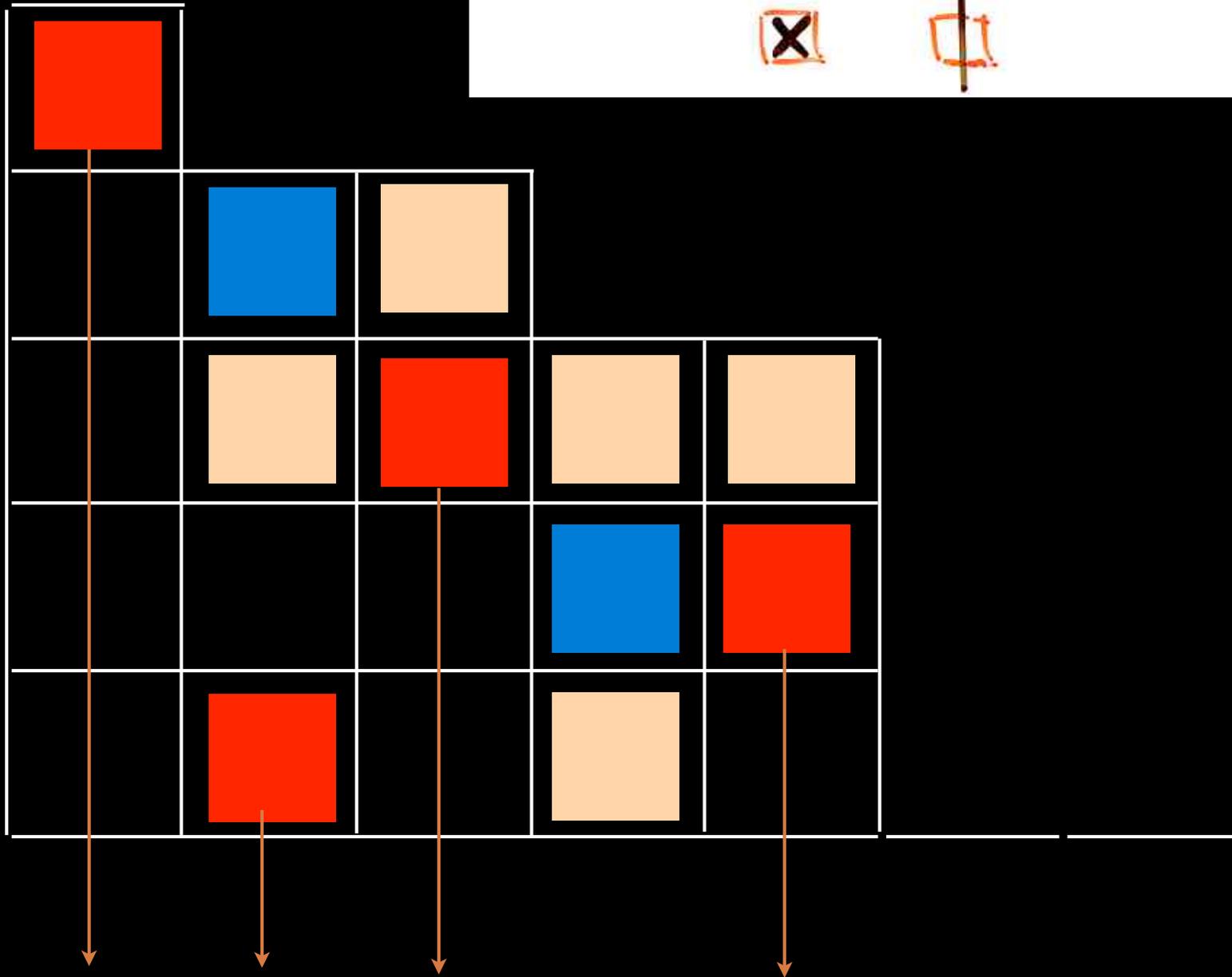
(v) mark the cells below a red



(vi) in each row where there exist empty cells, the rightmost of these cells is marked 



(vii) delete the marks

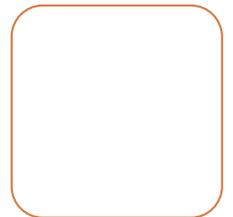


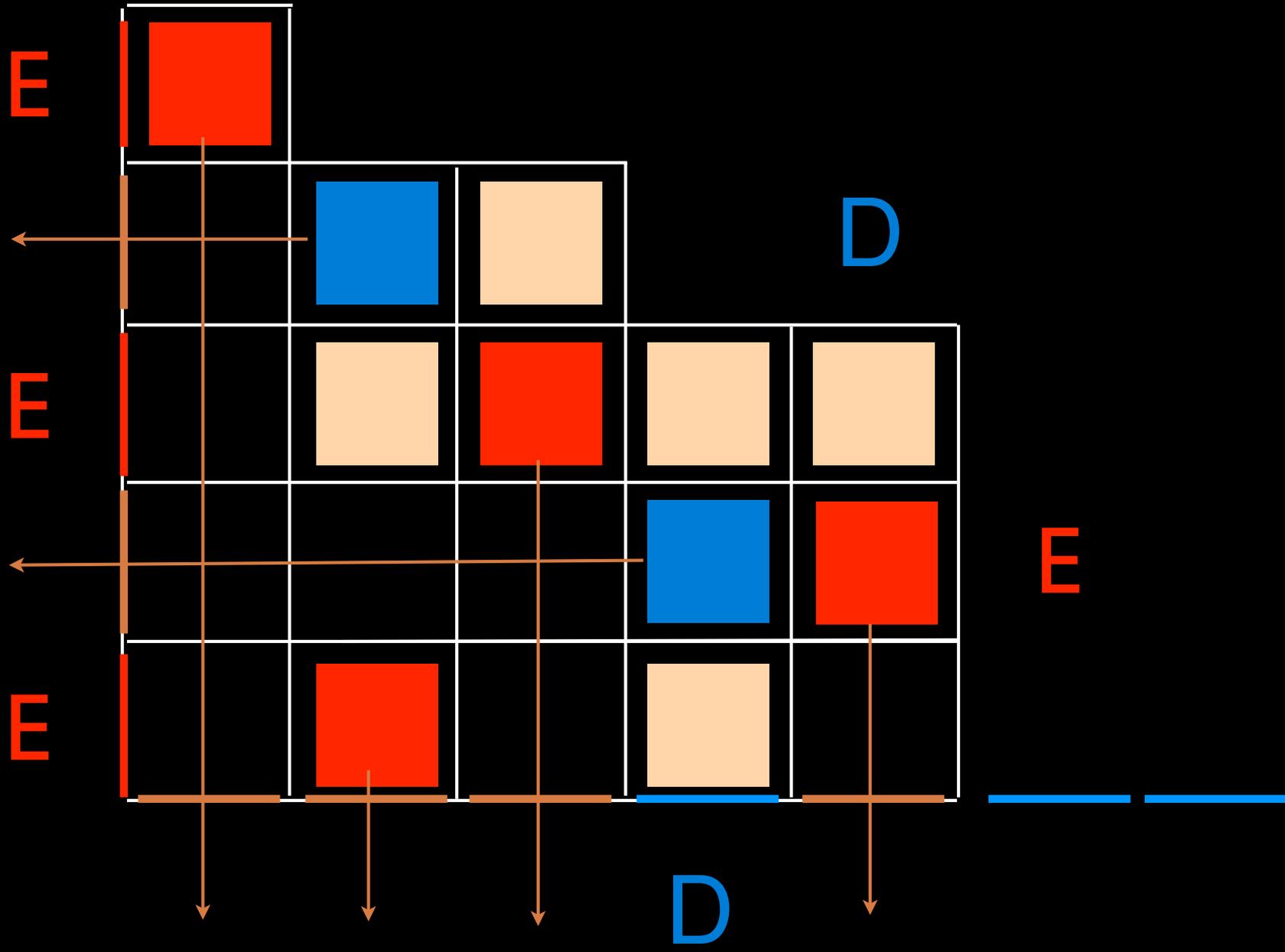
alternative tableau

■					
	■				
		■			
			■	■	
	■				

check

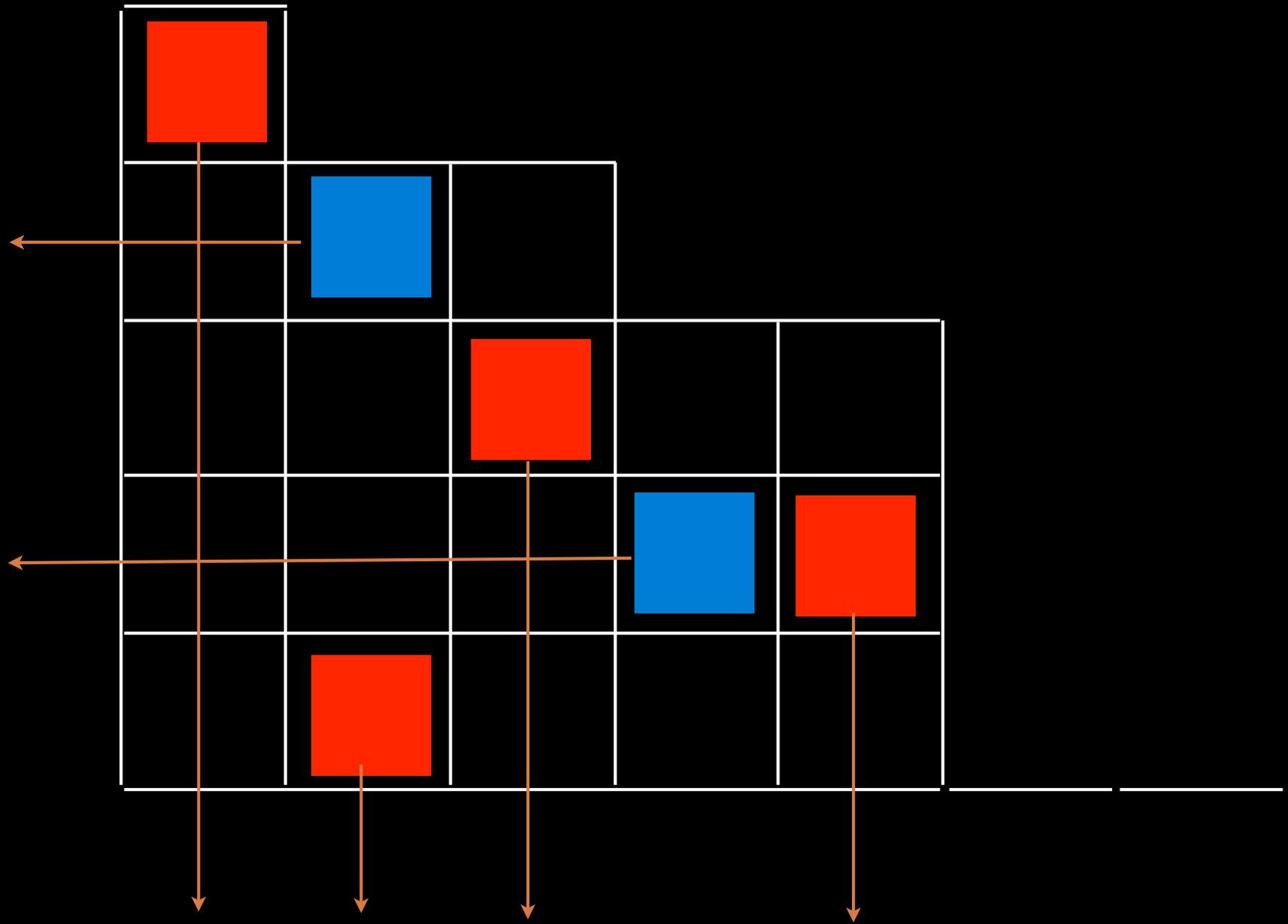
- $\psi$  (PT) is an alternative tableau
- $\psi = \varphi^{-1}$

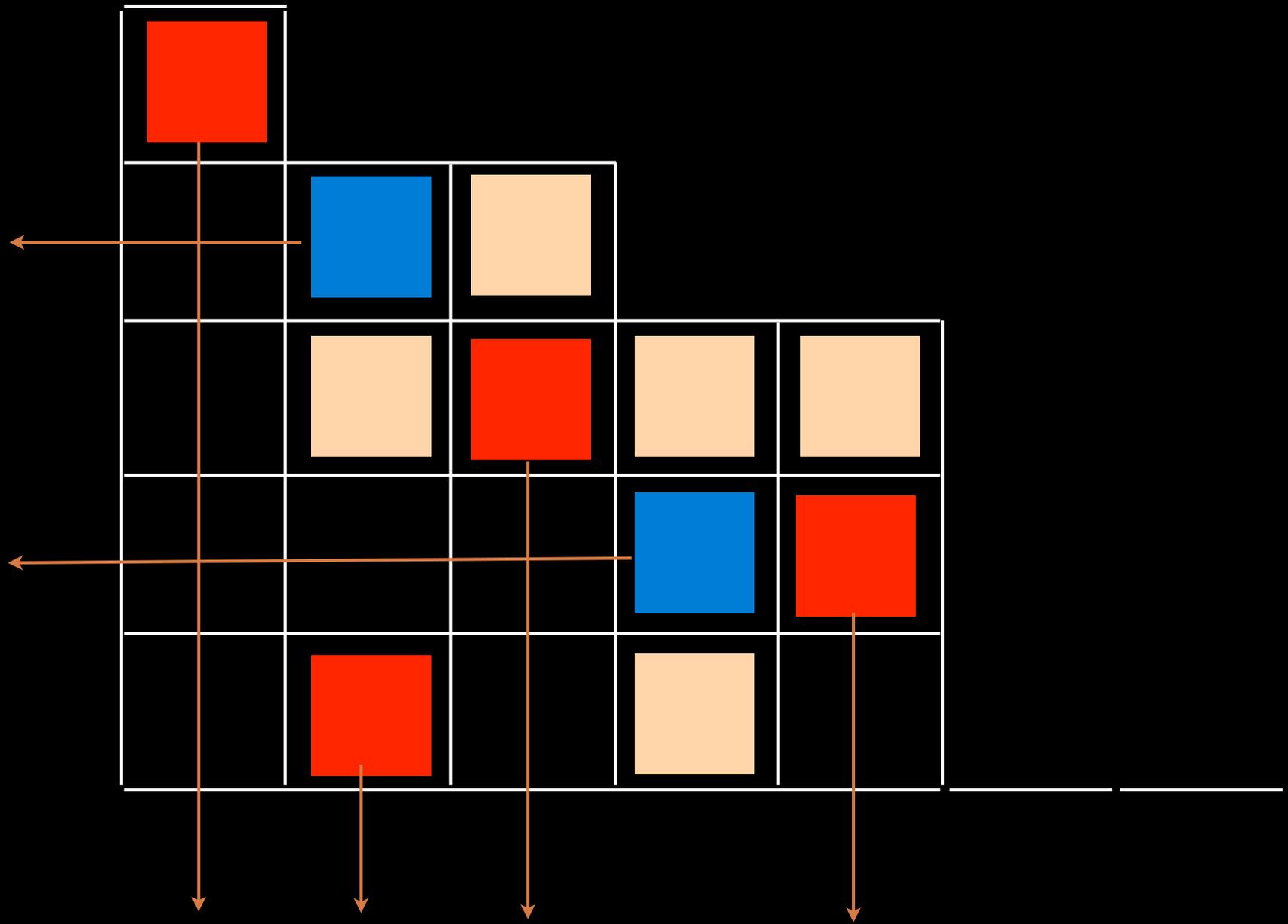




## 2. Tableaux alternatifs de Catalan: caractérisation

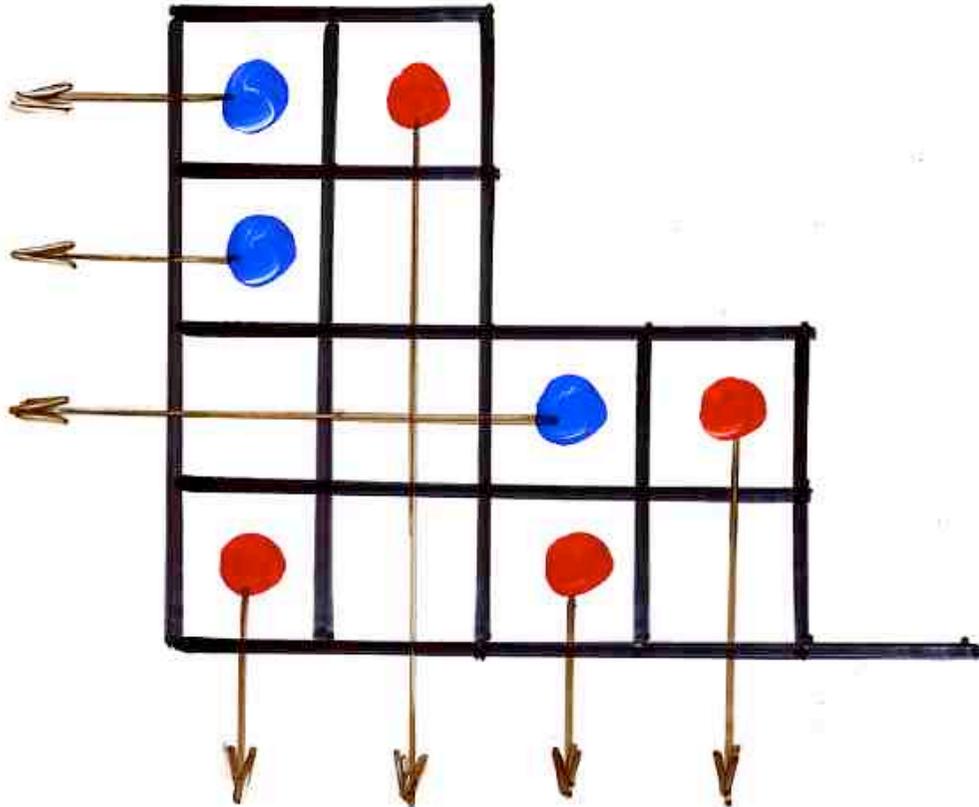
Montrer que la connaissance des cellules rouges (resp. bleues) d'un tableau alternatif de Catalan permet de retrouver ce tableau. Donner une caractérisation de ces cellules rouges d'un tableau alternatif de Catalan en termes de motif interdit (il y a ici une odeur de tableaux de permutations).

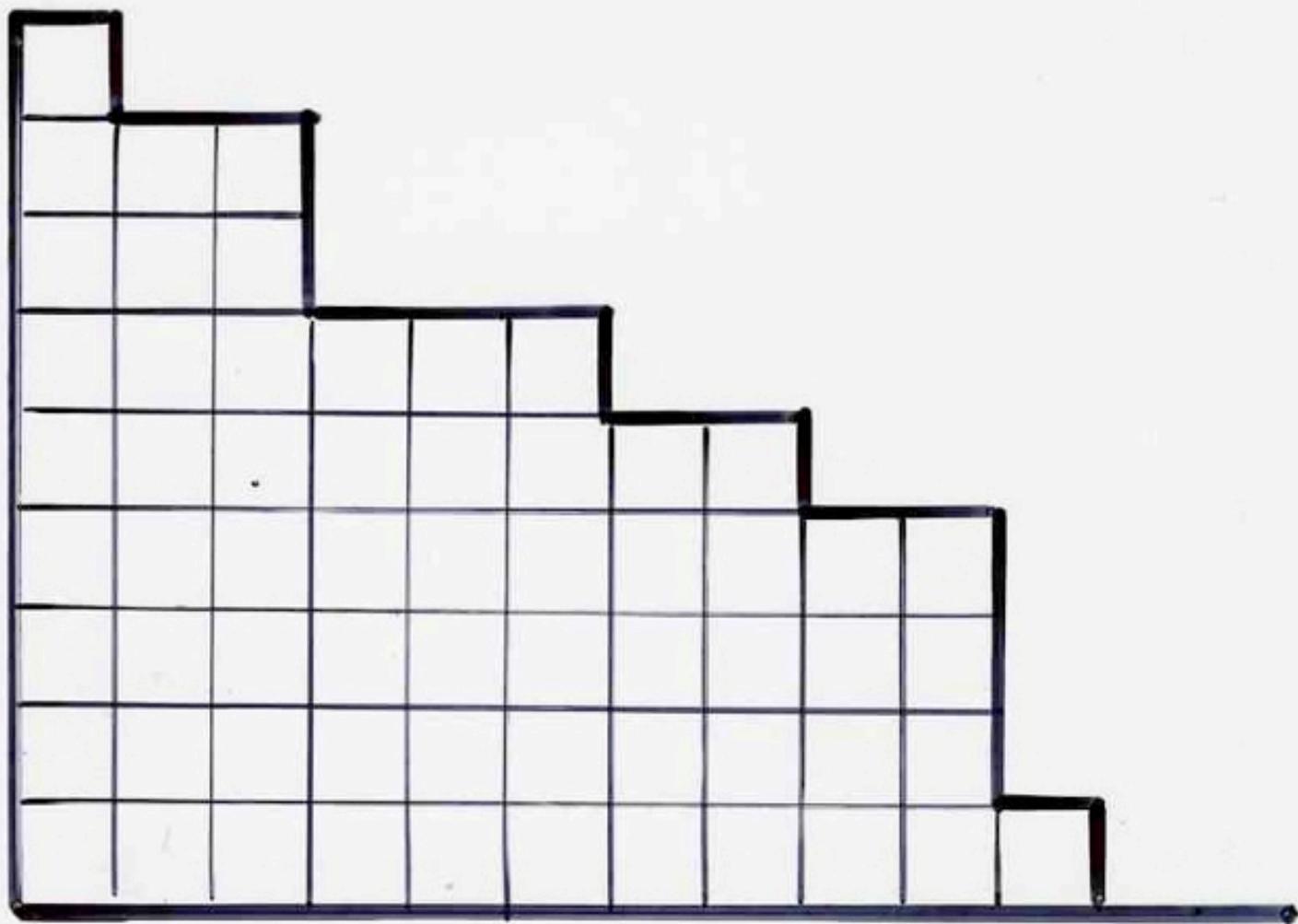


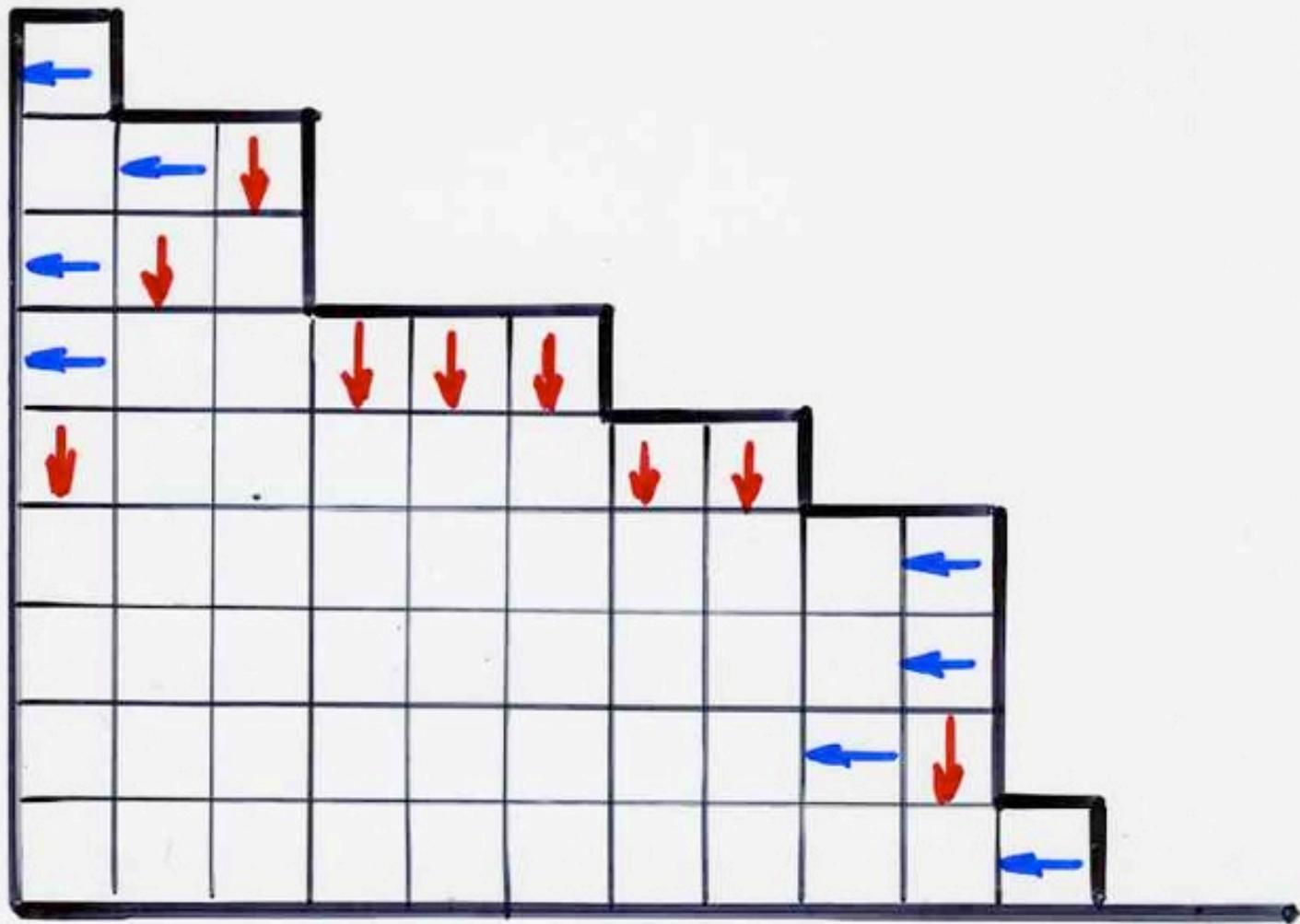


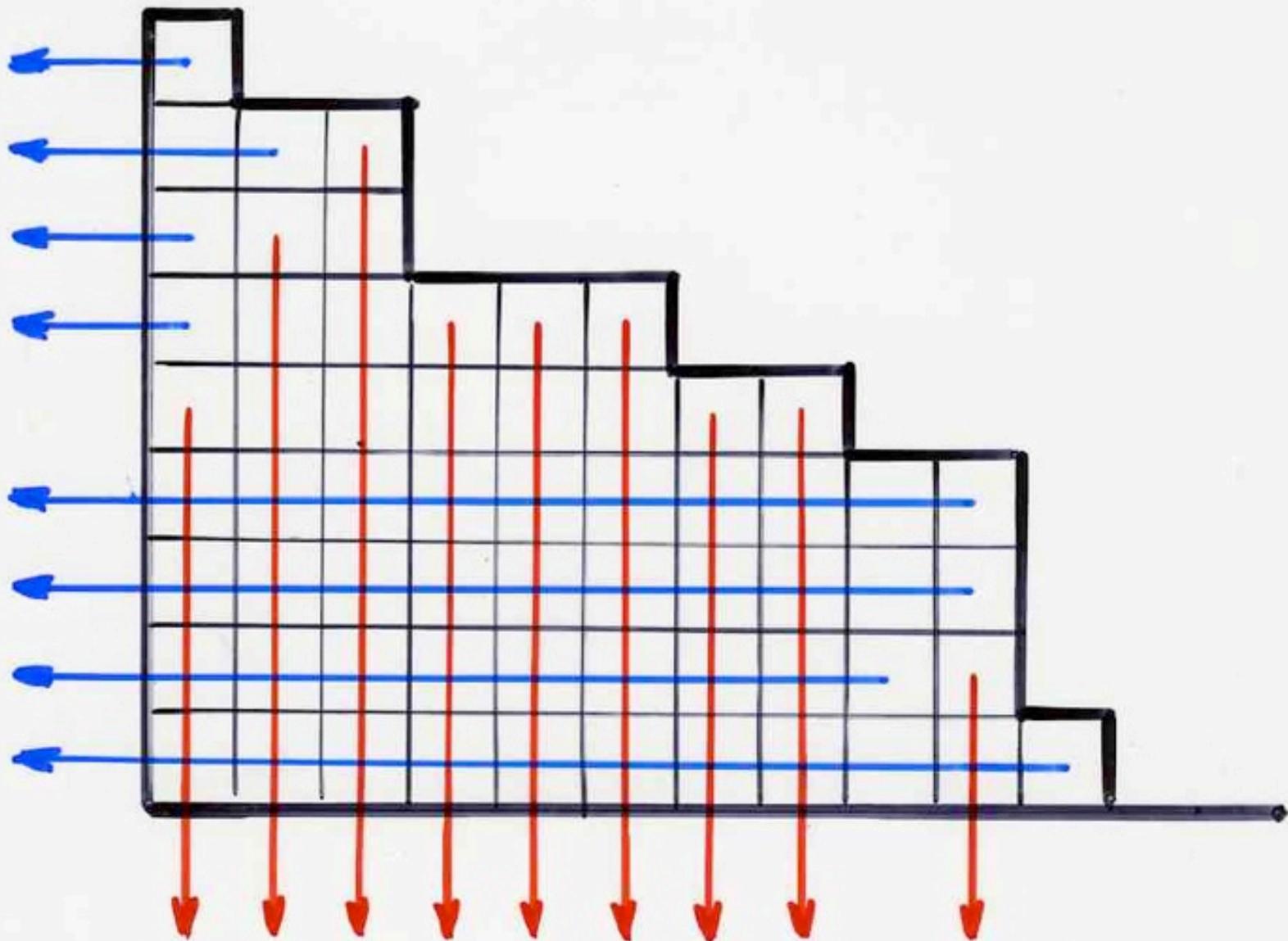
Def Catalan alternative tableau  $T$   
alt. tab. without cells  $\boxed{\times}$

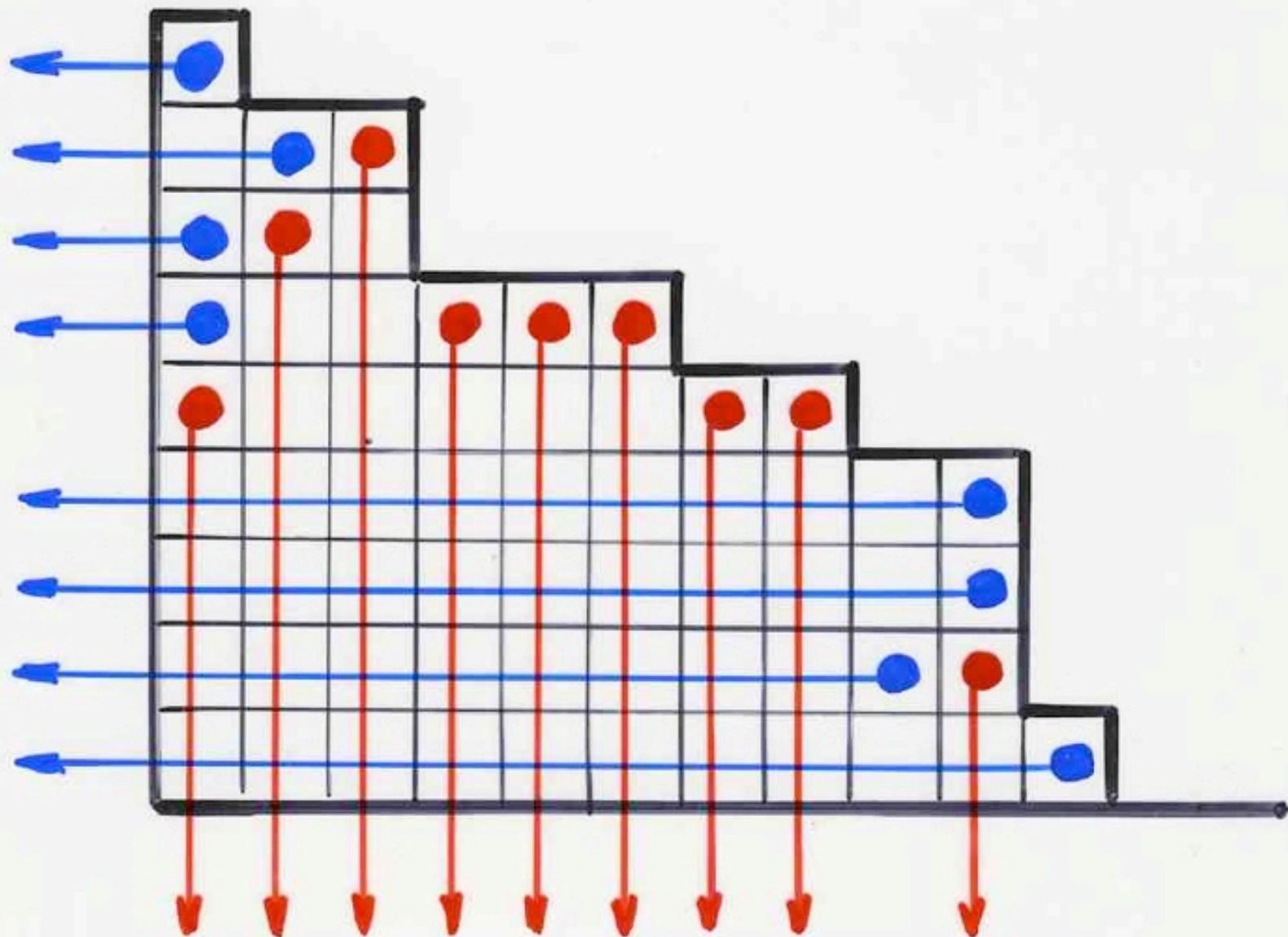
i.e. every empty cell is below a red cell or  
on the left of a blue cell





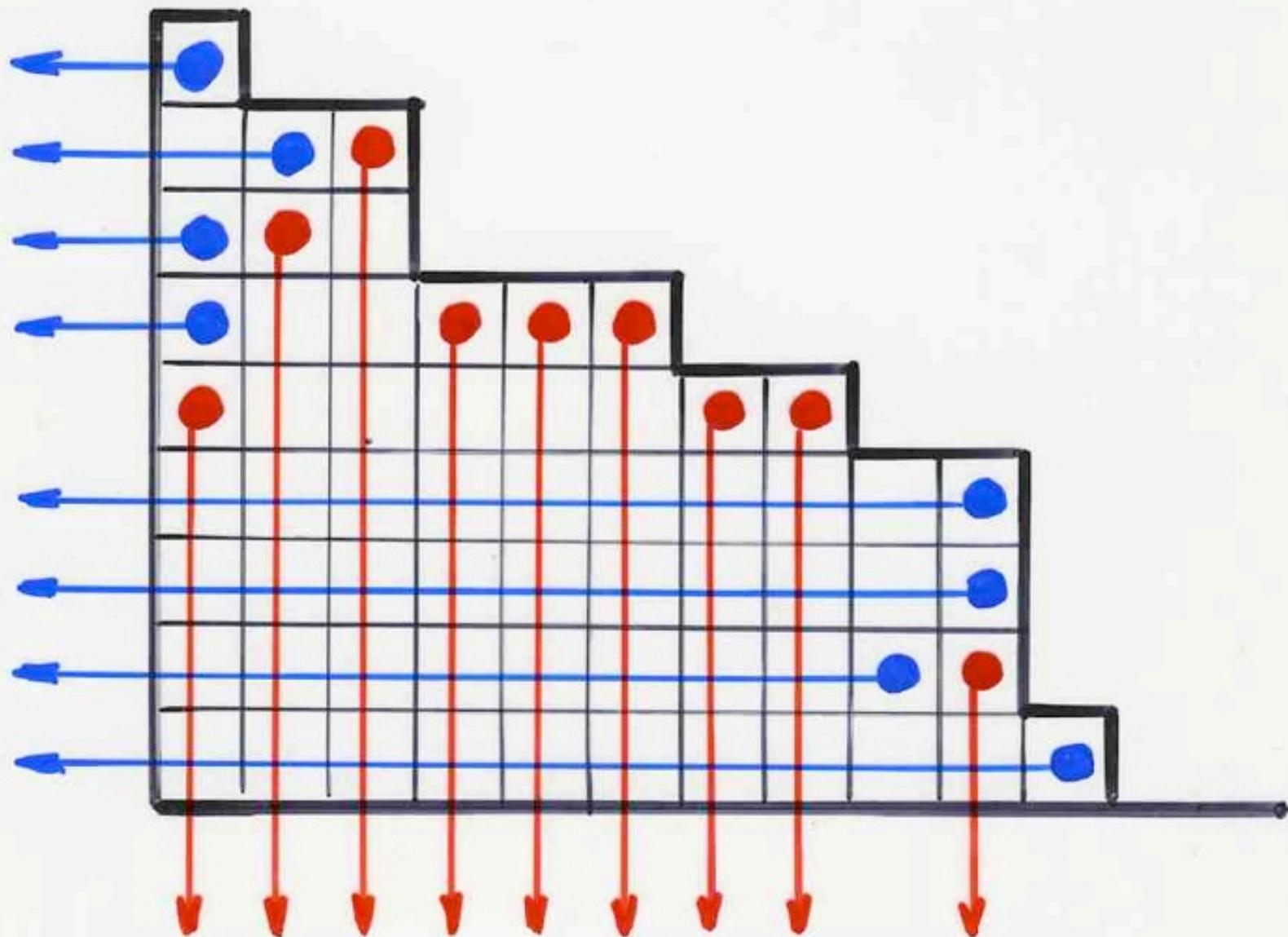


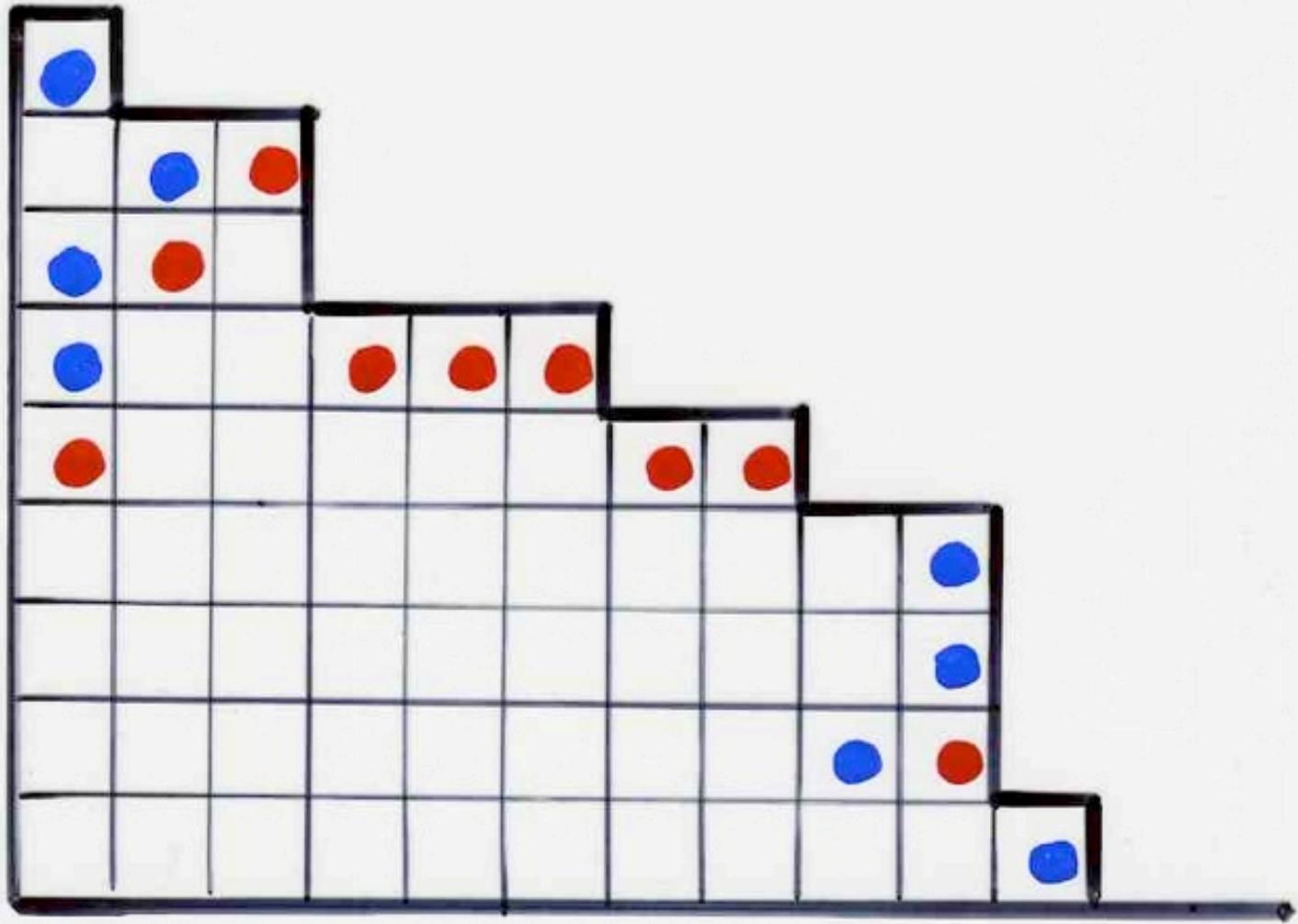






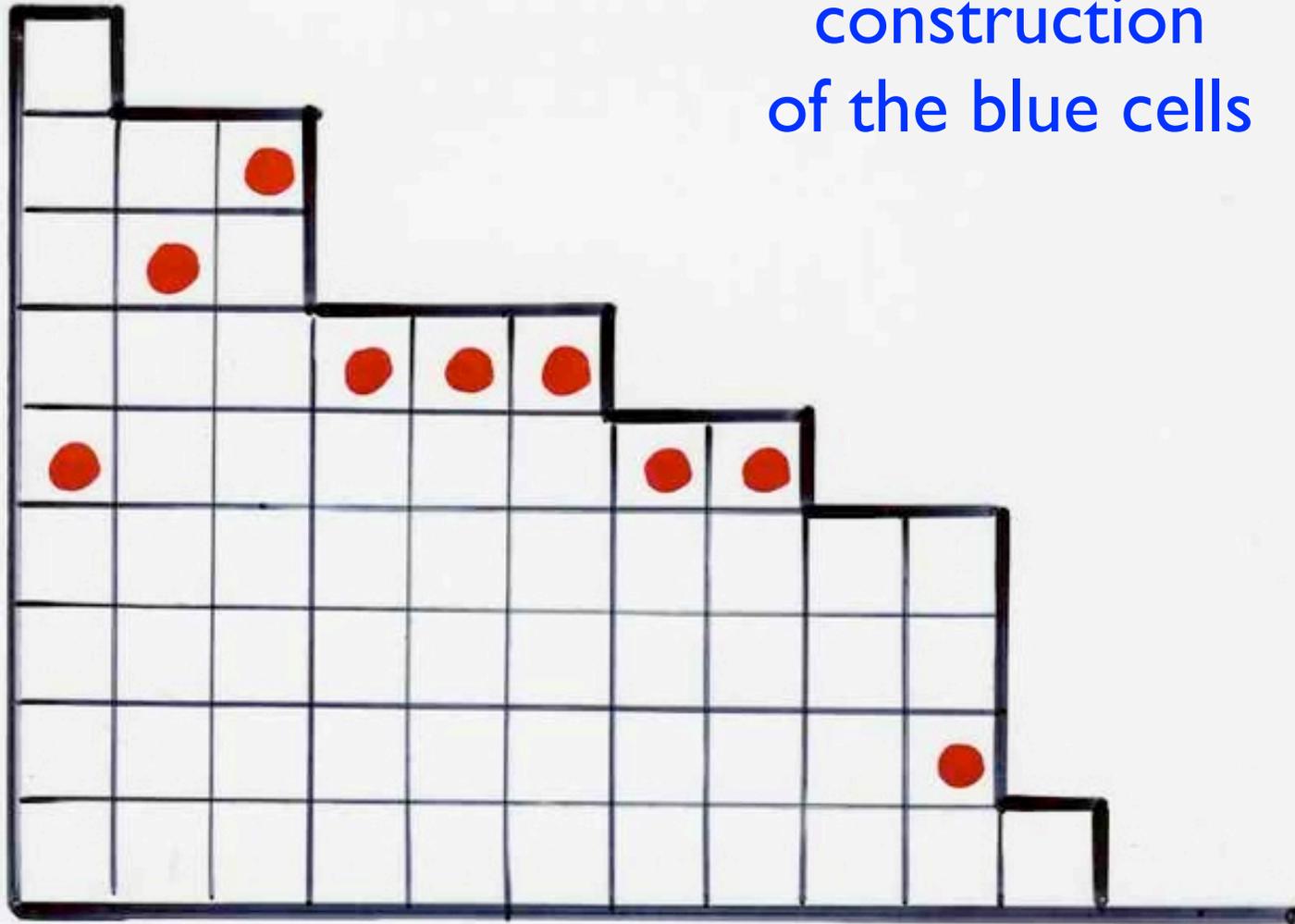
les cellules rouges  
caractérisent le tableau

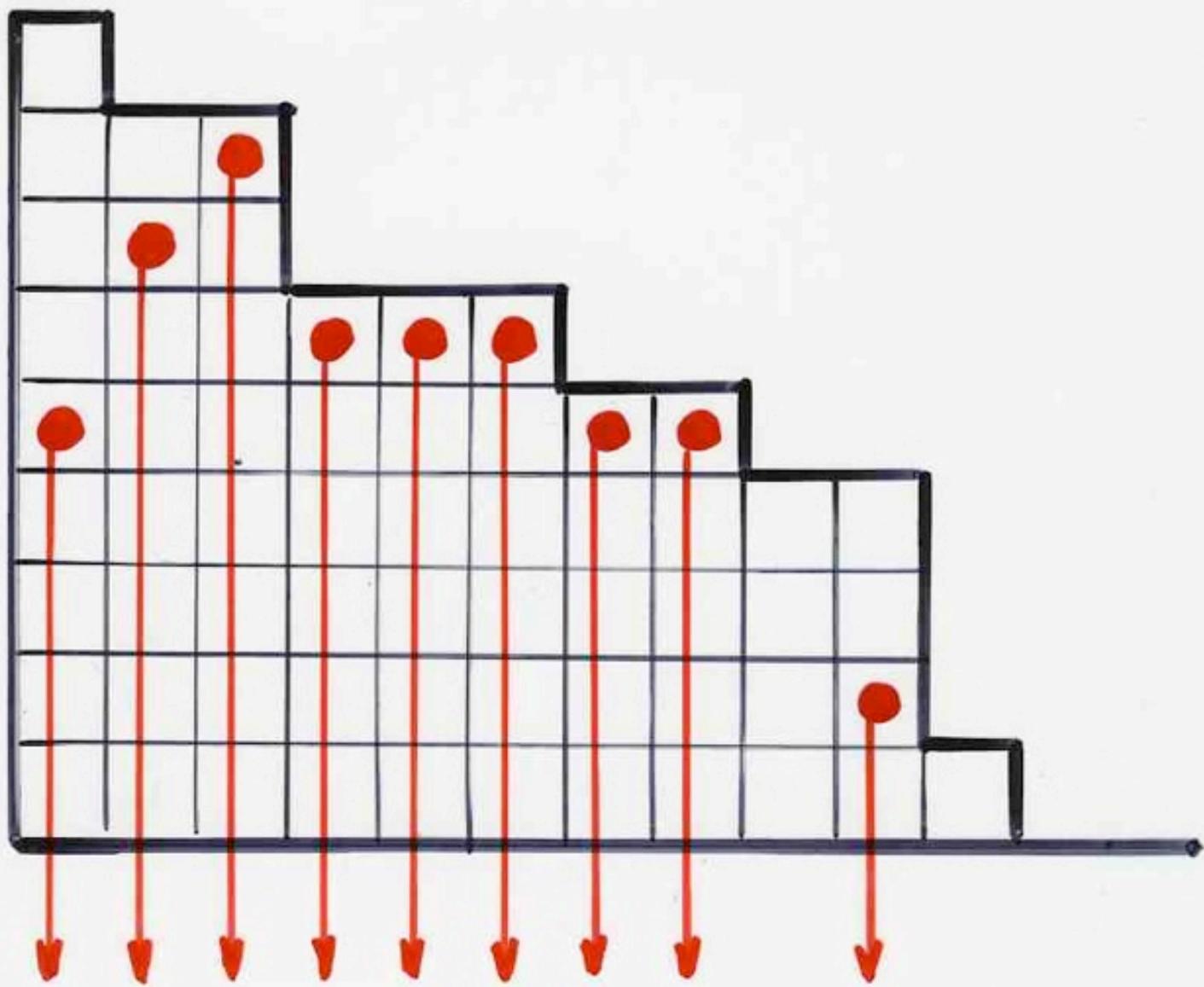


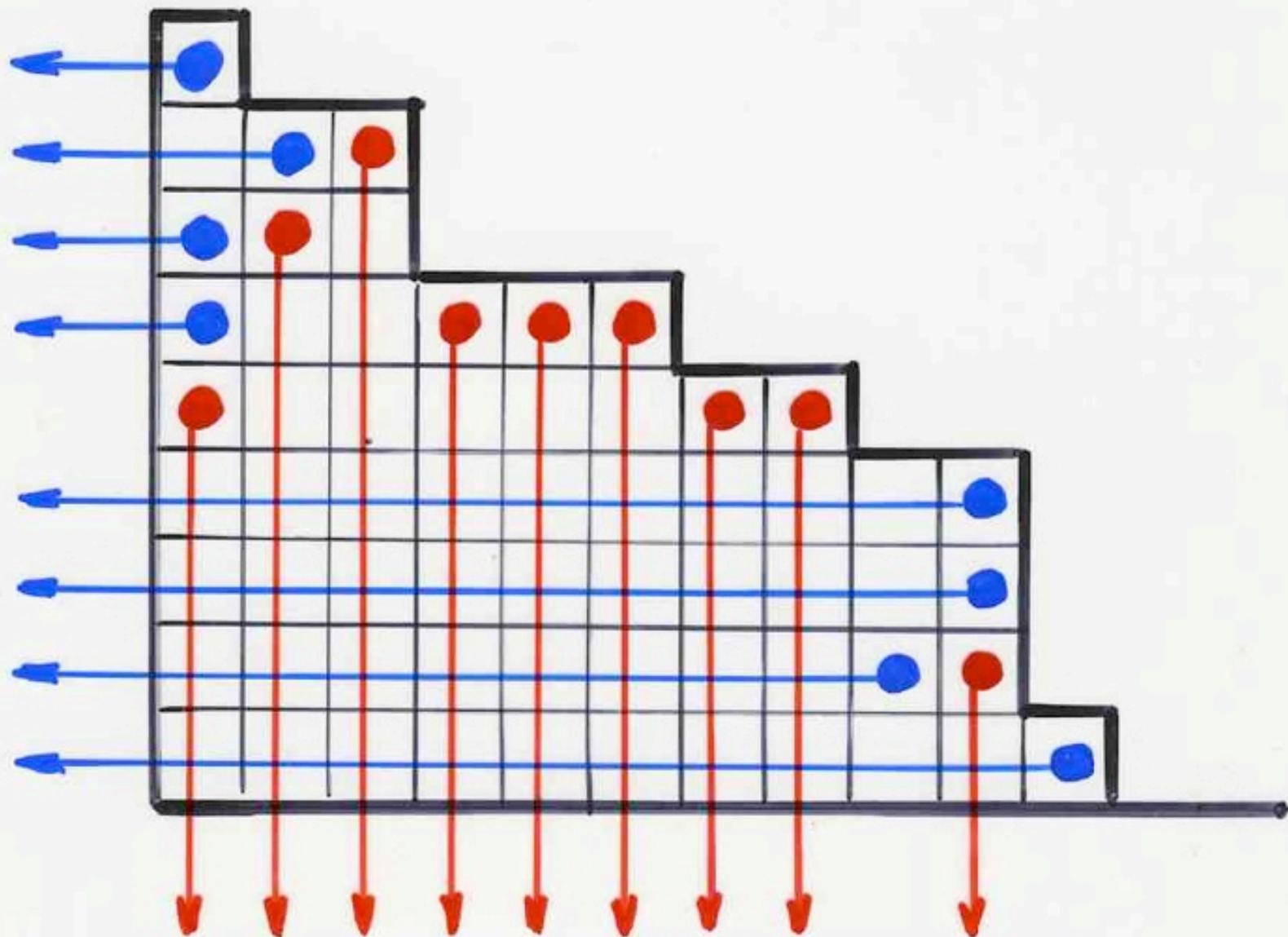


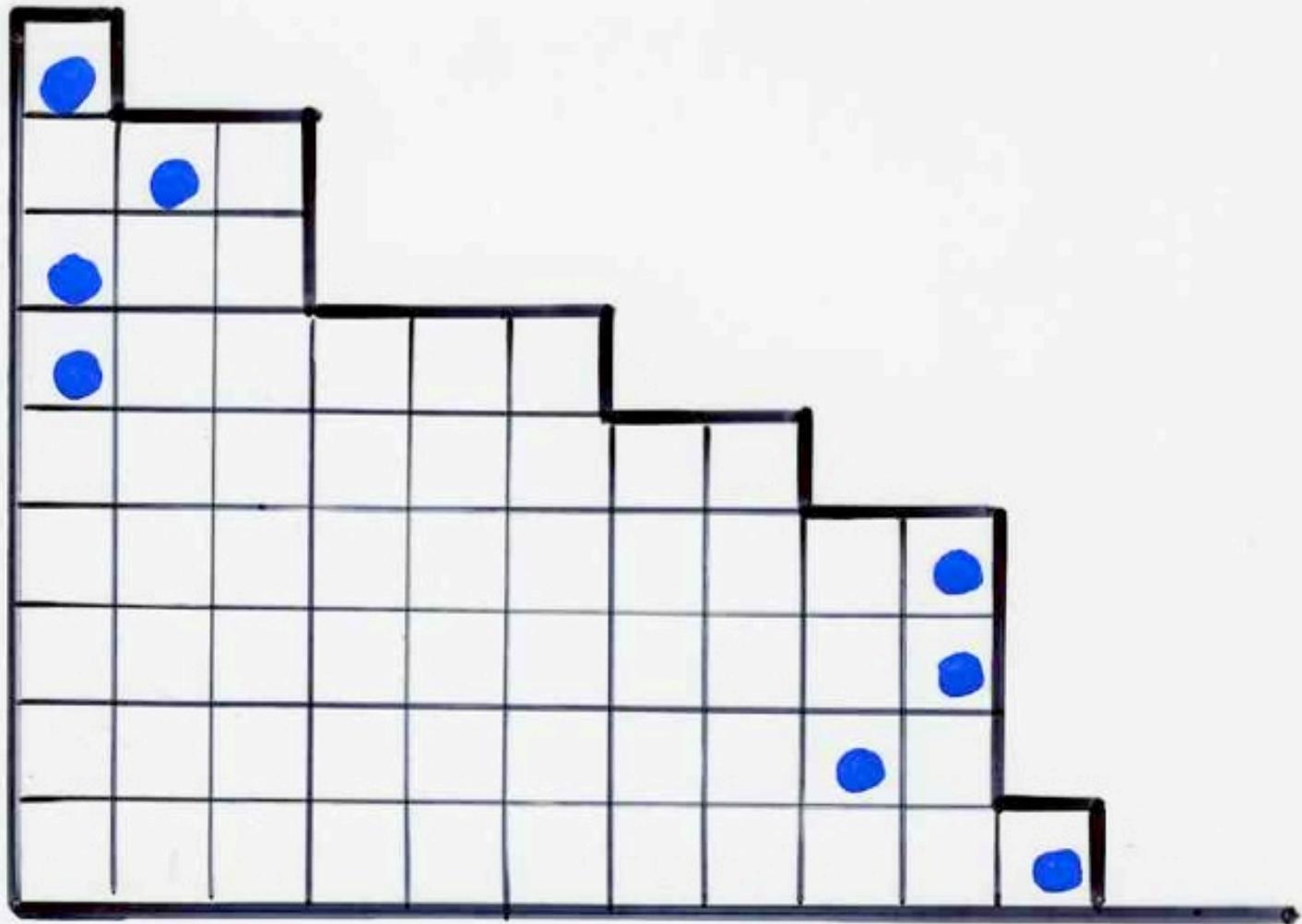


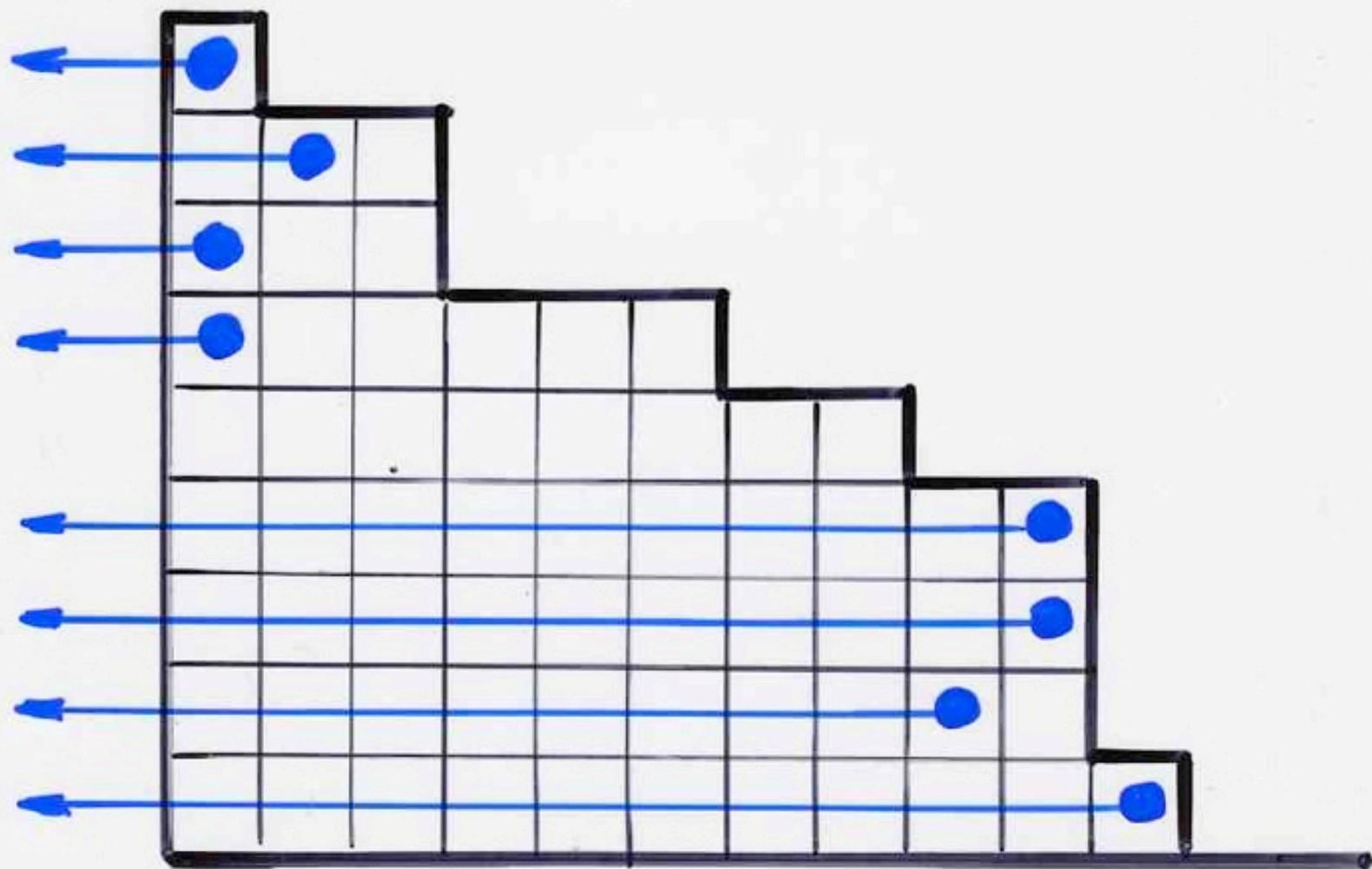
construction  
of the blue cells

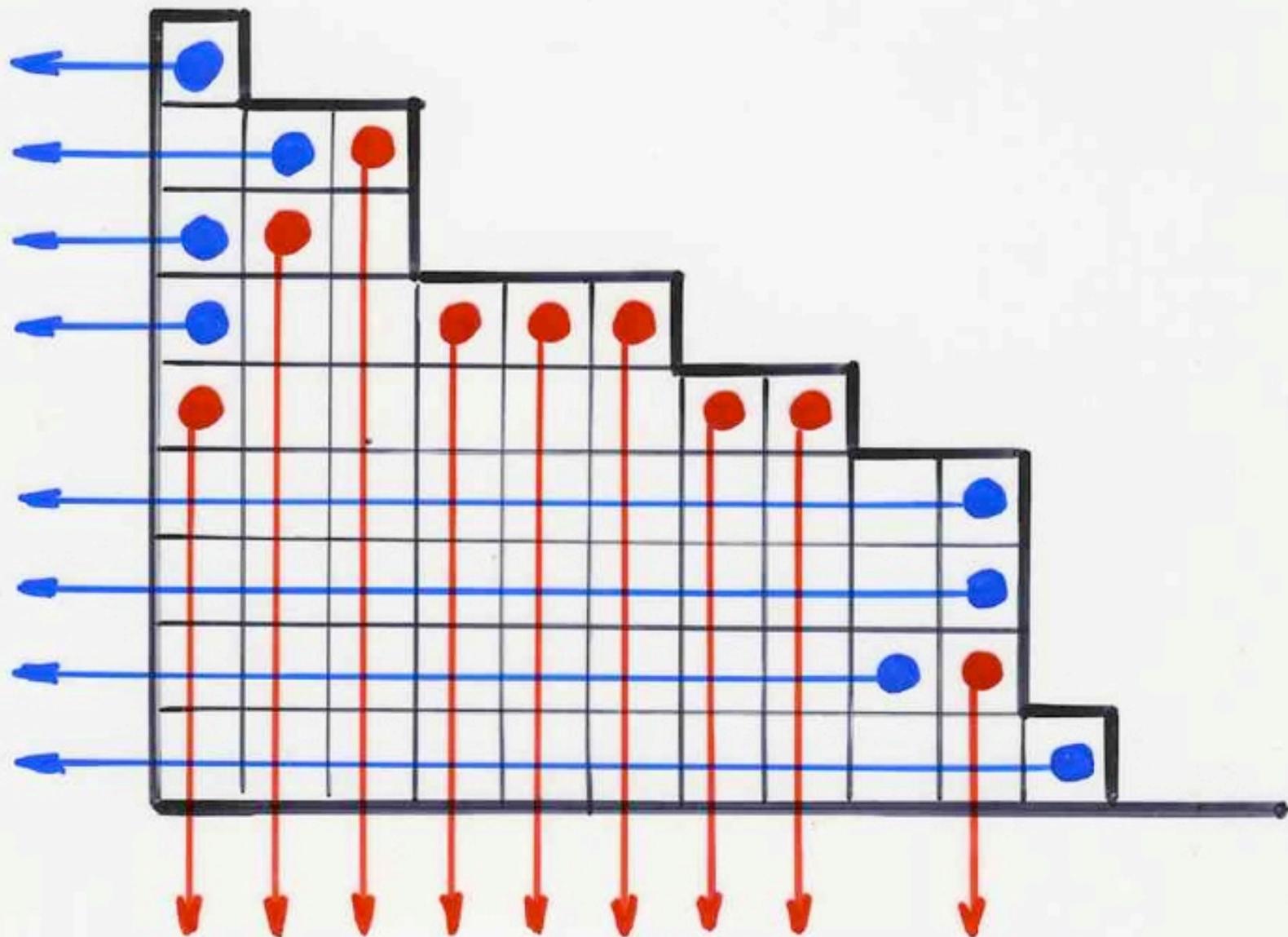


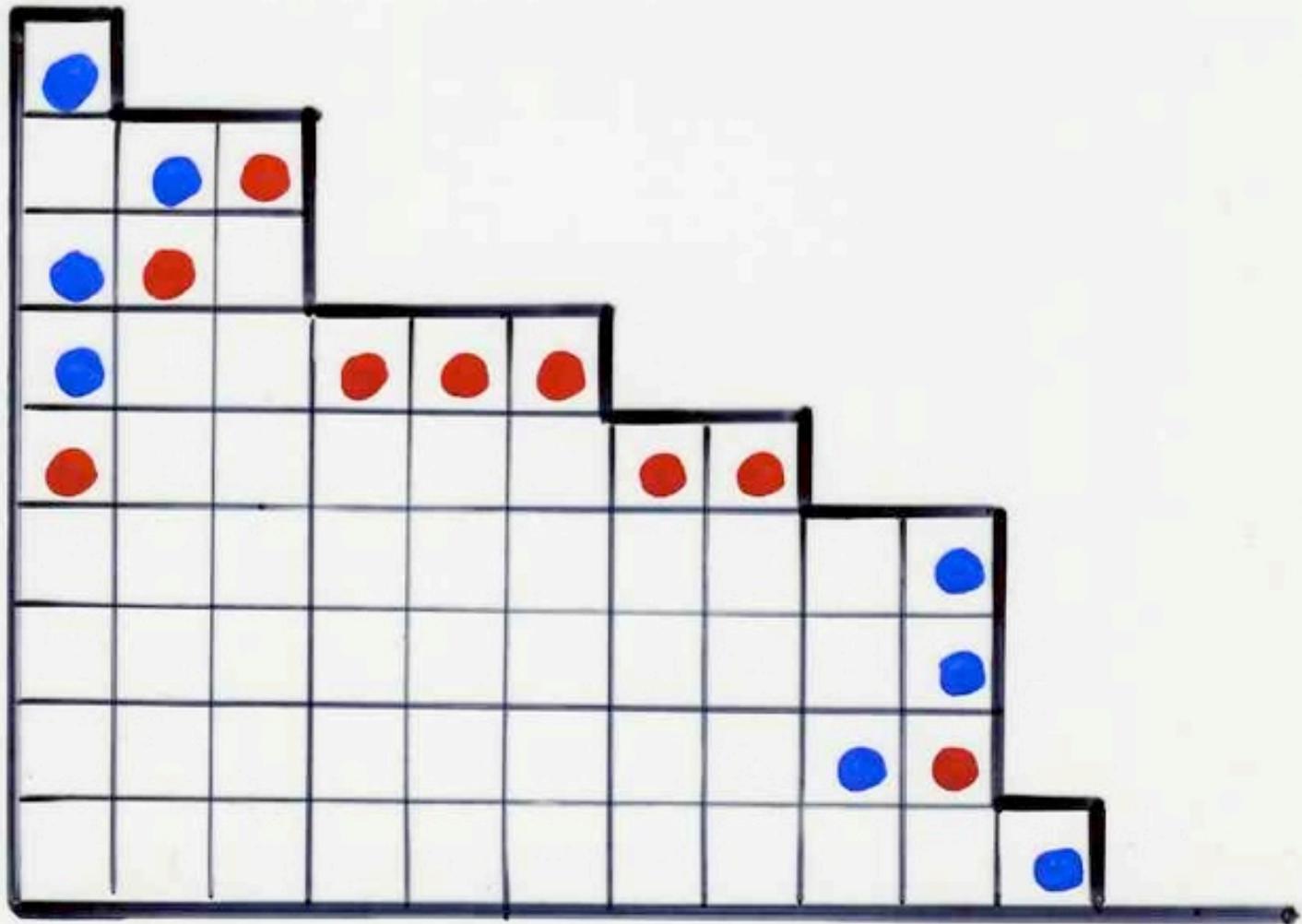




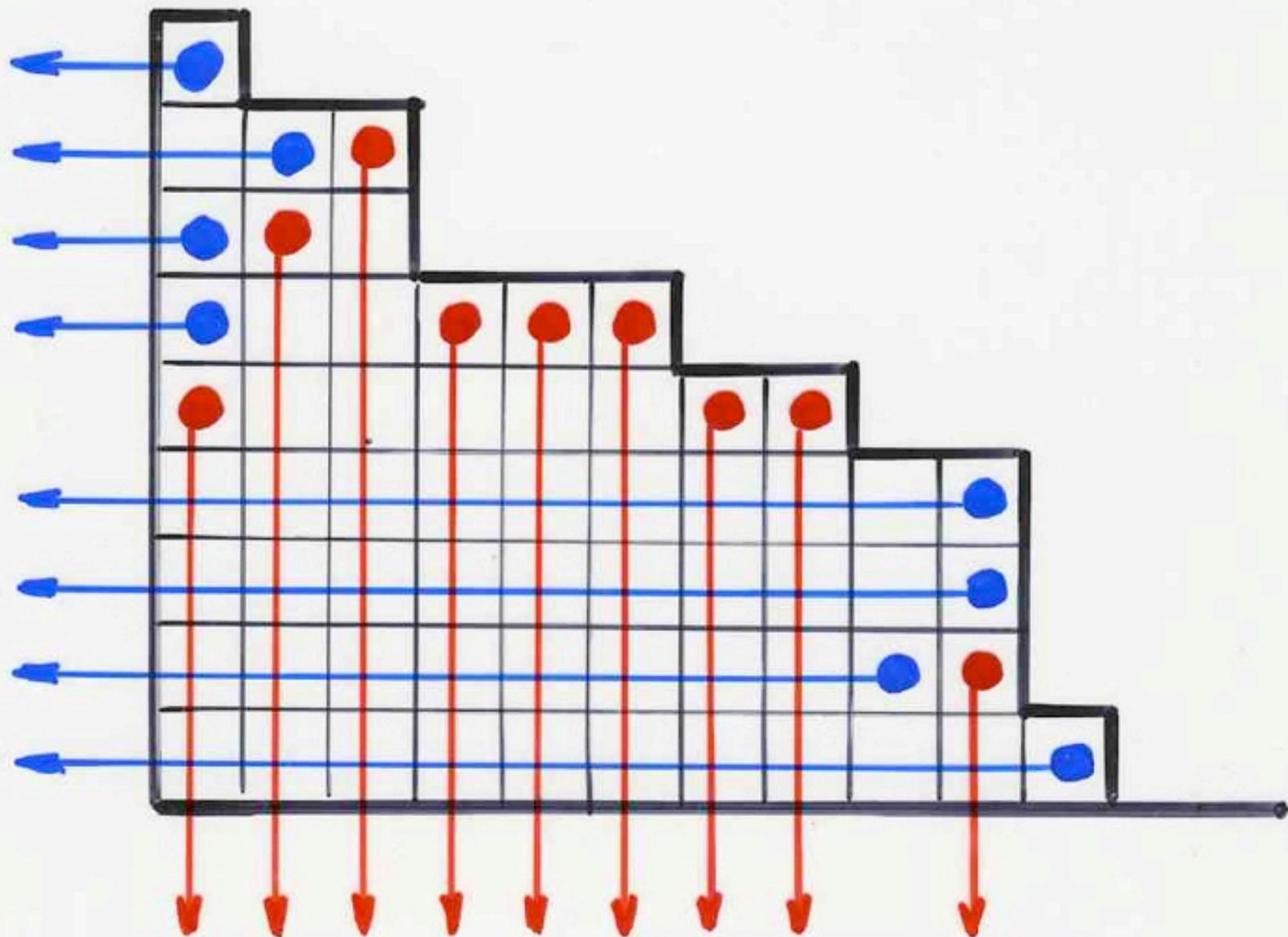




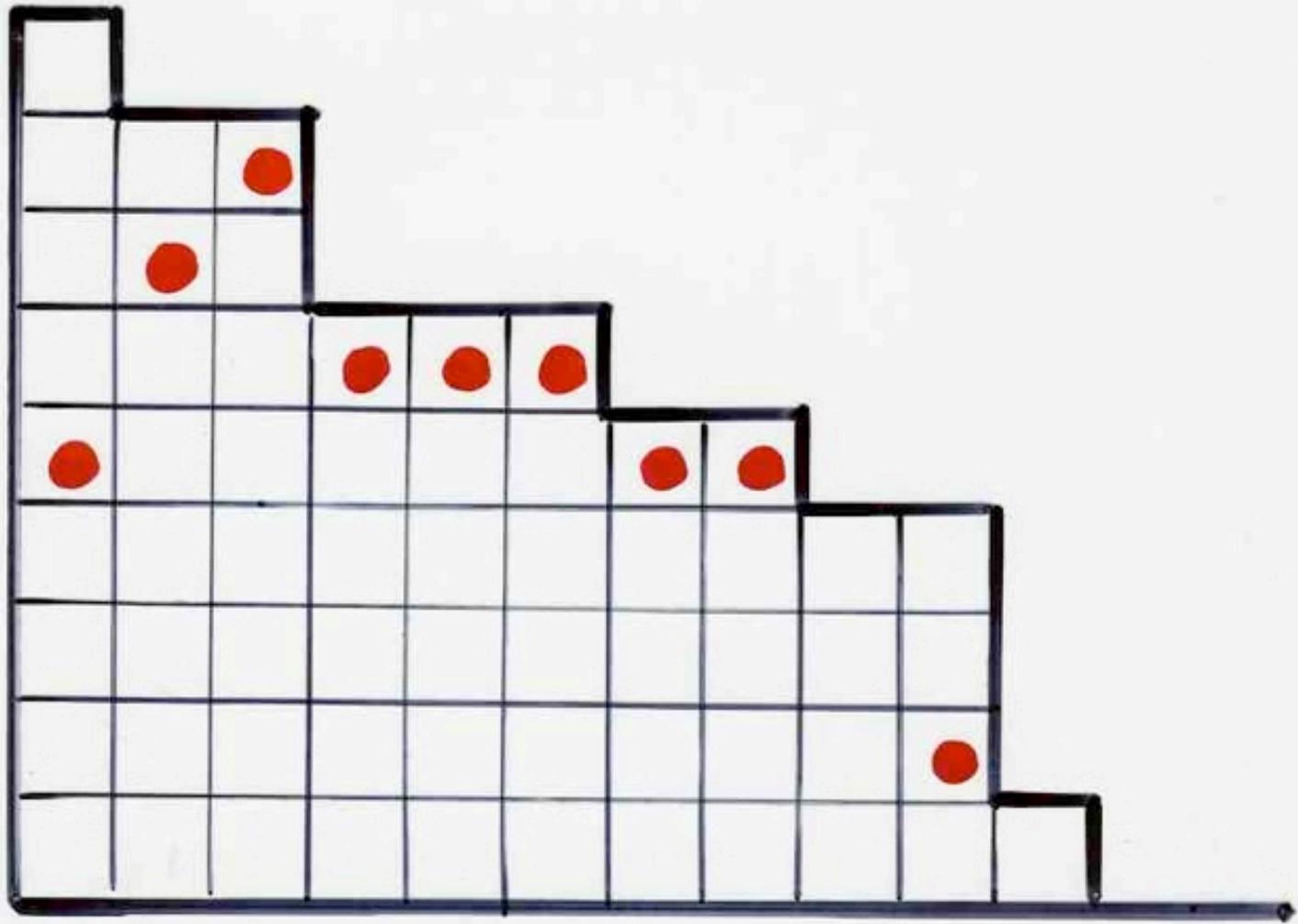




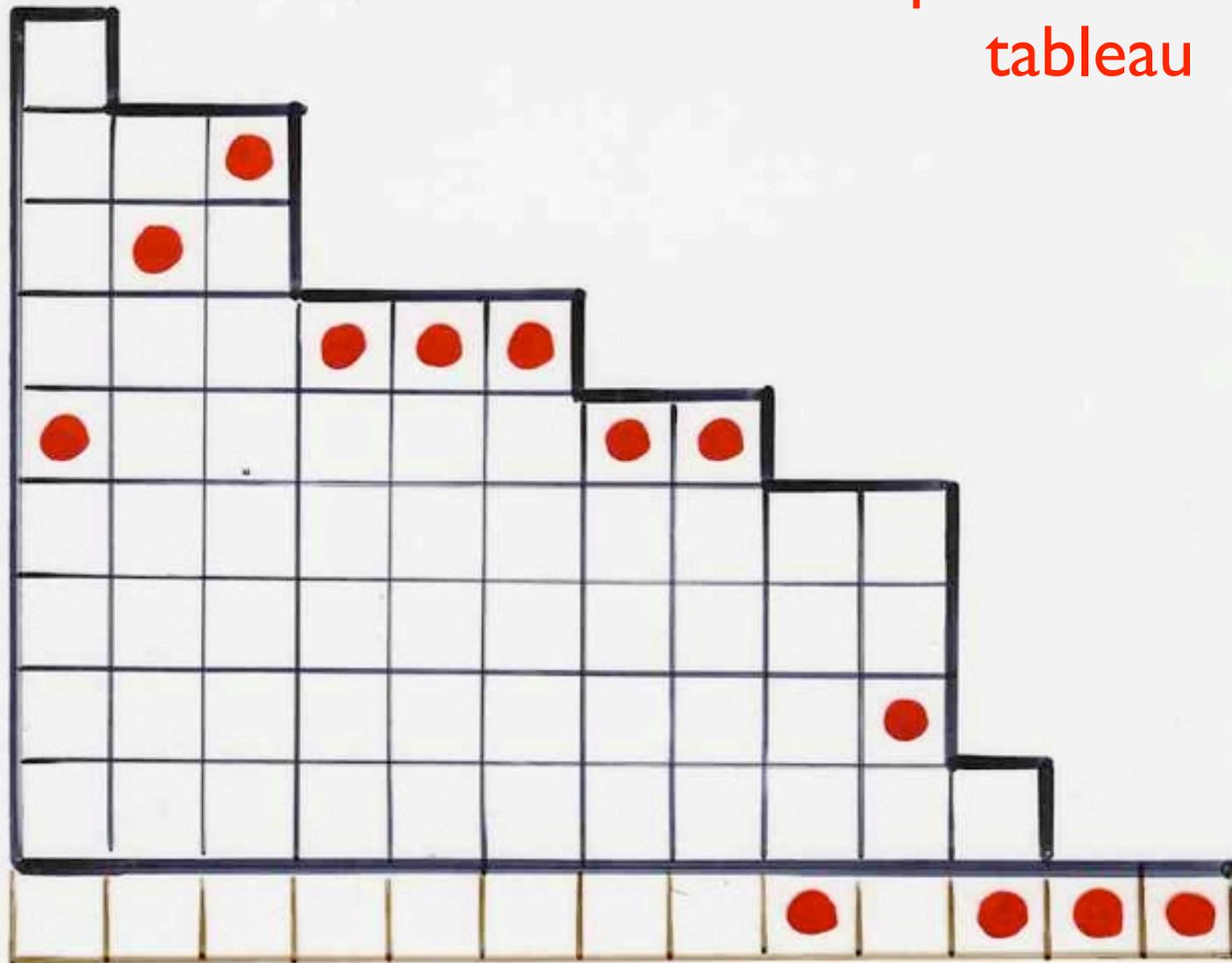
Caractérisation  
des tableaux alternatifs de Catalan

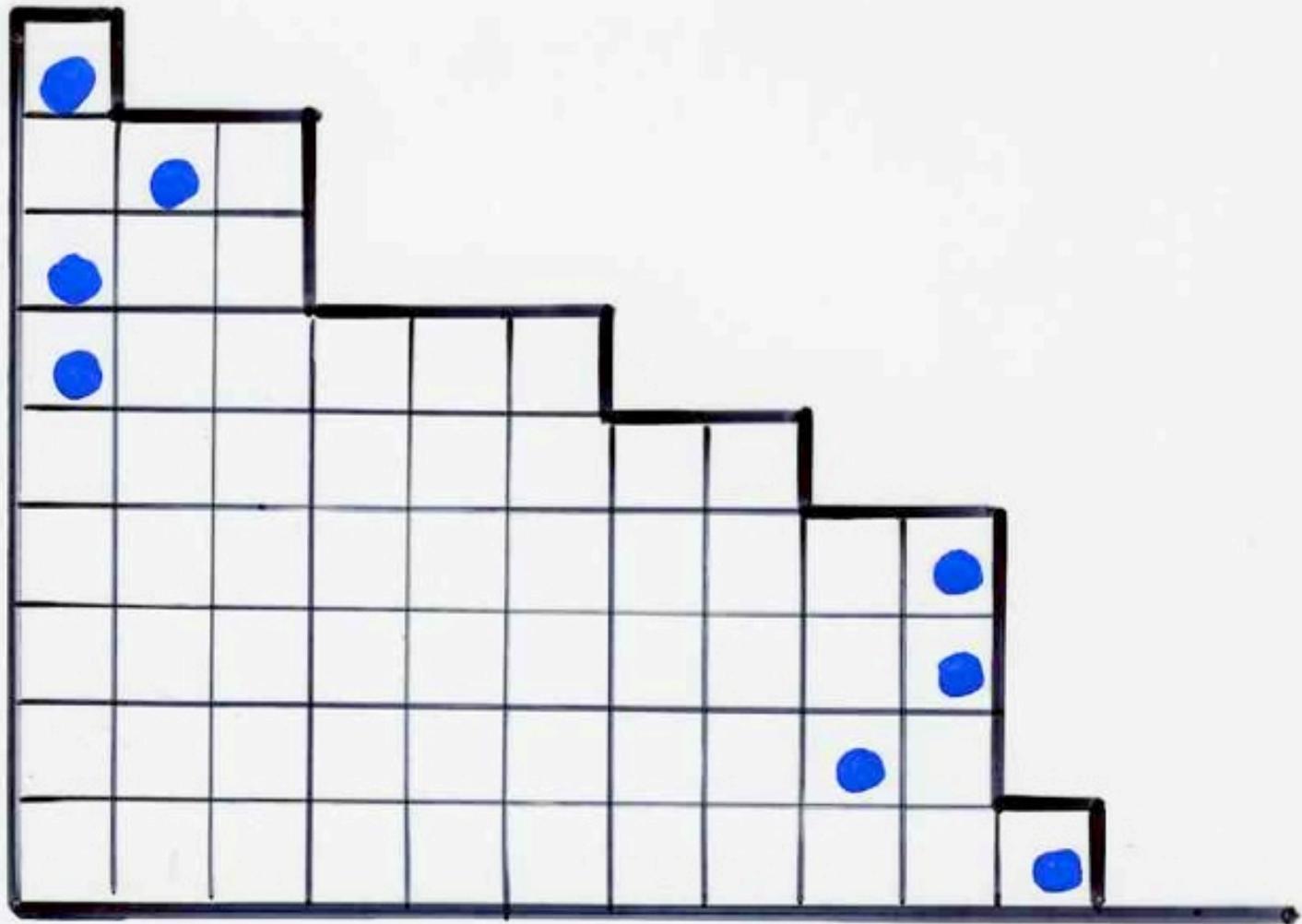






permutation  
tableau



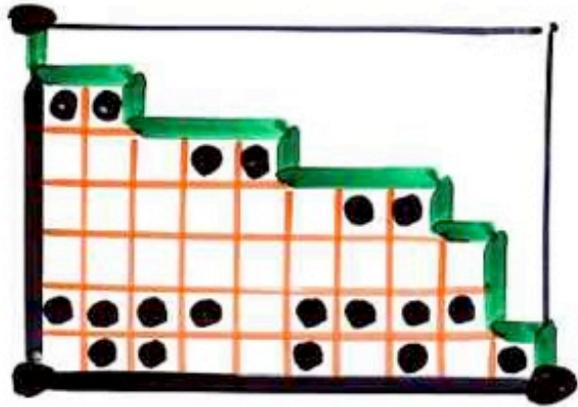




Catalan permutation tableaux

# Permutation Tableau

Ferrers diagram  $F \subseteq k \times (n-k)$   
rectangle



$\square = 0$      $\square \bullet = 1$

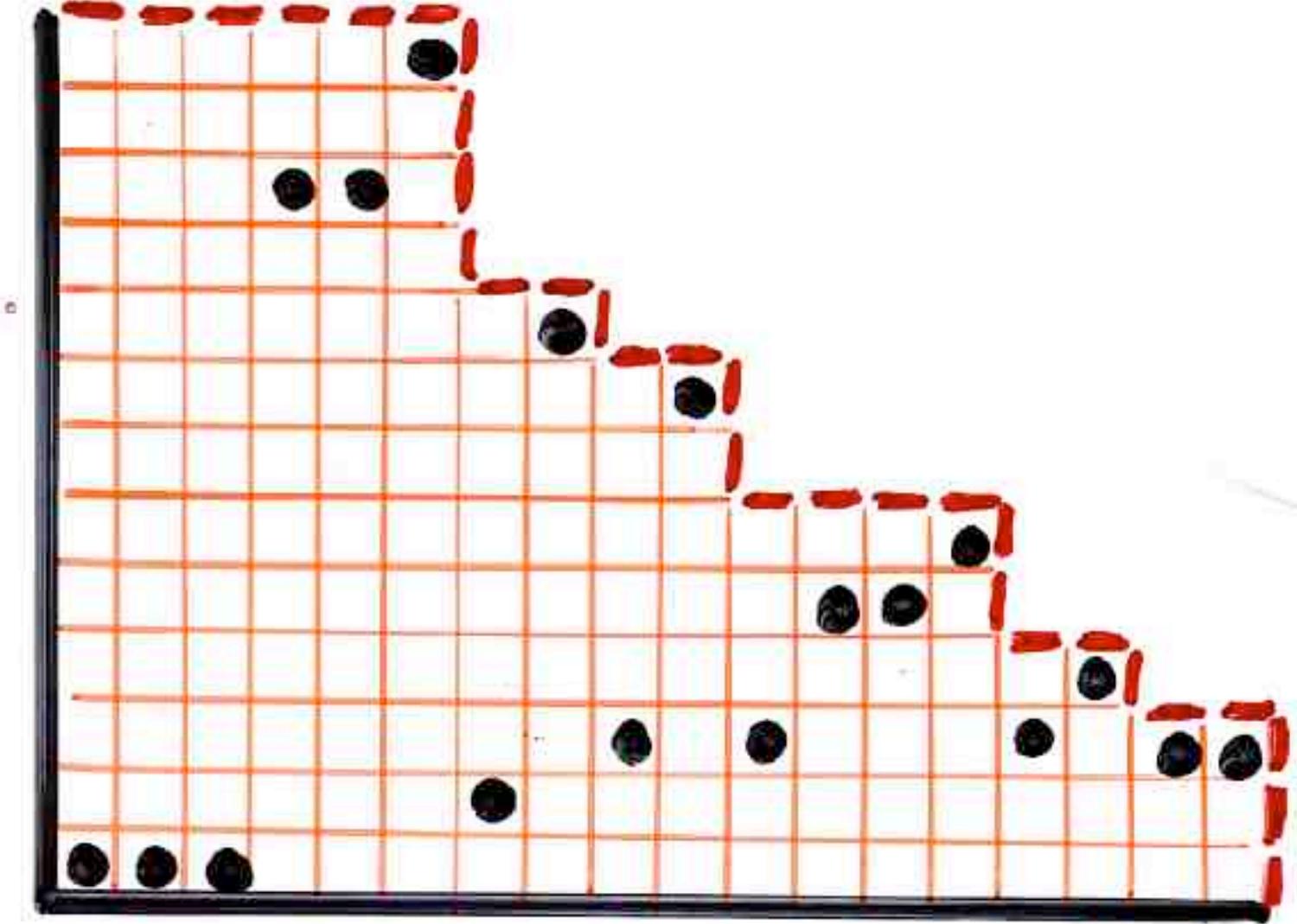
filling of the cells  
with 0 and 1

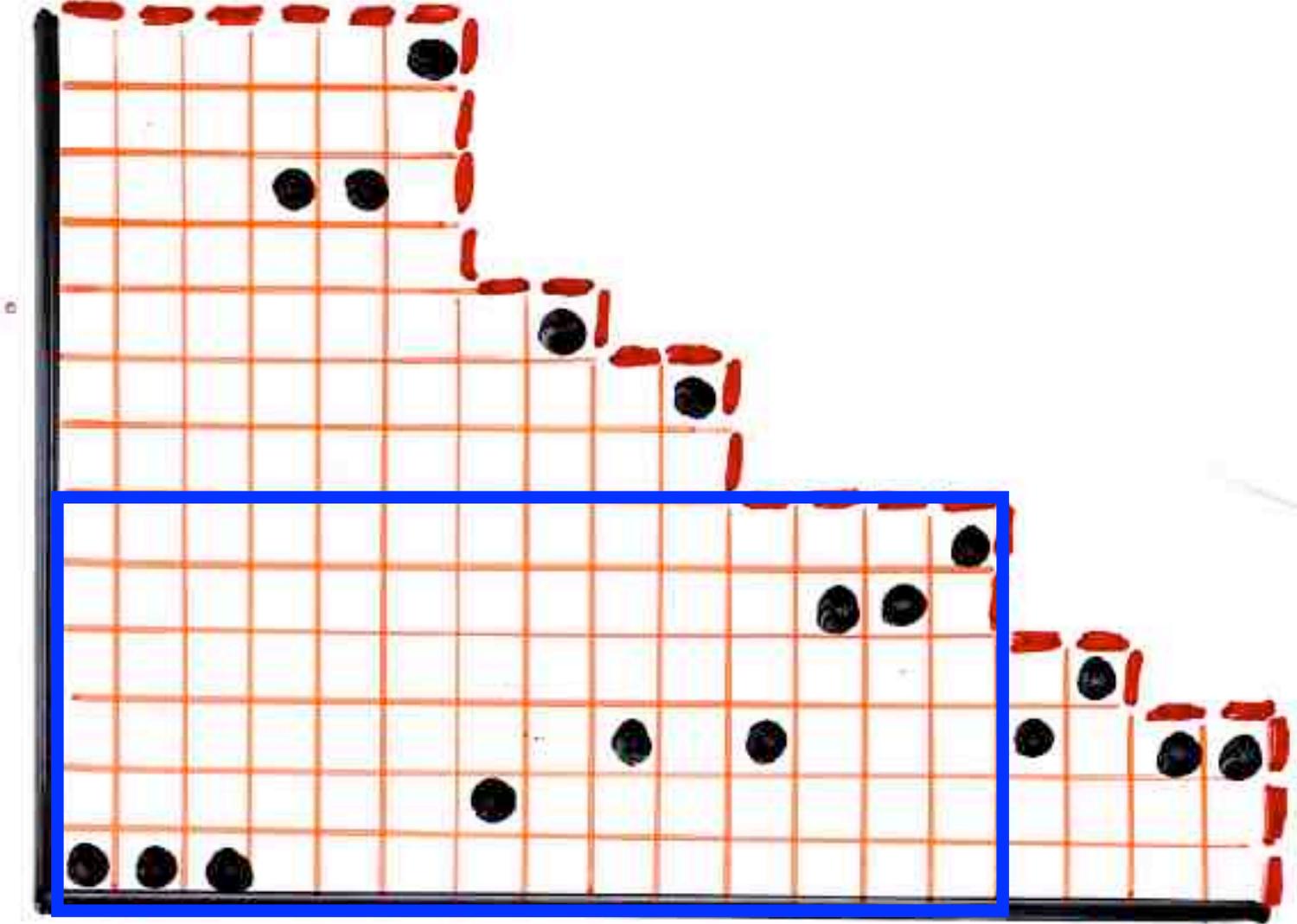
(i) in each column:  
at least one 1

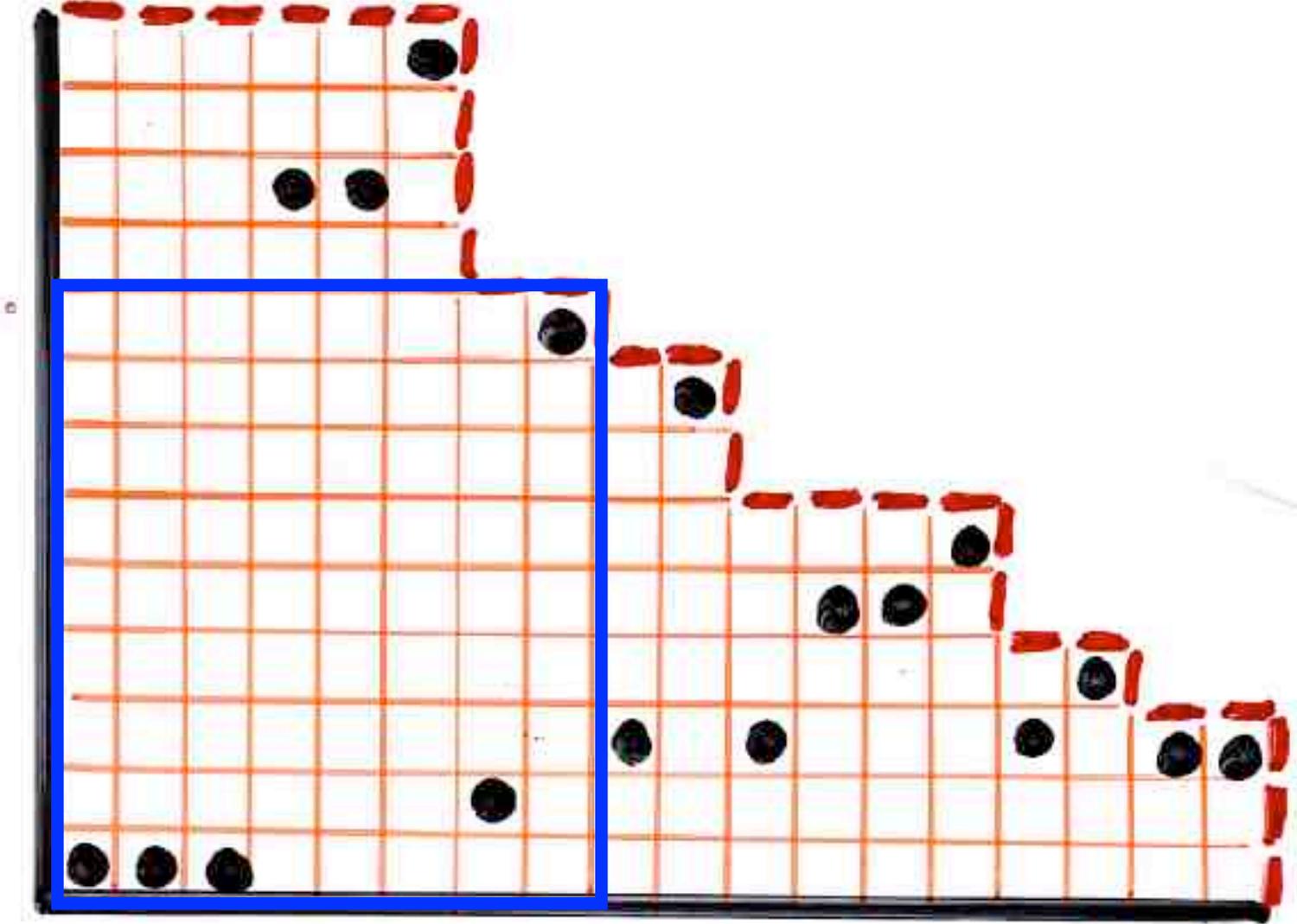
(ii)  $\begin{array}{c} 1 \text{ --- } 0 \\ \quad \quad \quad \vdots \\ \quad \quad \quad 1 \end{array}$  forbidden

Catalan permutation  
tableau

(iii) only one 1 in each column







### 3. Fonction de partition du TASEP

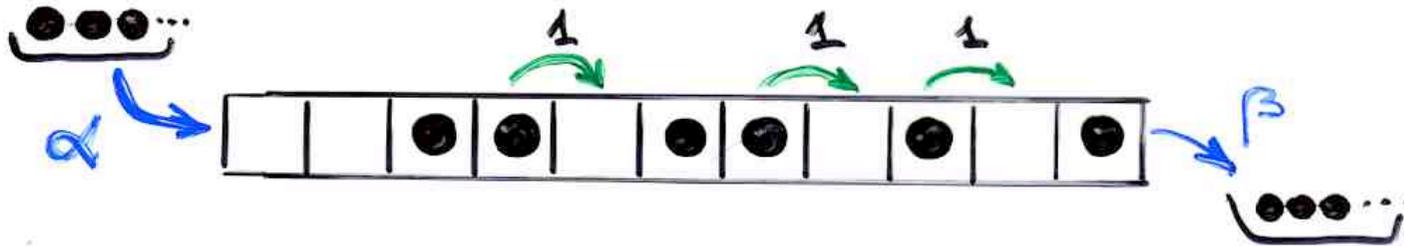
D'après une bijection (XGV) entre les tableaux alternatifs de Catalan et les arbres binaires, la fonction de partition  $Z_n$  du TASEP est la série génératrice des arbres binaires de taille  $n$  selon les paramètres "longueur des branches principales gauche et droite". De plus il est bien connu que la distribution du paramètre "longueur de la branche gauche principale" des arbres binaires est donnée par le "ballot number"

$\frac{i}{2n-i} \binom{2n-i}{n}$  . Montrer alors bijectivement que  $Z_n$  est donnée par:

$$Z_n = \sum_{i=1}^n \frac{i}{2n-i} \binom{2n-i}{n} \frac{\alpha^{-(i+1)} - \beta^{-(i+1)}}{\alpha^{-1} - \beta^{-1}}$$

# TASEP

"Totally asymmetric exclusion process"

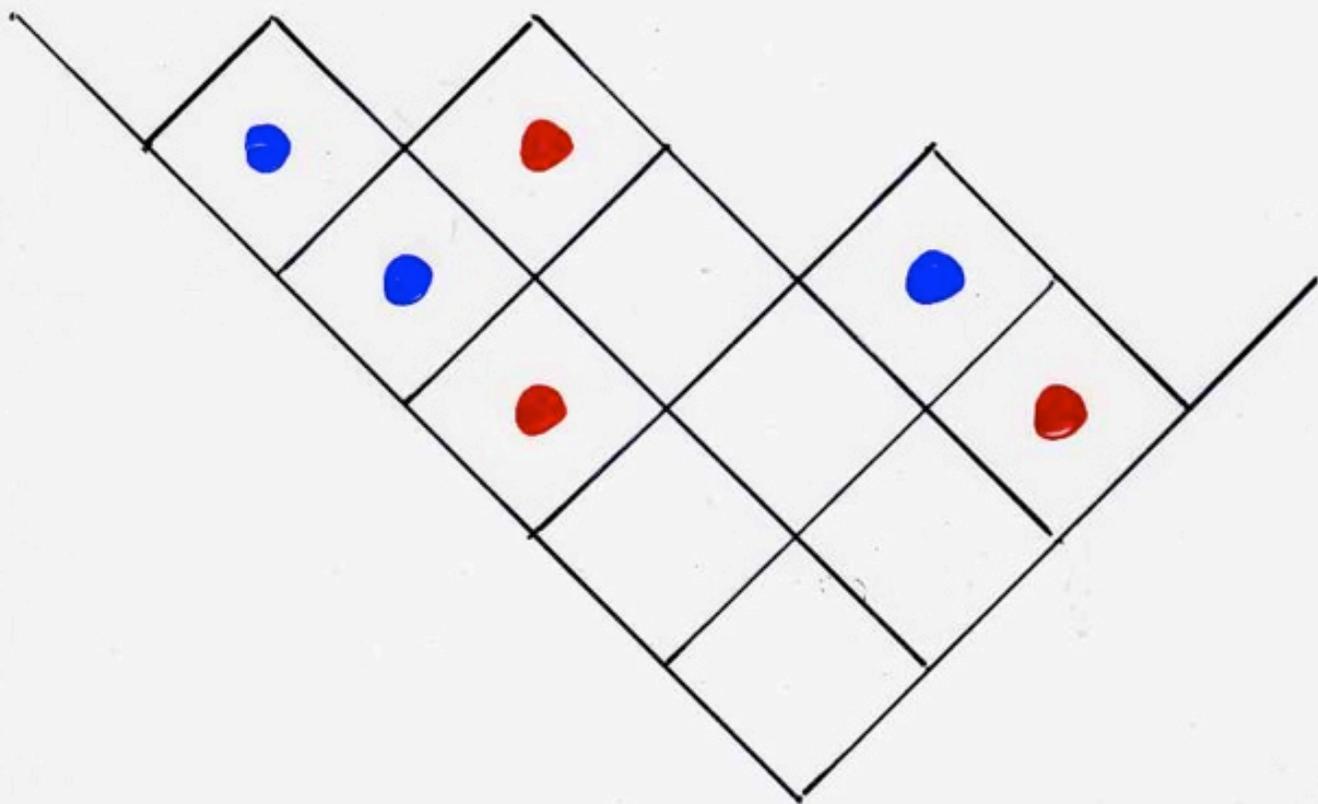


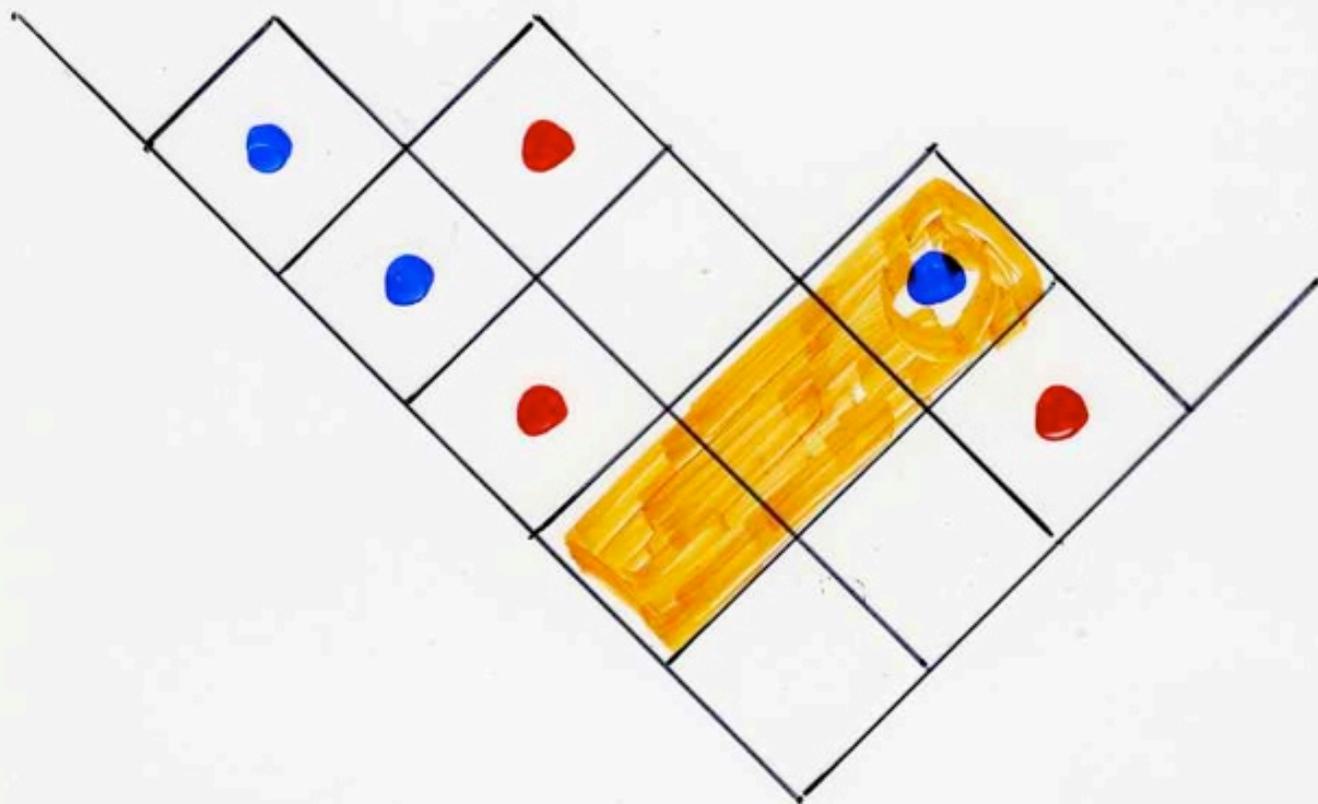
Bijection  
alternative Catalan tableaux  
binary trees

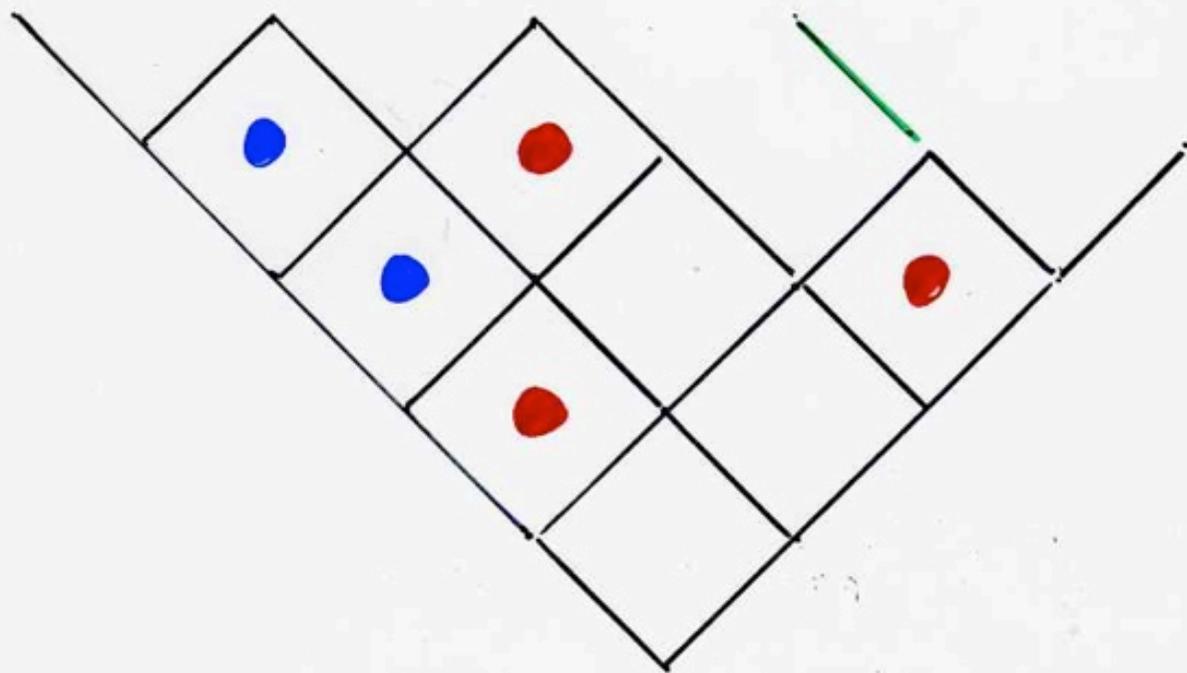
XGV., Catalan tableaux and the asymmetric exclusion process, in Proc. FPSAC'07 (Formal Power Series and Algebraic Combinatorics), Tienjin, Chine, 2007, 12 pp.

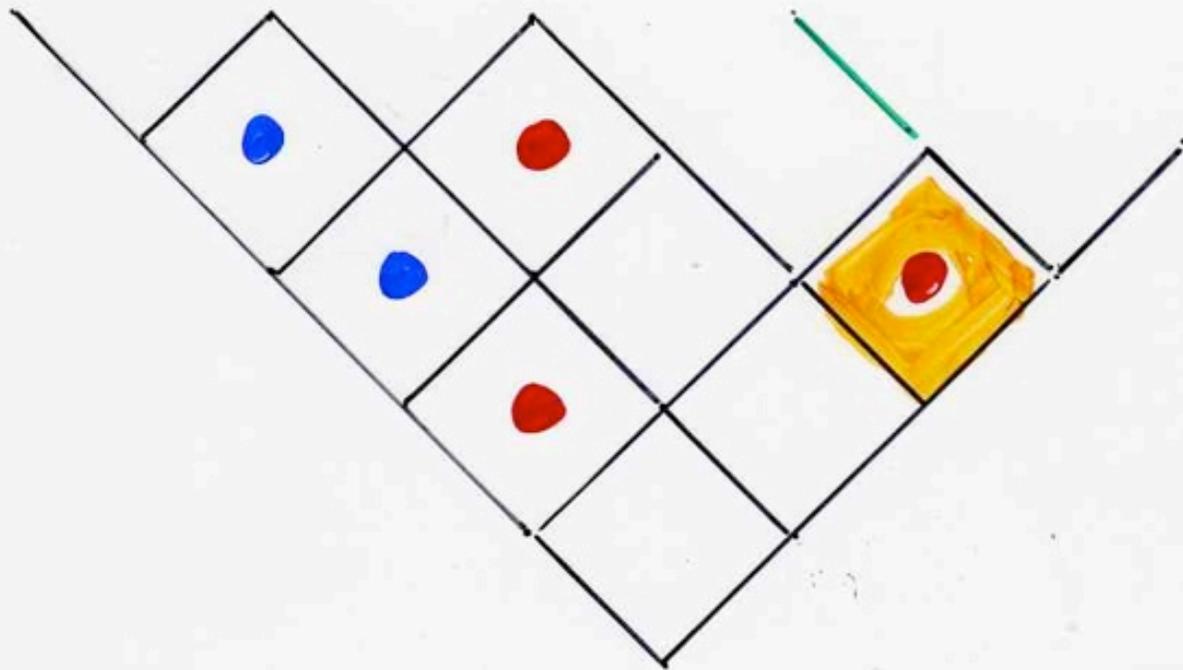
<http://www.fpsac.cn/PDF-Proceedings/Talks/87.pdf>

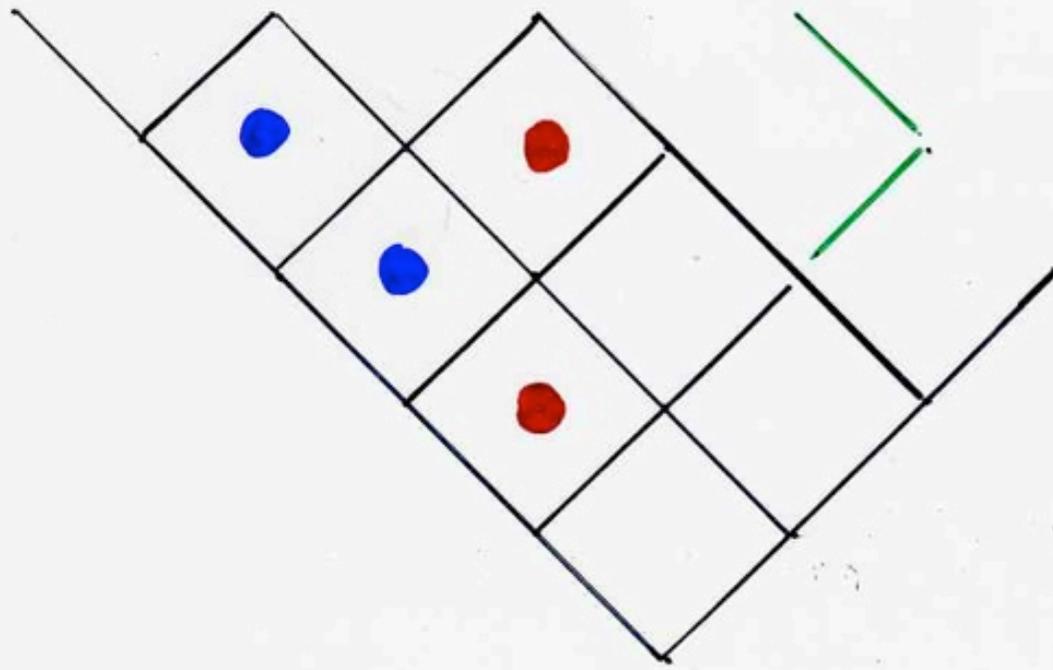
[arXiv:0905.3081](https://arxiv.org/abs/0905.3081) [math.CO]

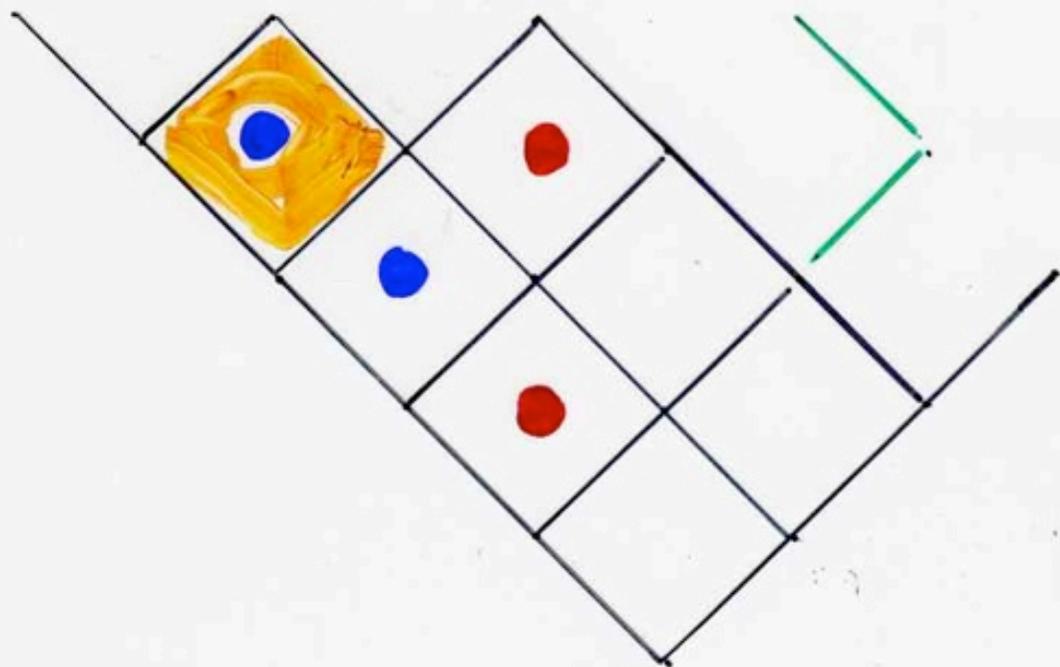


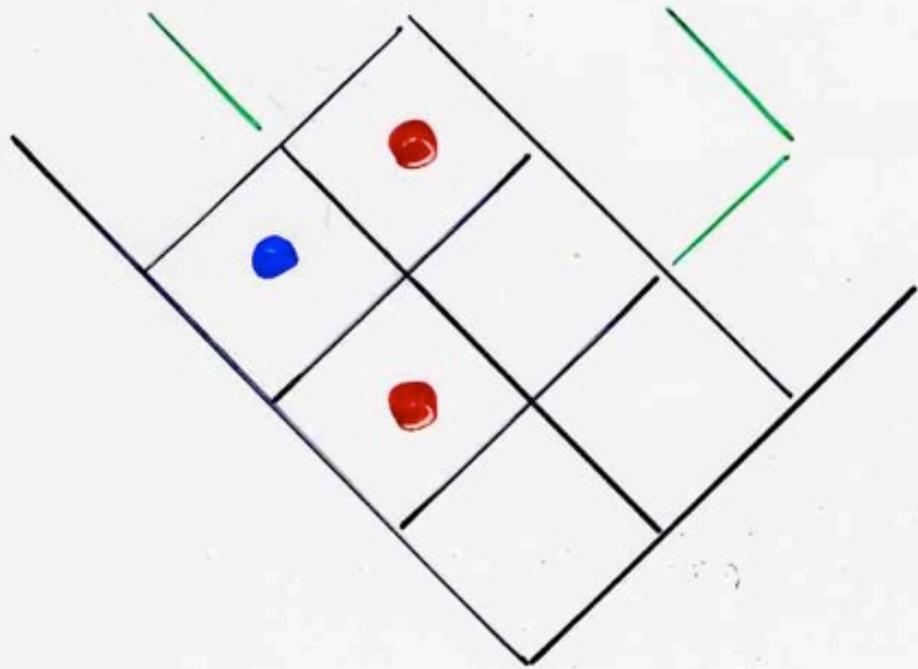


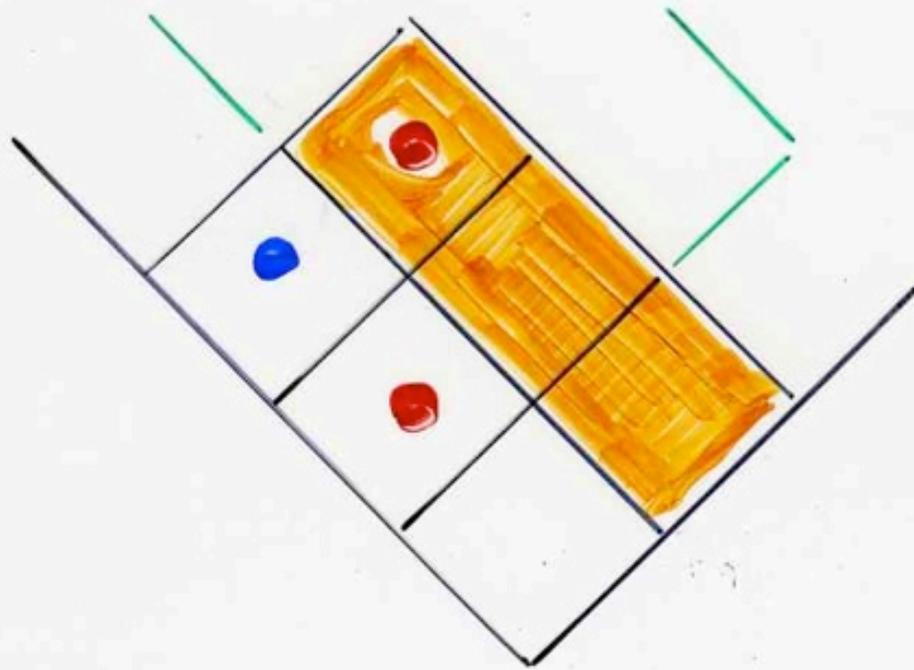


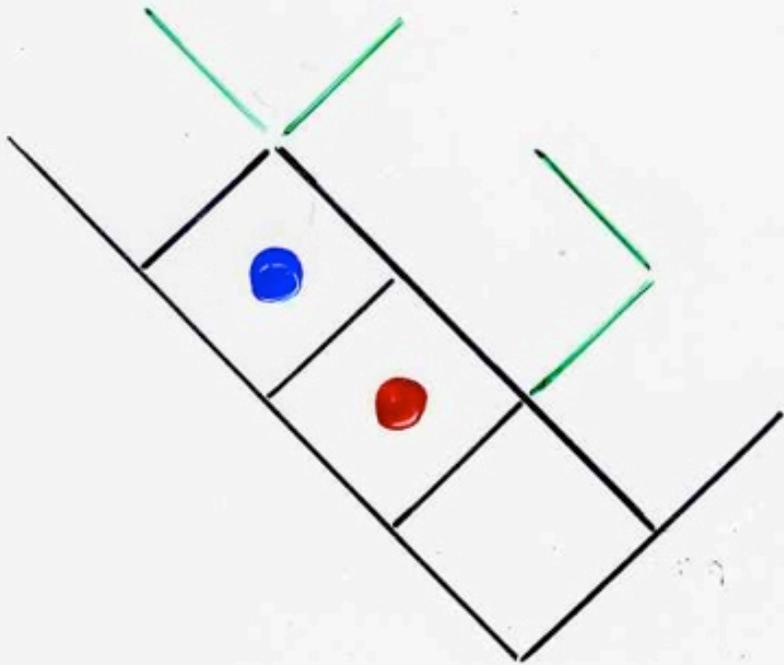


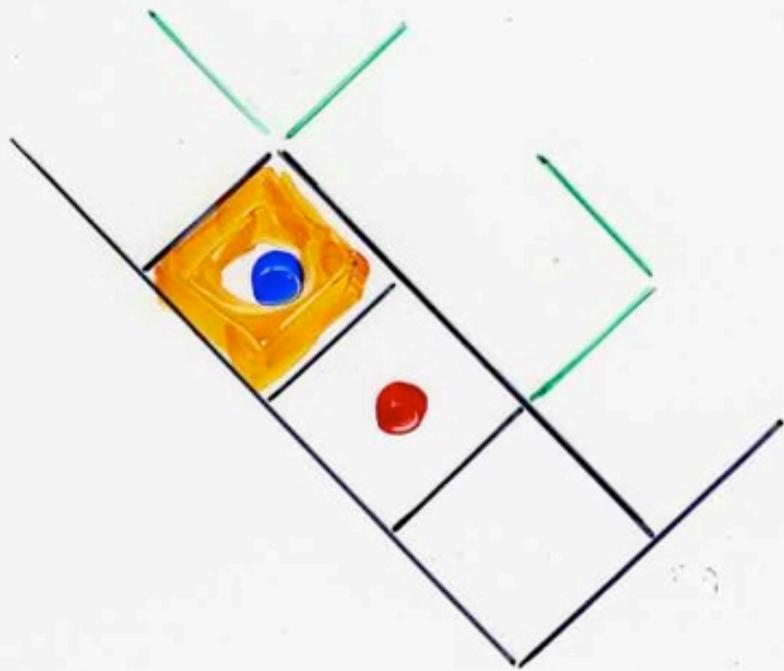


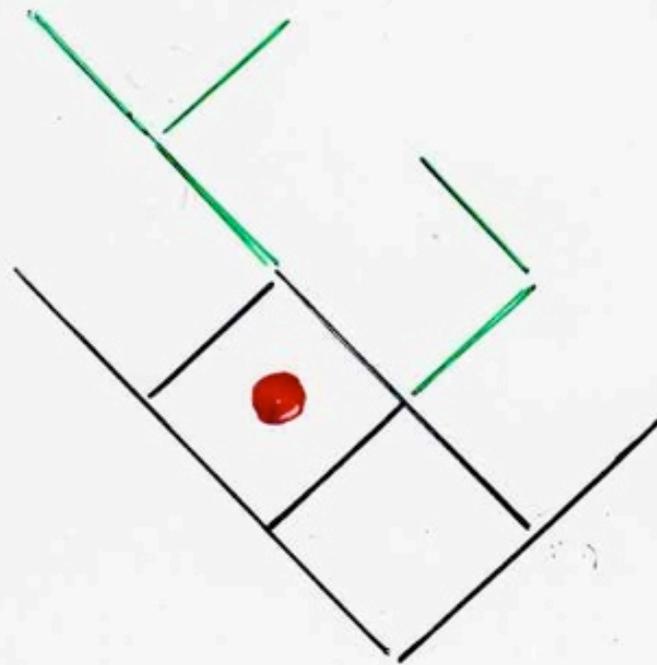


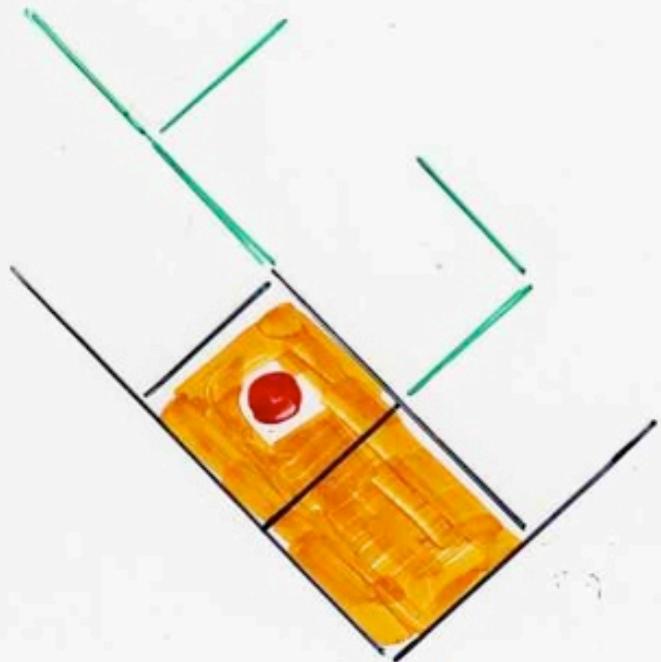


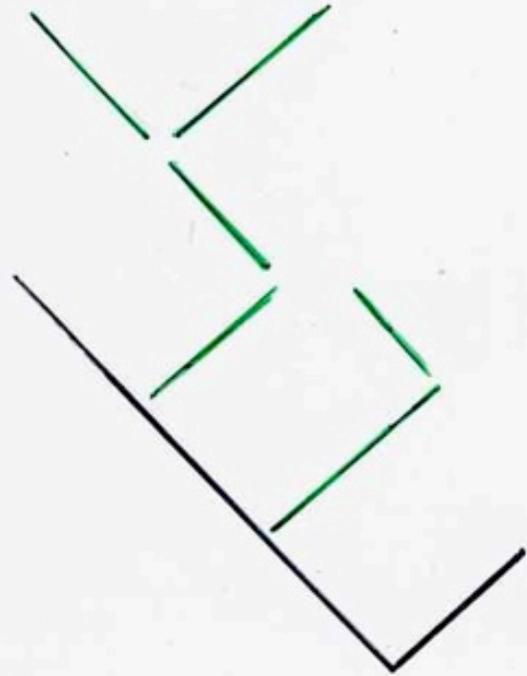






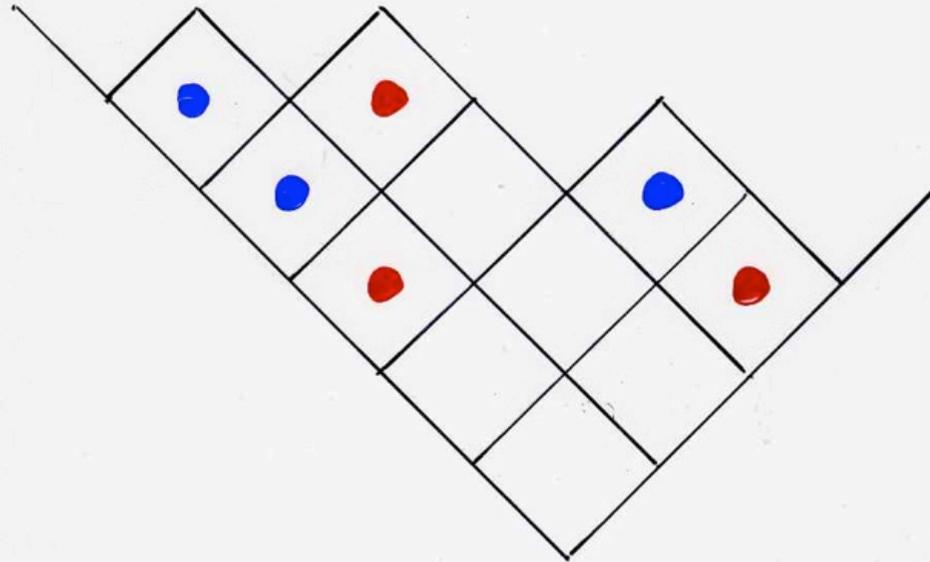
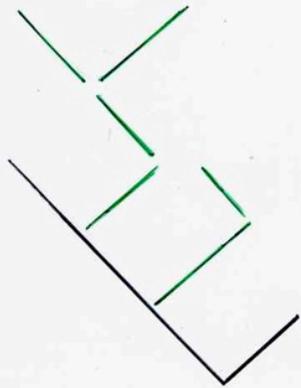




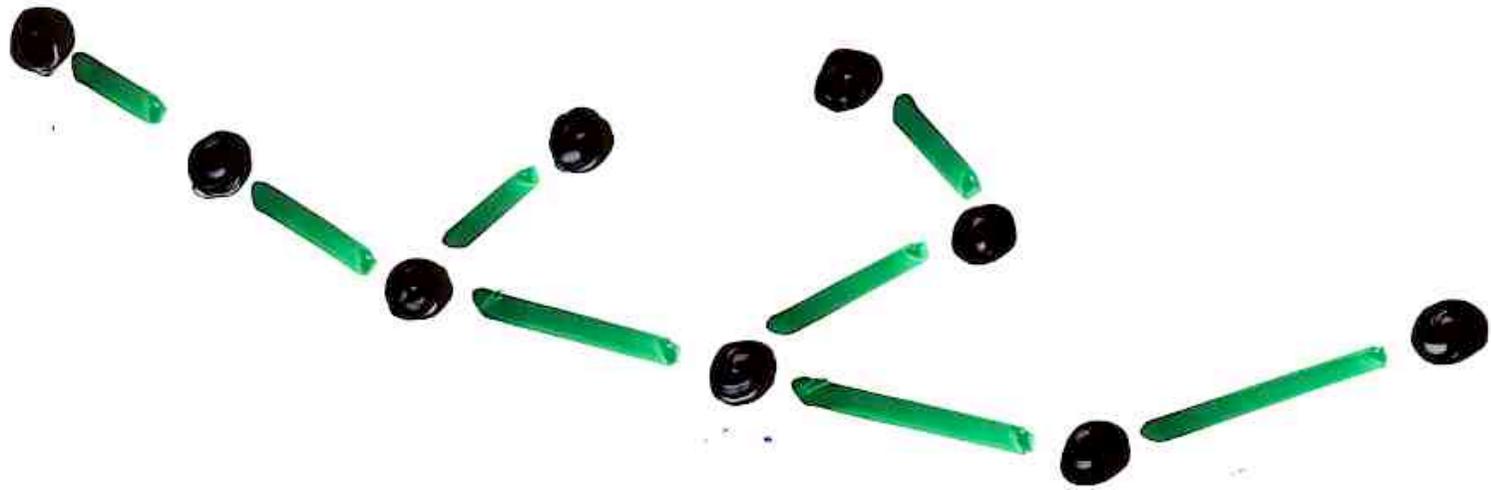


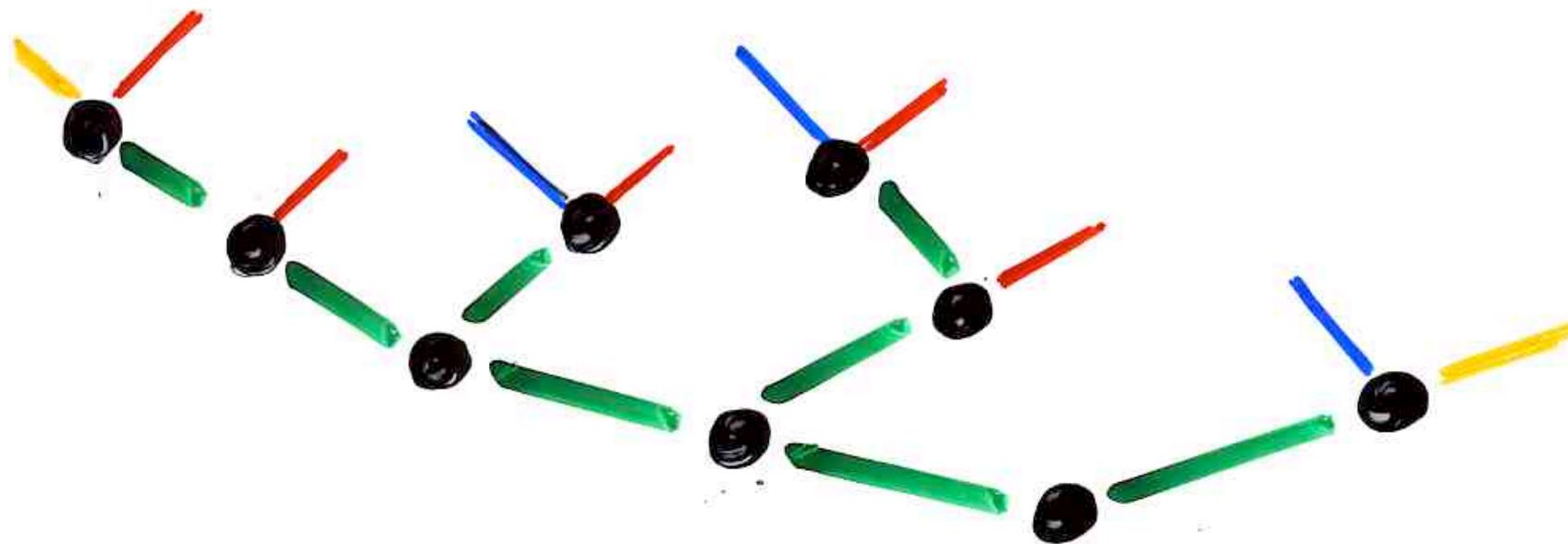
Bijection

tableaux  
alternatifs  
de Catalan  $\longleftrightarrow$  arbres  
binaires  
taille  $n$   $n$  arêtes



profil (bord)  
du diagramme  
de Ferrers  $\longleftrightarrow$  comopée





canopy of a binary tree

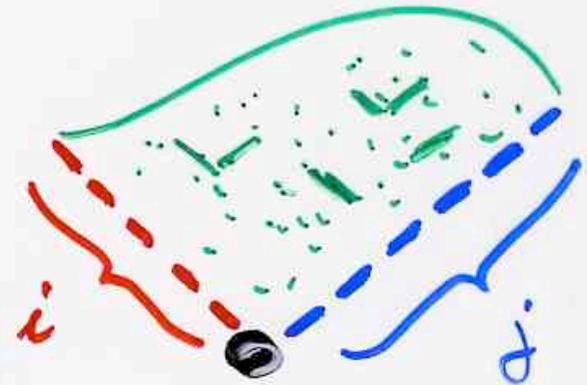
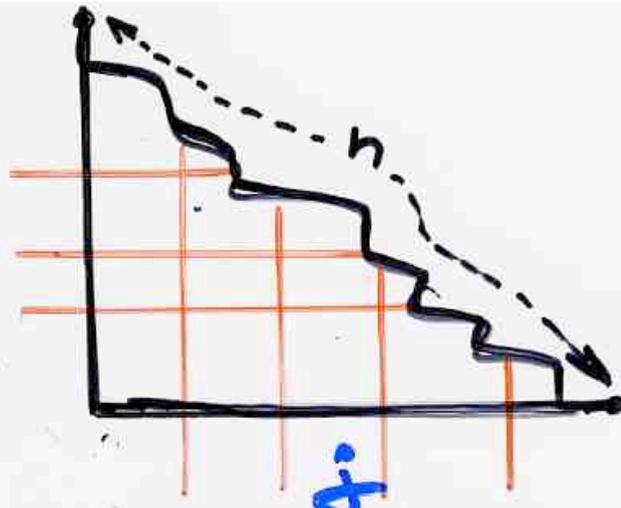
$$c(B) = - - + - + - - +$$

Bijection

tableaux  
alternatifs  
de Catalan <sup>taille n</sup> ↔ arbres  
binaires <sup>n arêtes</sup>

profil (bord)  
du diagramme  
de Ferrers ↔ comopée

nb de  
lignes  
(sans  $\leftarrow$ )  
libres  $i$



nb de colonnes  
(sans  $\downarrow$ )  
libres  $j$

Fonction de partition  
du TASEP

$$\lambda = (\tau_1, \dots, \tau_n)$$

$$P_n(\lambda; \alpha, \beta) = \frac{1}{Z_n} \sum_{\mathcal{B}} \alpha^{lb(\mathcal{B})} \beta^{rb(\mathcal{B})}$$

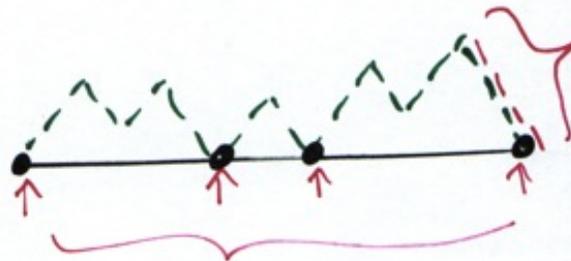
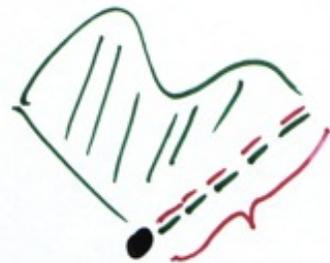
binary trees  
canopy  $\lambda$

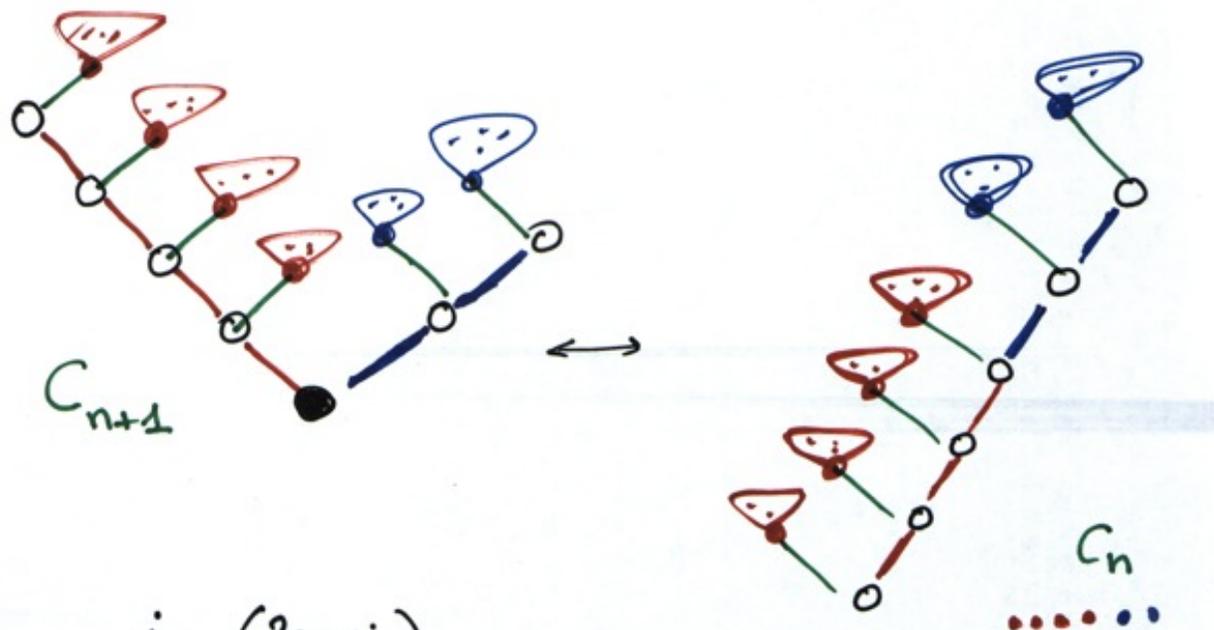


$$Z_n = \sum_{i=1}^n \frac{i}{2n-i} \binom{2n-i}{n} \frac{\alpha^{(i+1)} - \beta^{(i+1)}}{\alpha - \beta}$$

partition function

"ballot" numbers





$$\frac{i}{2n-i} \binom{2n-i}{n}$$

$$\left[ \alpha^{(i)} + \alpha^{(i)} \beta + \dots + \alpha \frac{(i-1)}{\beta + \beta^{(i)}} \right]$$

$$\sum_{n=0}^{\infty} Z_n t^n = \frac{1}{(1-\alpha f(t))} \times \frac{1}{(1-\beta f(t))}$$

$$f(t) = t \left( \begin{array}{l} \text{generating function} \\ \text{of Catalan numbers} \end{array} \right)$$
$$= \frac{1 - \sqrt{1-4t}}{2}$$

# 4. TASEP et tableaux de Catalan

relation avec l'interprétation  
de Shapiro-Zeilberger par paires de chemins

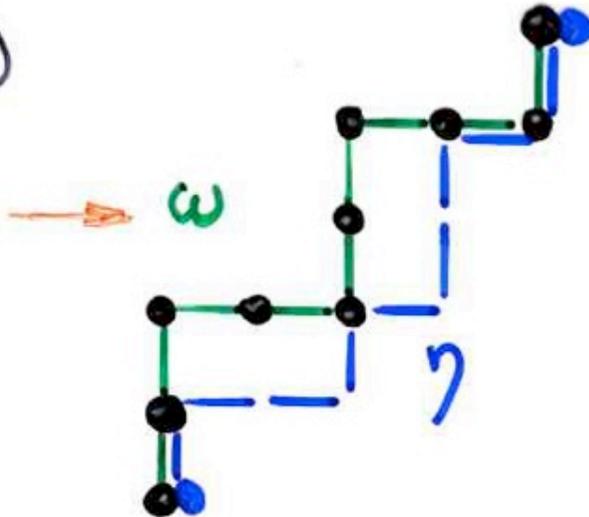
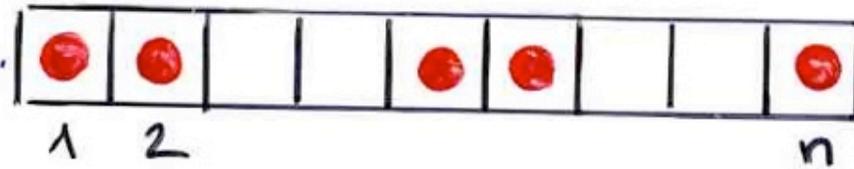
Donner une bijection entre les tableaux alternatifs de Catalan et les paires de chemins de l'interprétation combinatoire Shapiro-Zeilberger du TASEP. (plus difficile, commencer par un rectangle).

XGV., Catalan tableaux and the asymmetric exclusion process, in Proc. FPSAC'07 (Formal Power Series and Algebraic Combinatorics), Tienjin, Chine, 2007, 12 pp.

<http://www.fpsac.cn/PDF-Proceedings/Talks/87.pdf>

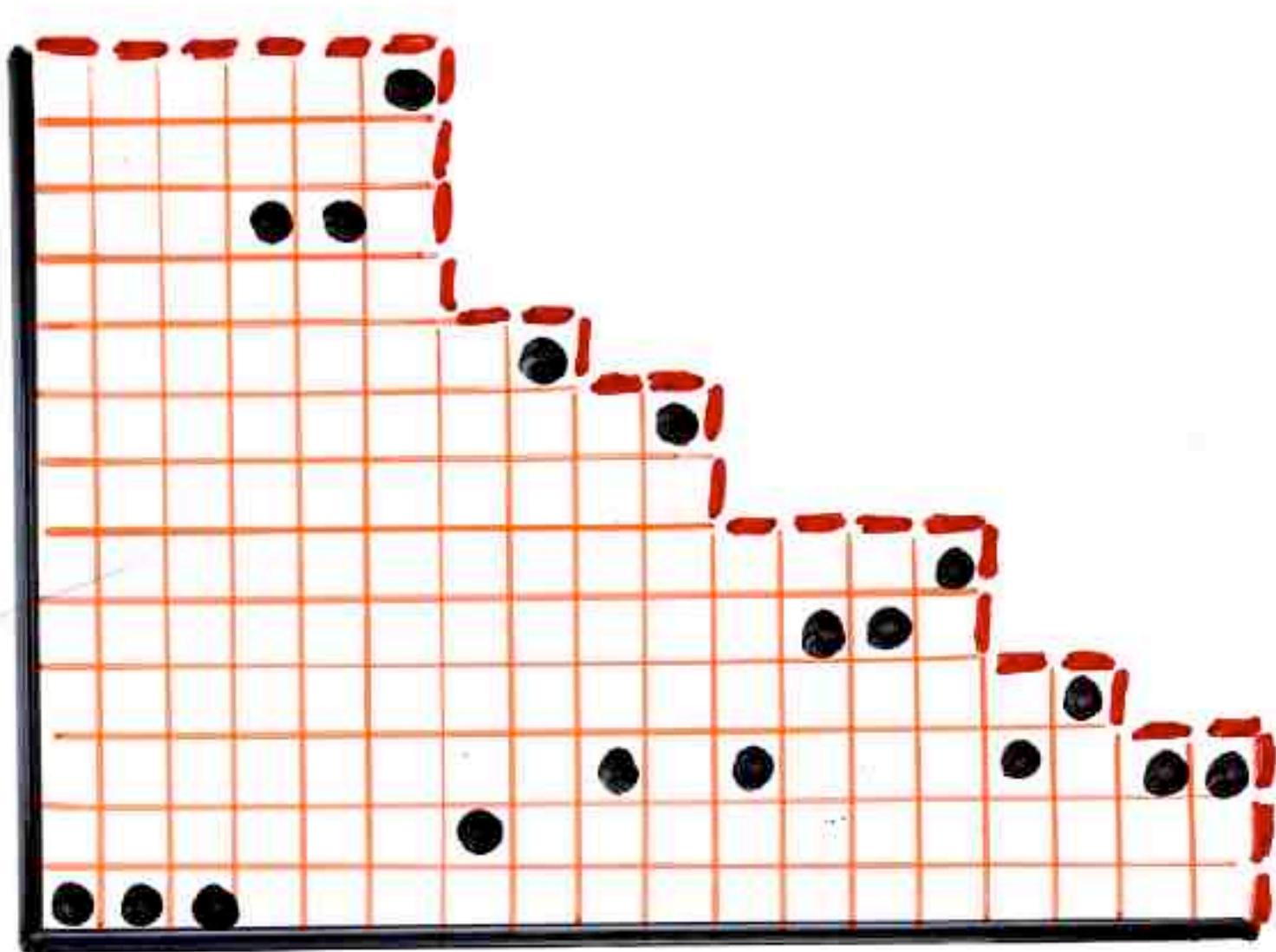
[arXiv:0905.3081](https://arxiv.org/abs/0905.3081) [math.CO]

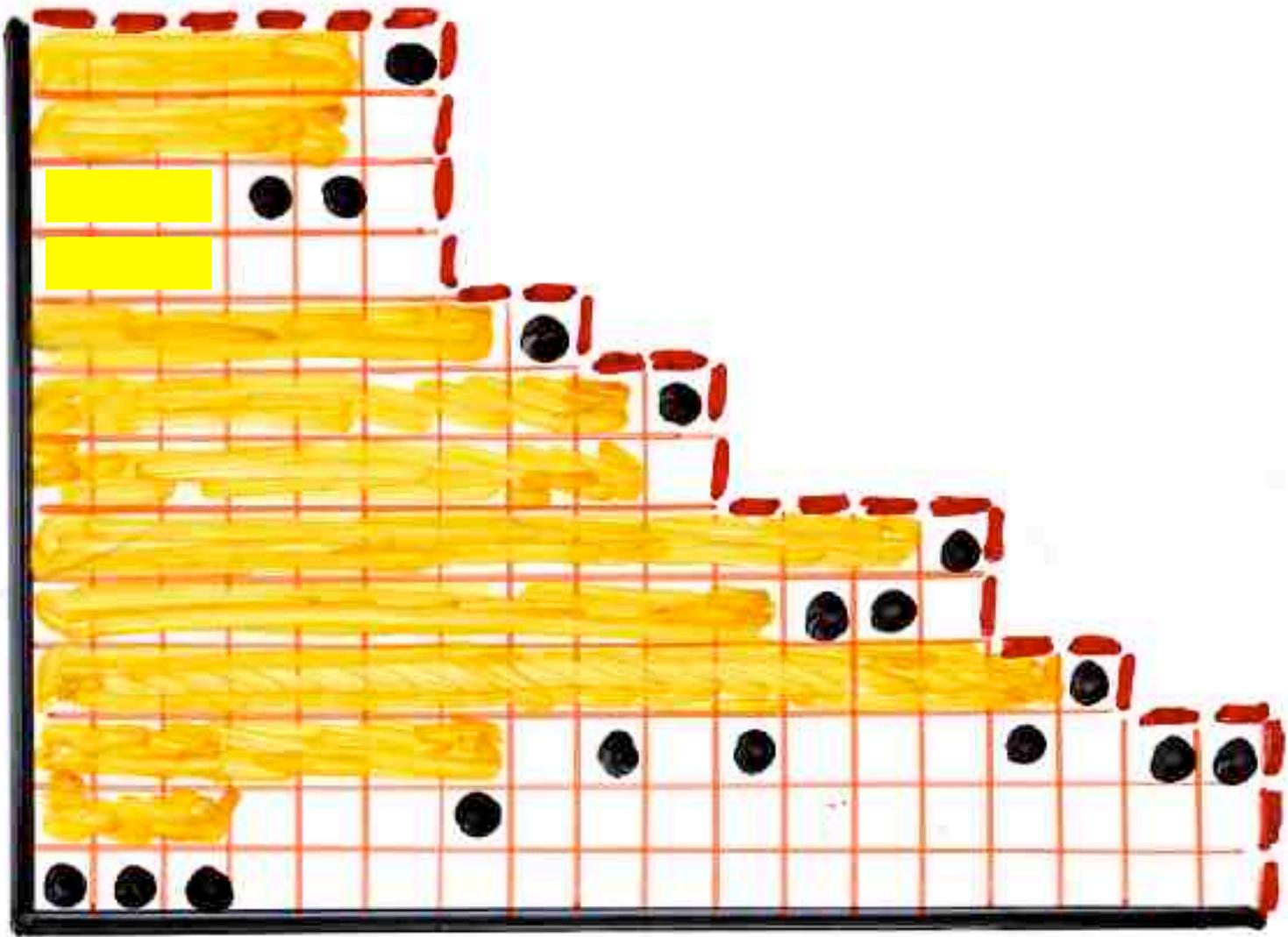
state  $s = (\tau_1, \dots, \tau_n)$

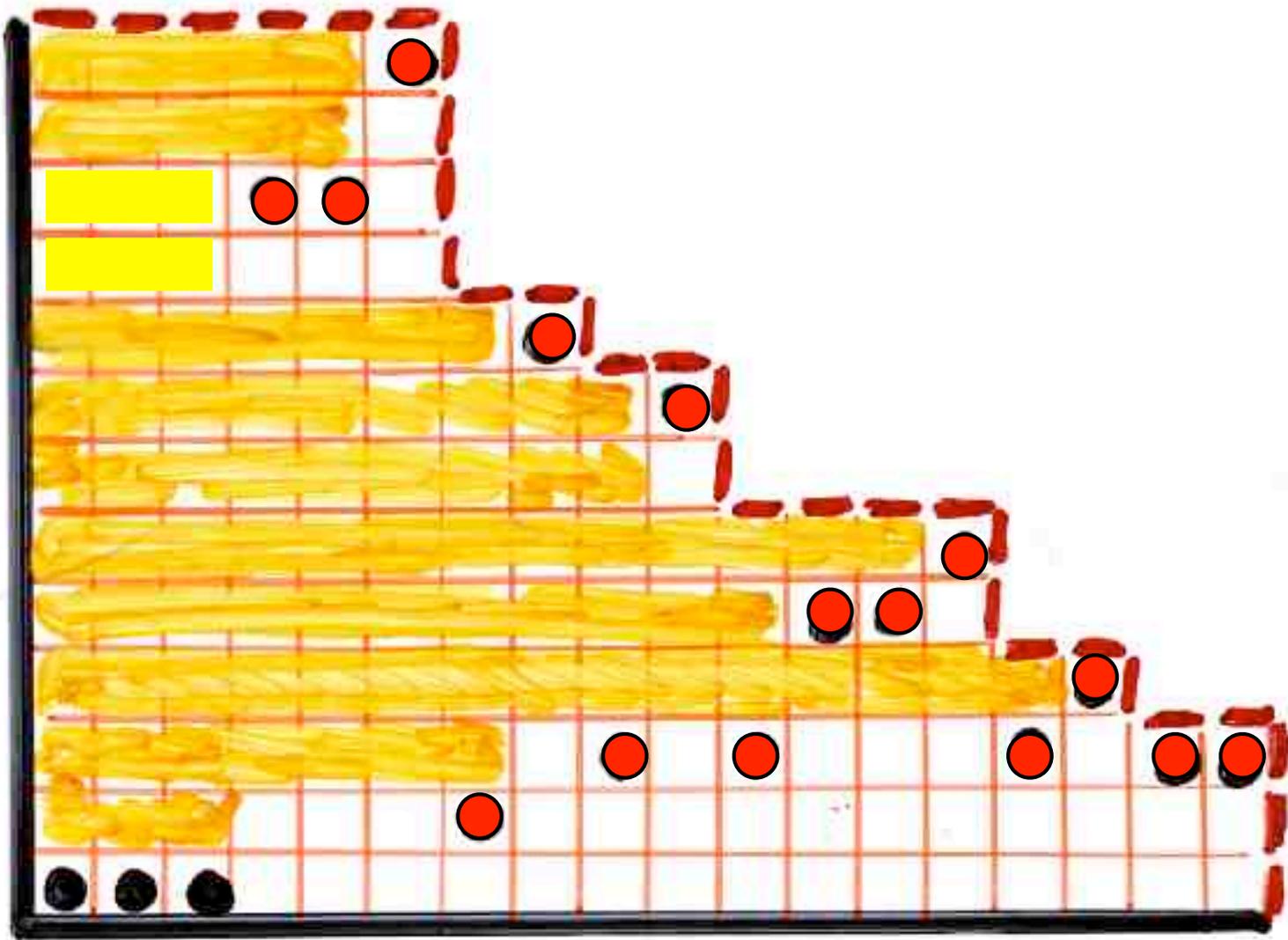


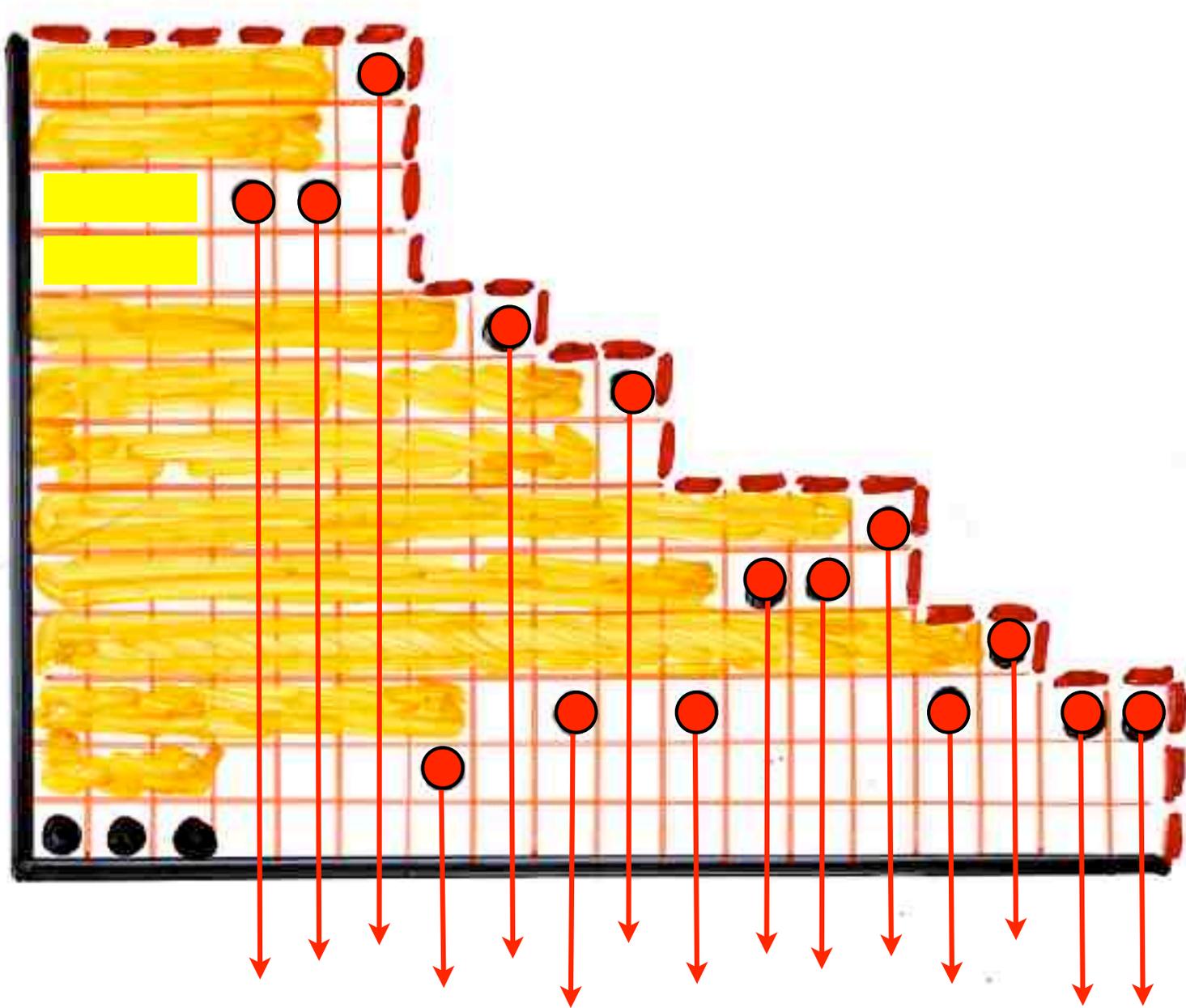
$$P_n(s) = \frac{1}{C_{n+1}} \left( \begin{array}{l} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

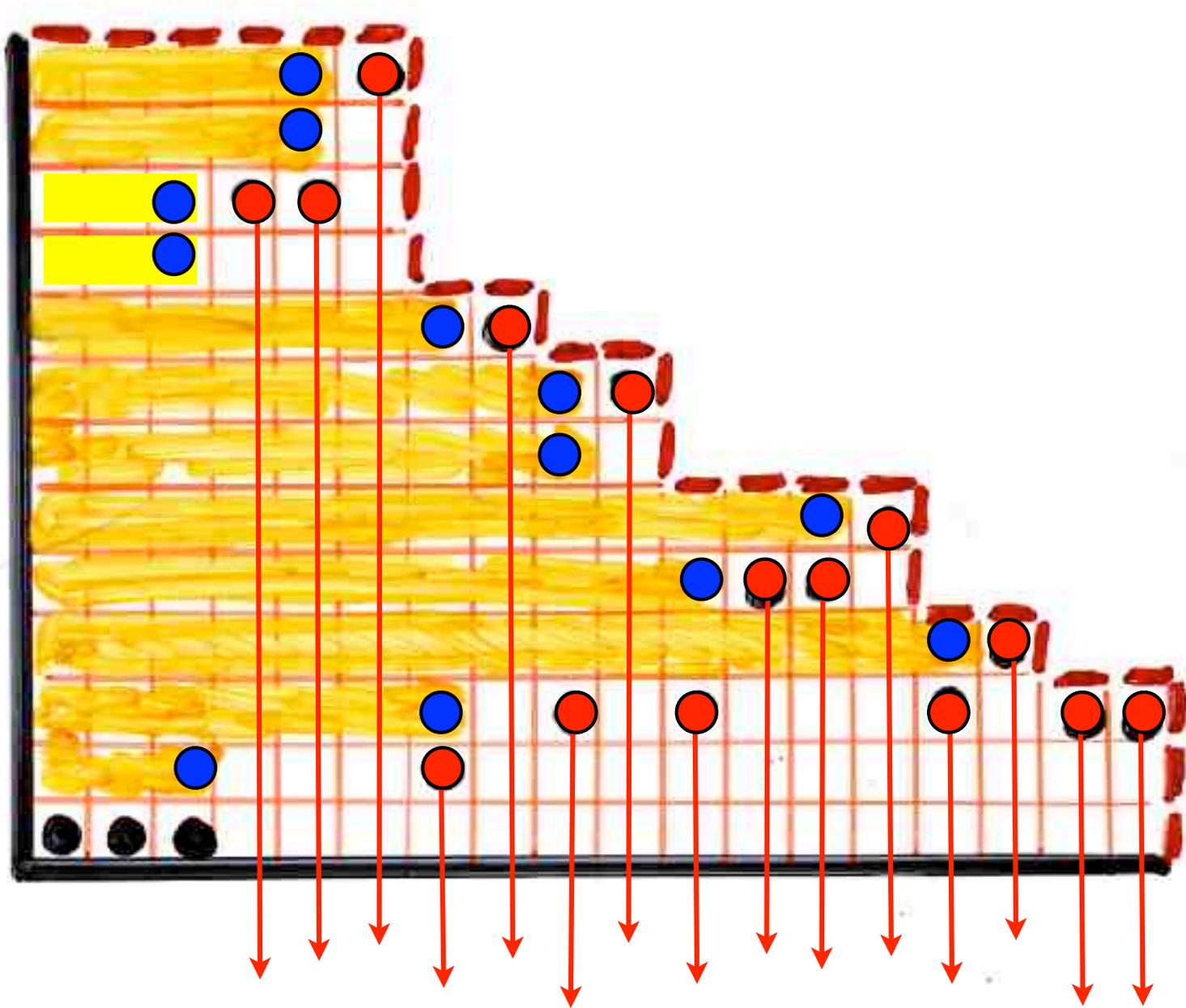
Shapiro, Zeilberger, 1982

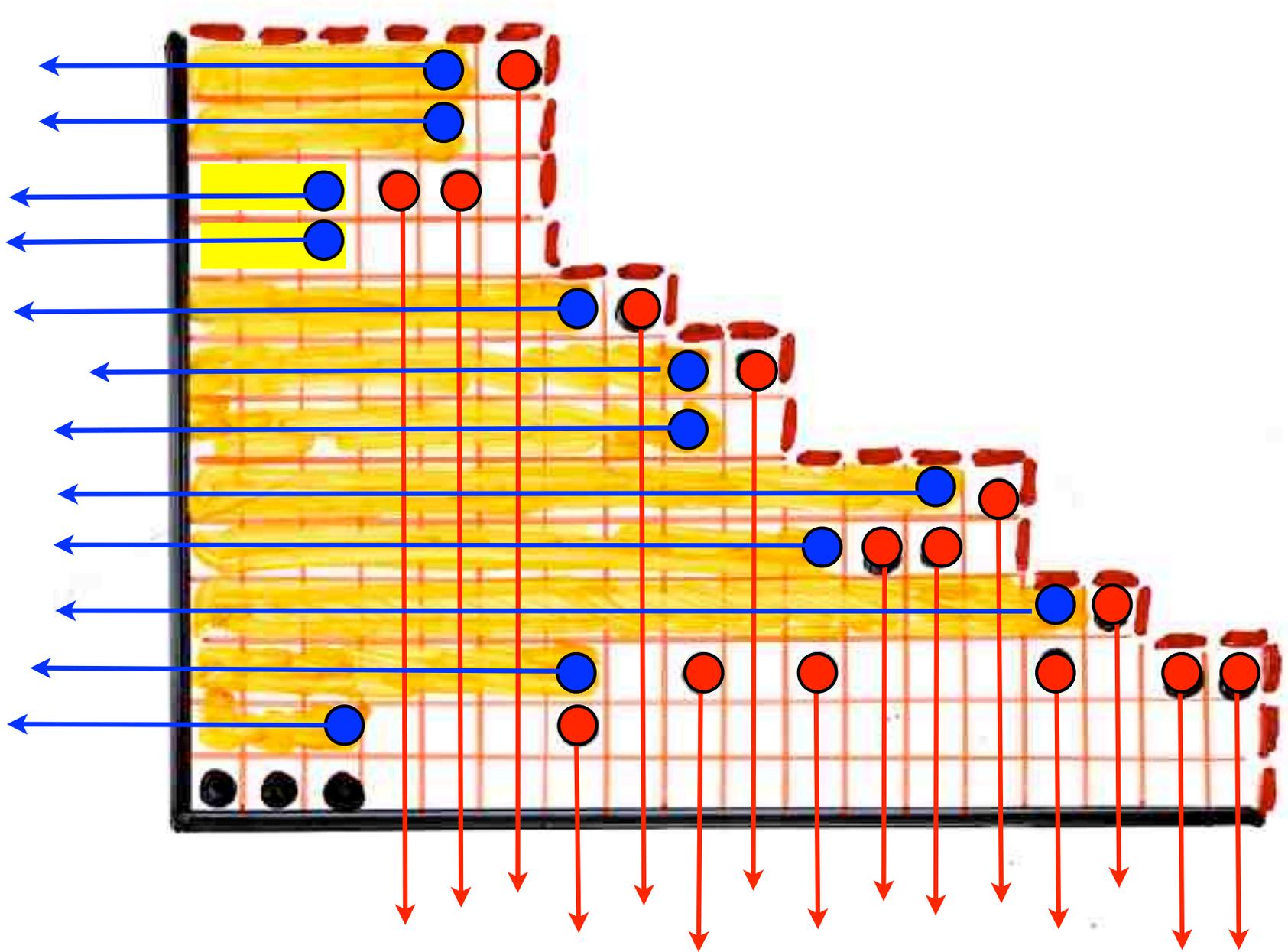


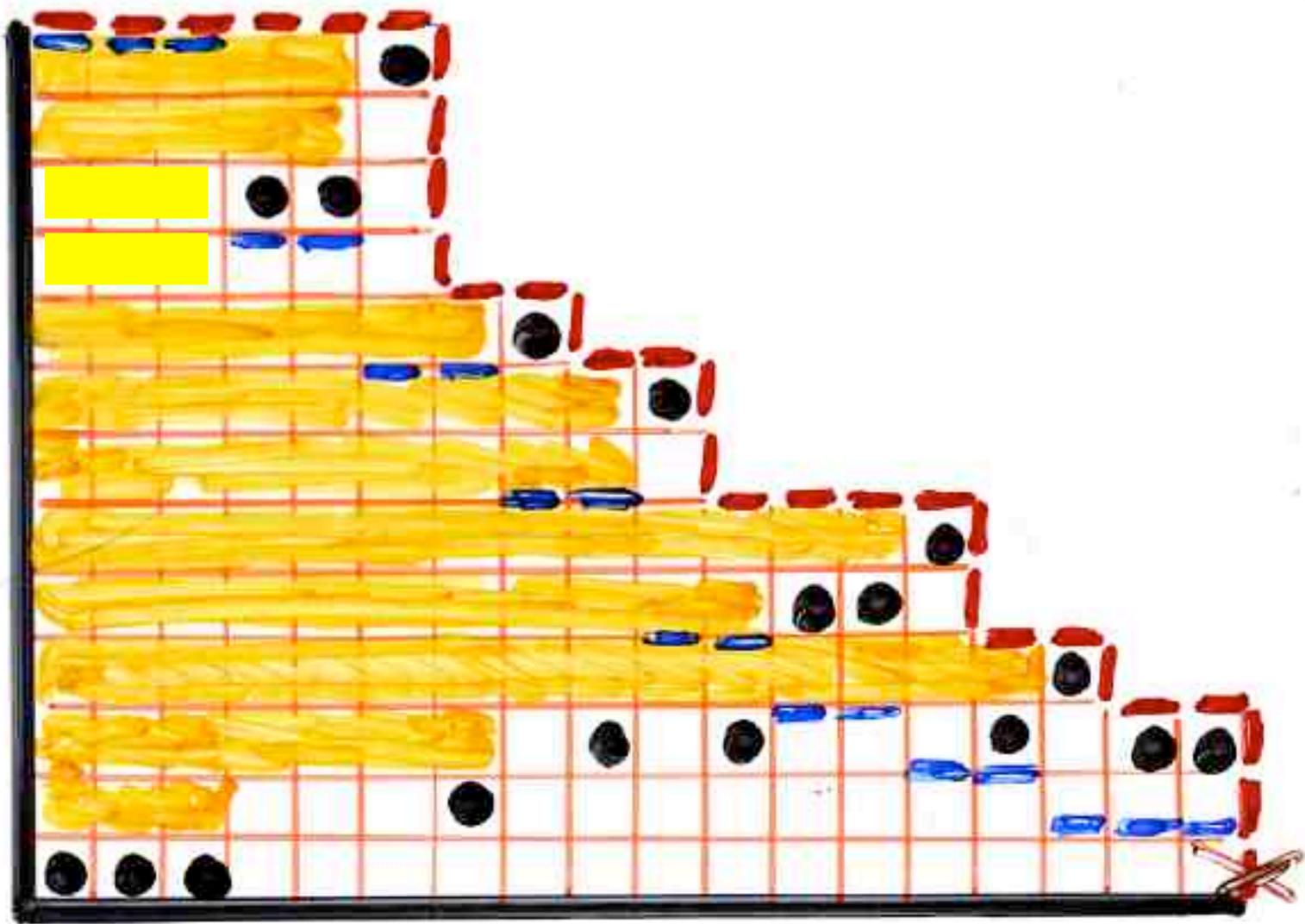


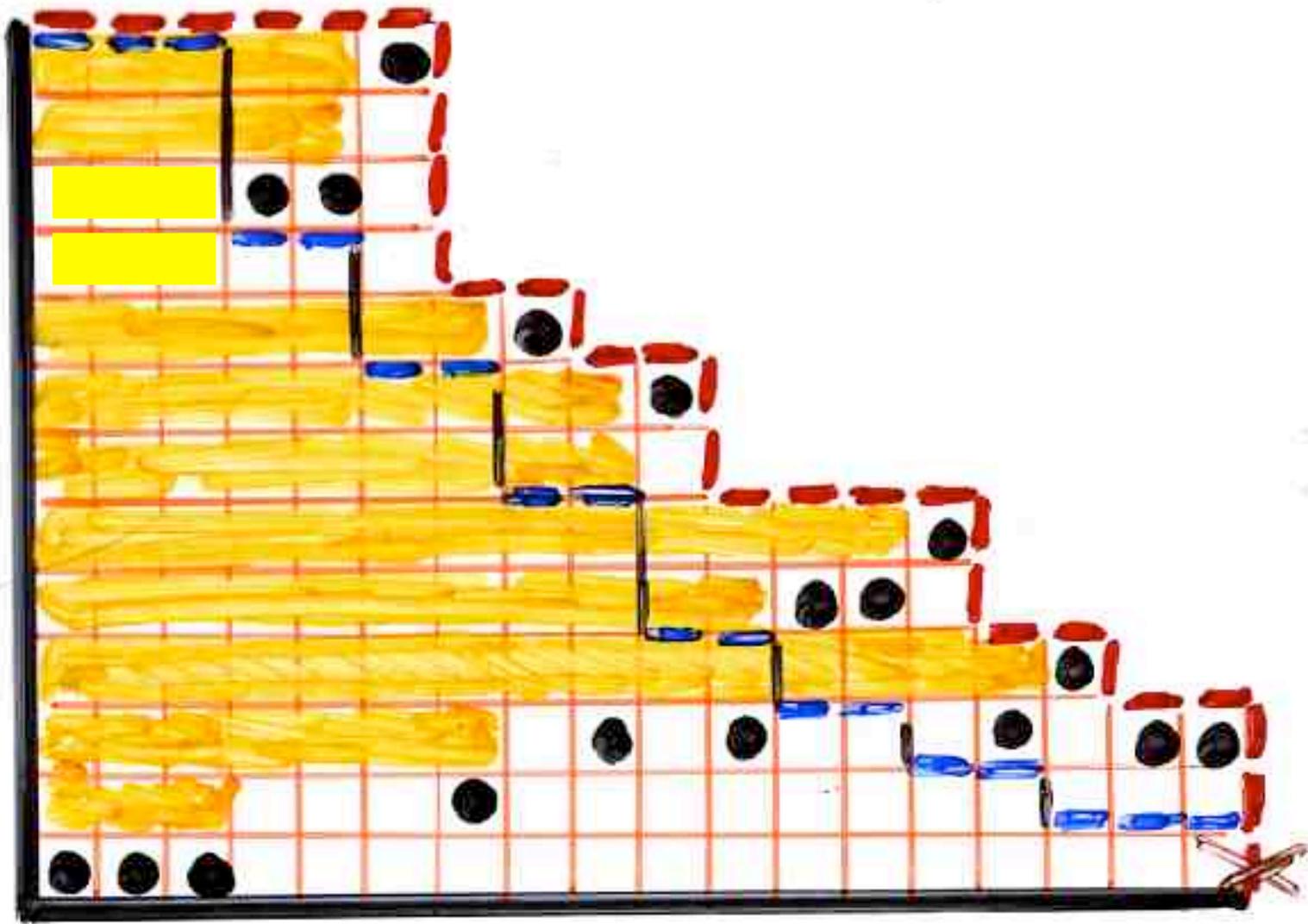










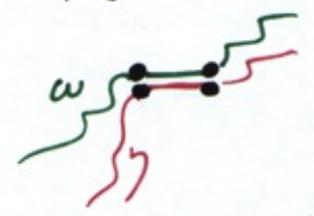


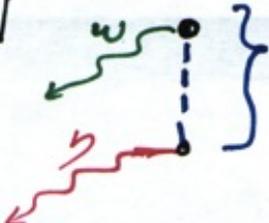
$\Delta = (\tau_1, \dots, \tau_n) \rightarrow$   $\omega$  path   
 state

$$P_n(\Delta; \alpha, \beta) = \frac{1}{Z_n} \sum_{\eta} \alpha^{f(\omega, \eta)} \beta^{g(\omega, \eta)}$$

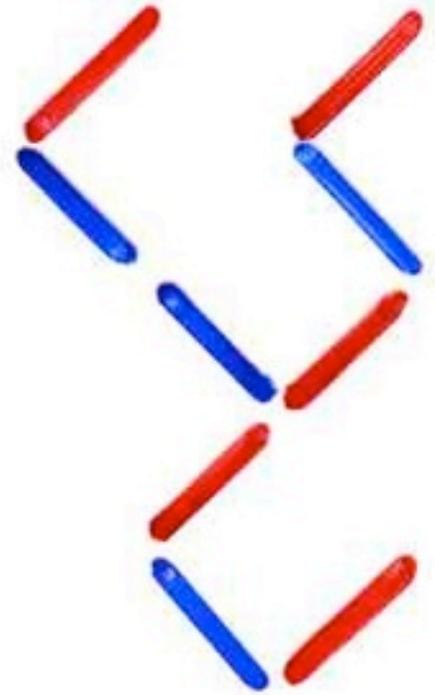
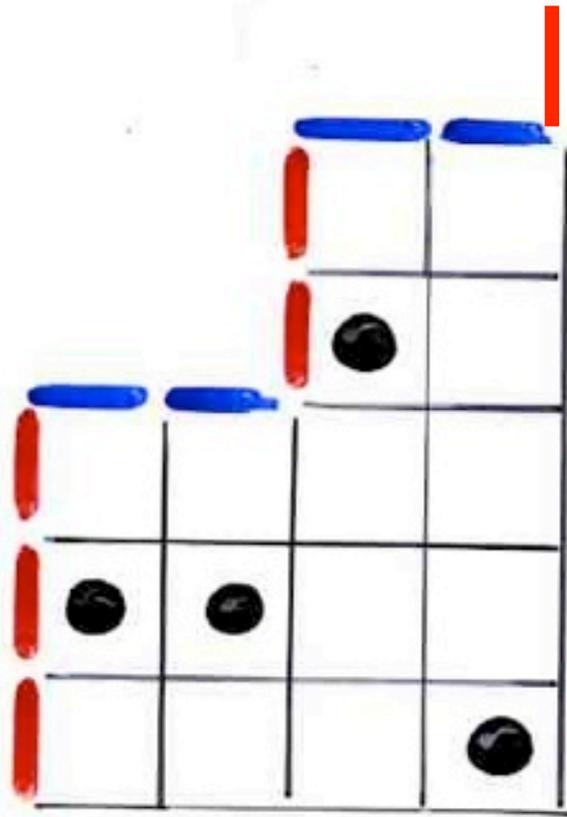


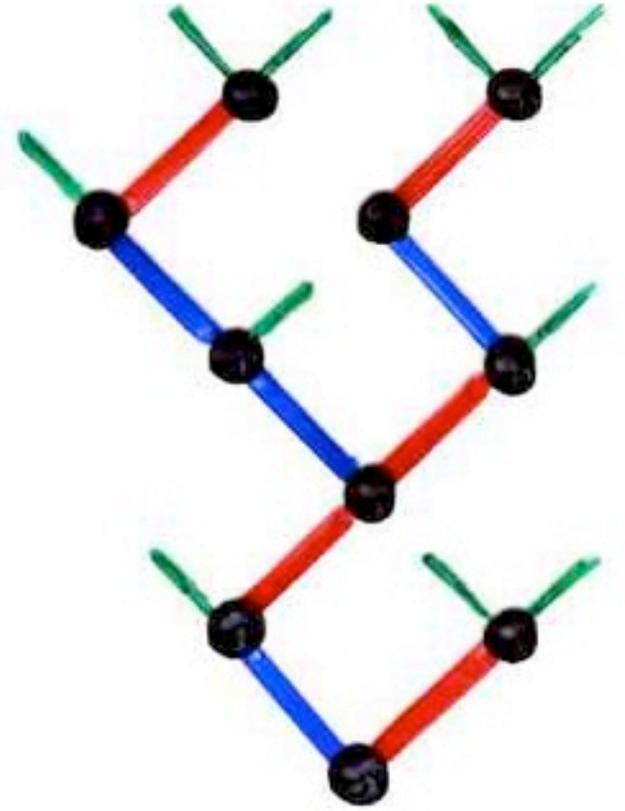
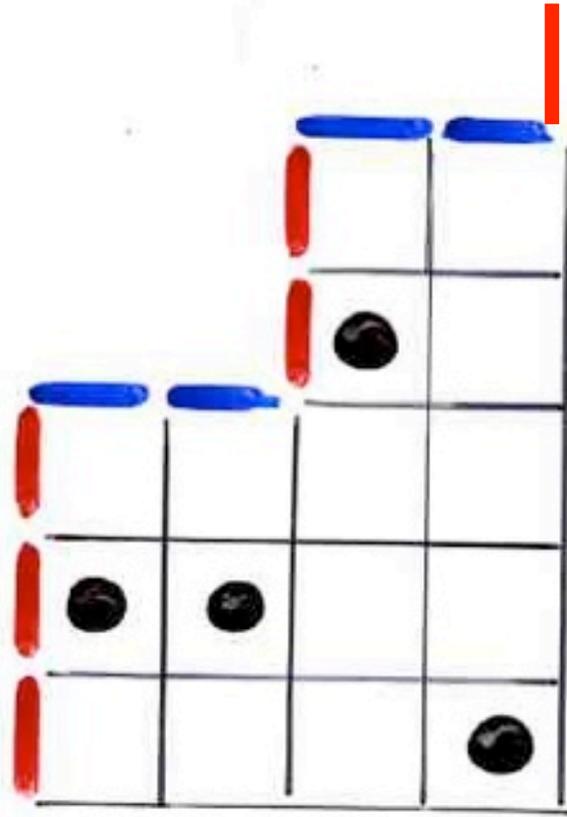
$f(\omega, \eta) = \text{nb of "contacts" horizontal}$

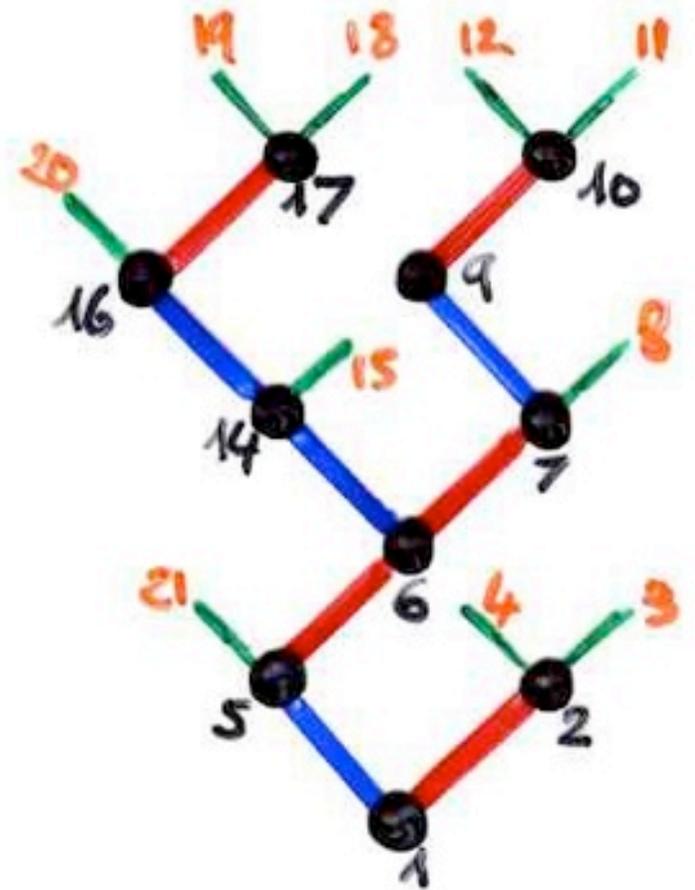
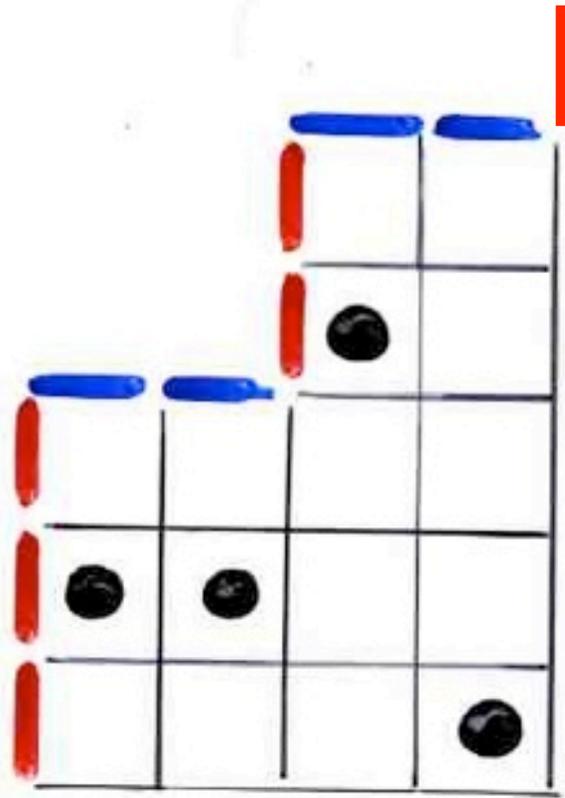


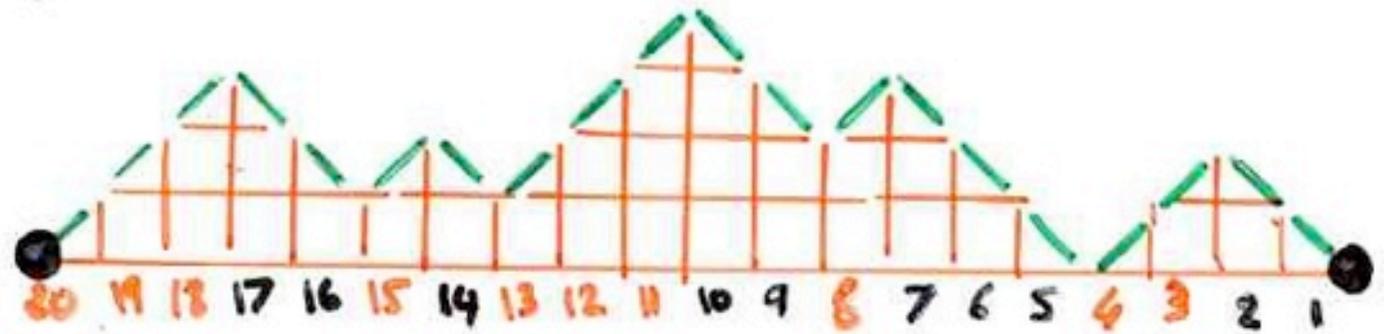
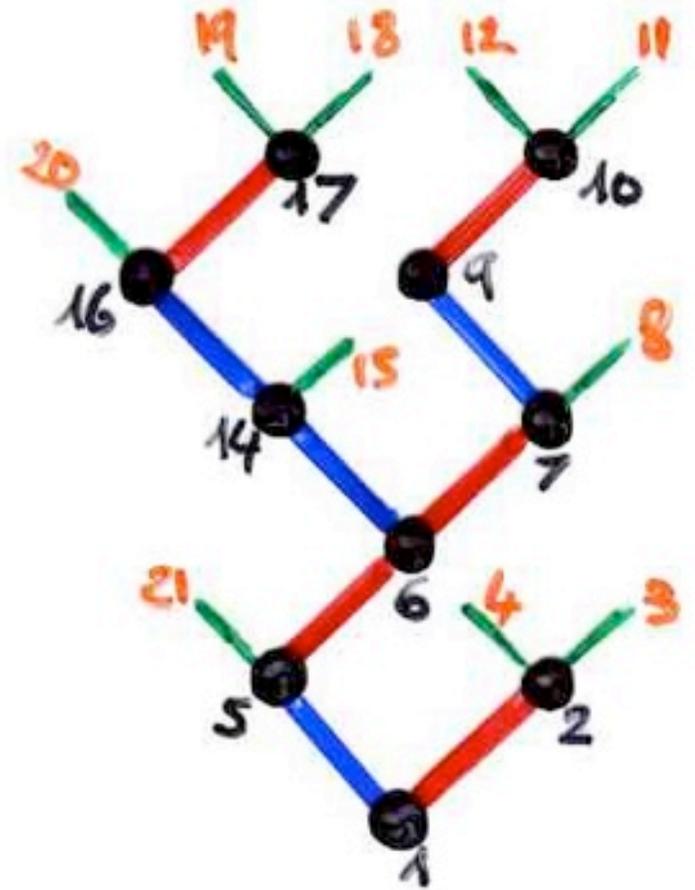
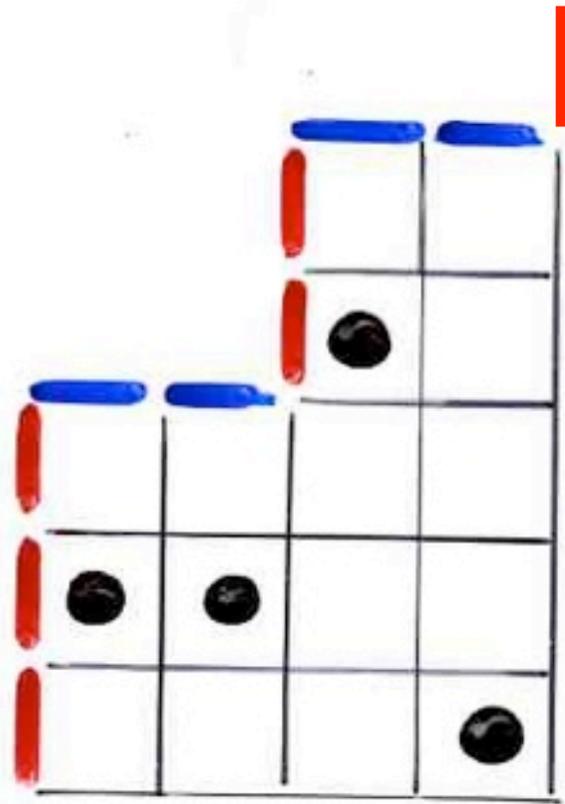
$g(\omega, \eta) = \text{nb of steps } N$ 


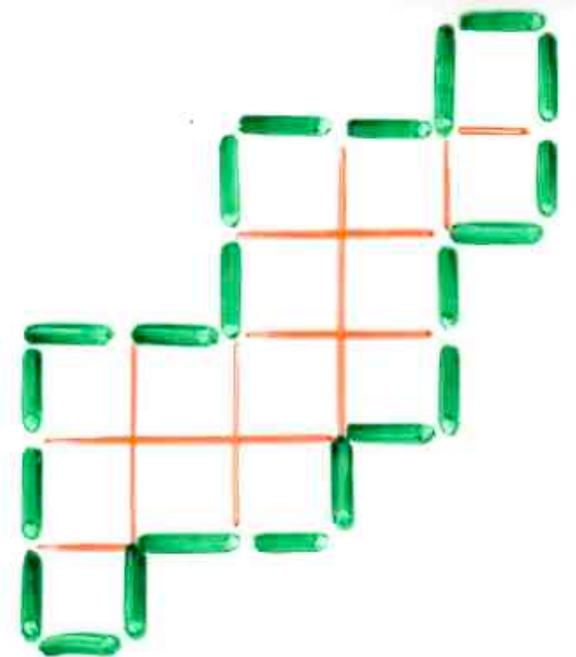
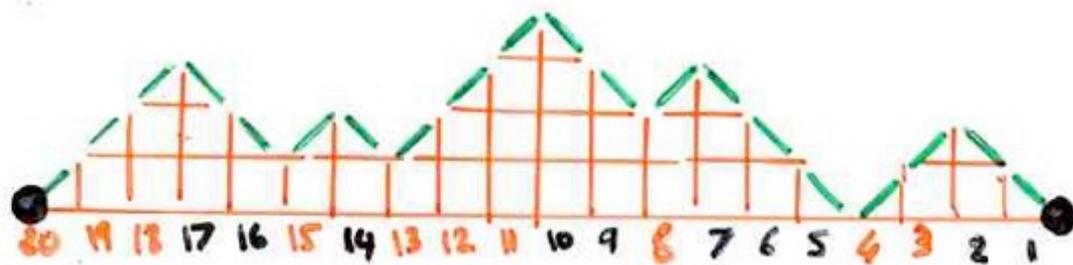
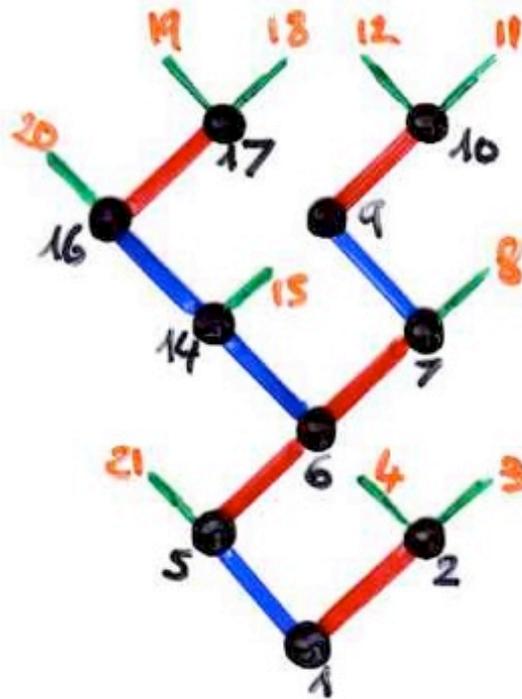
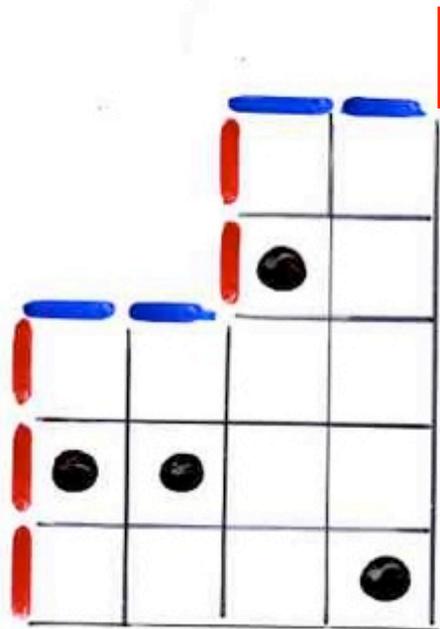
complements

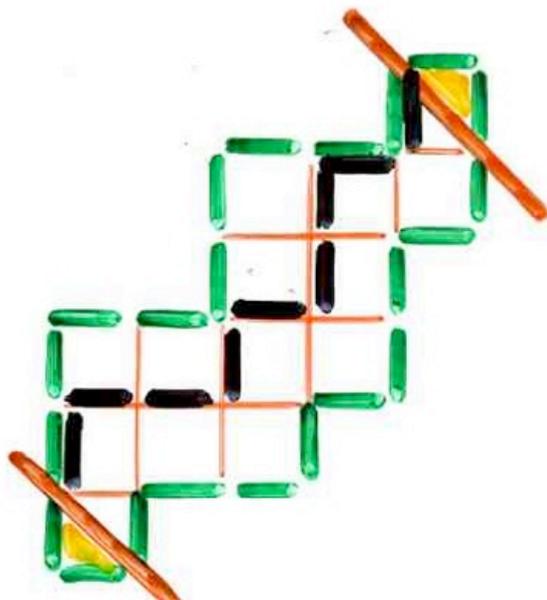
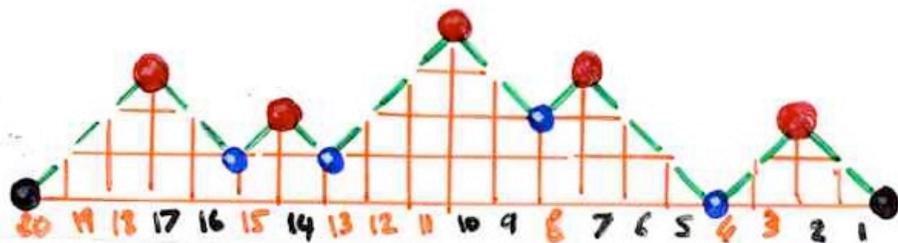
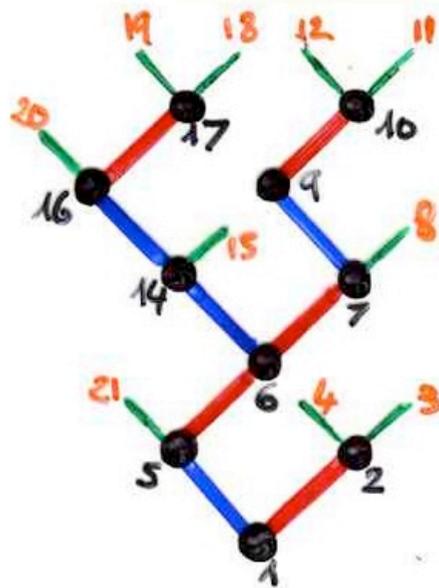
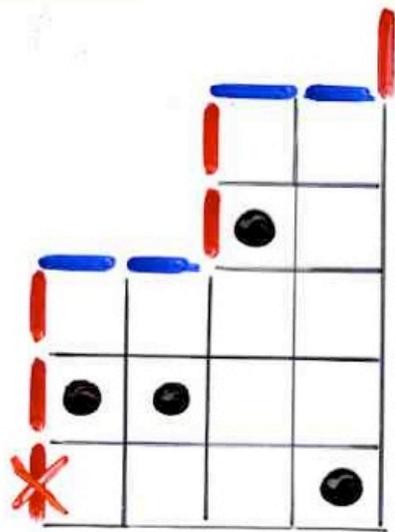


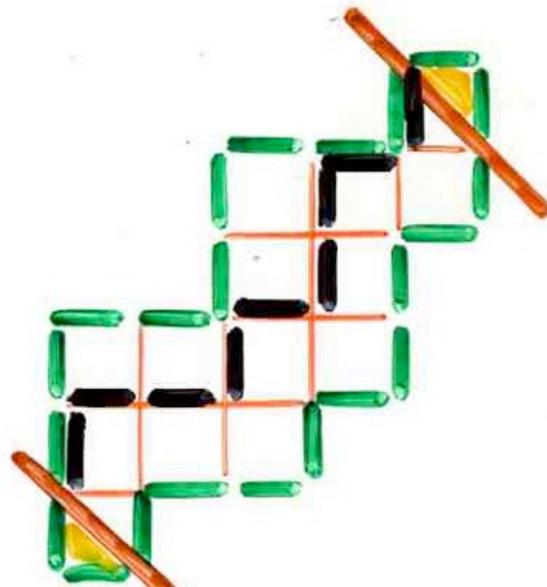
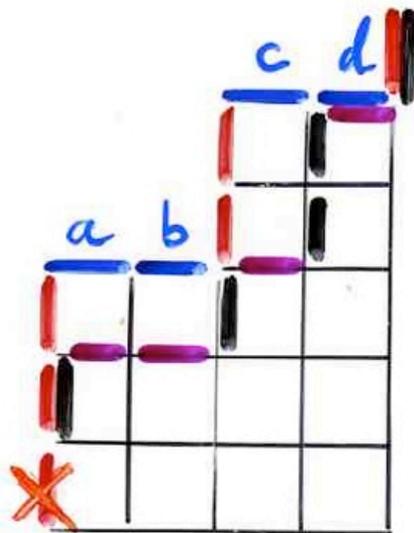
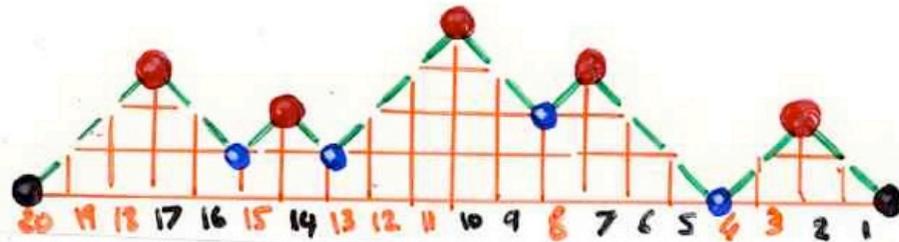
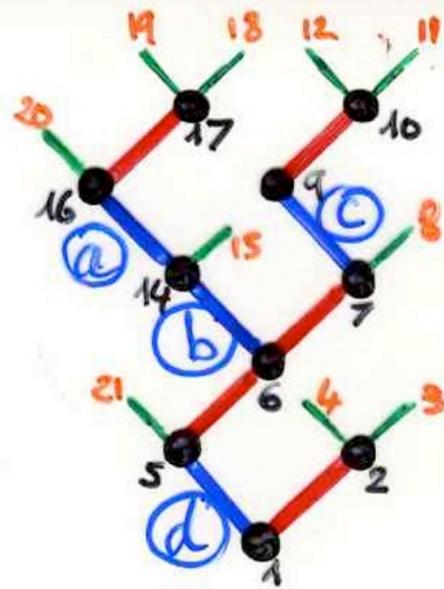
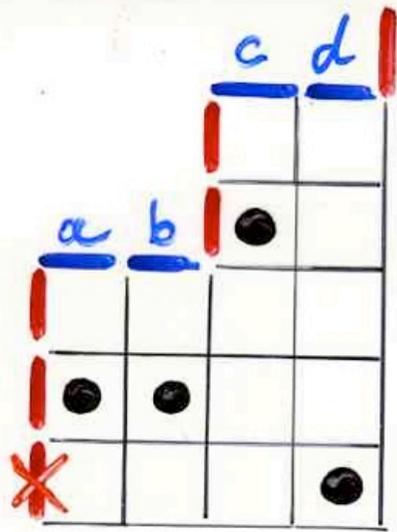


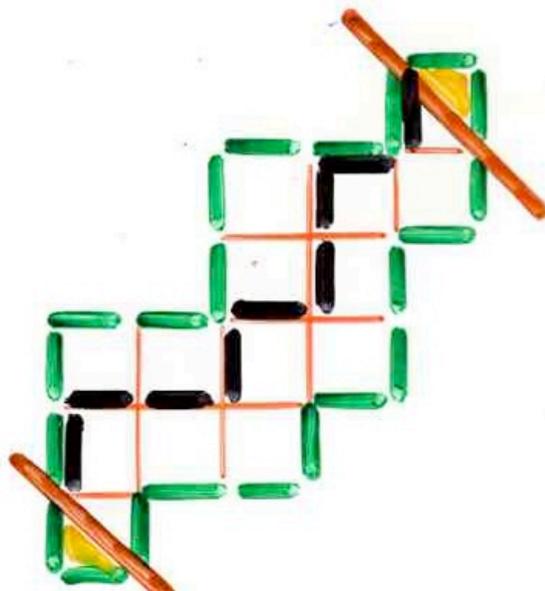
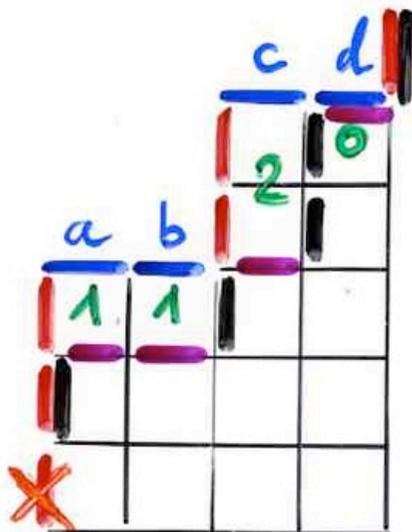
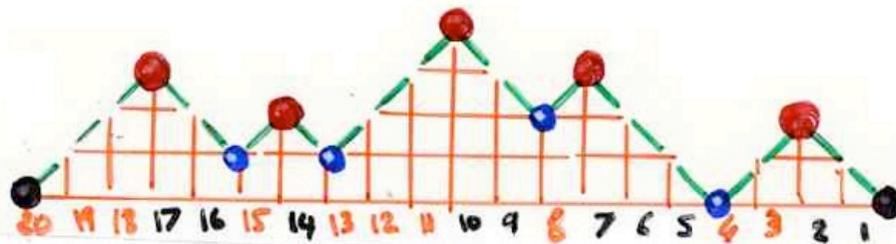
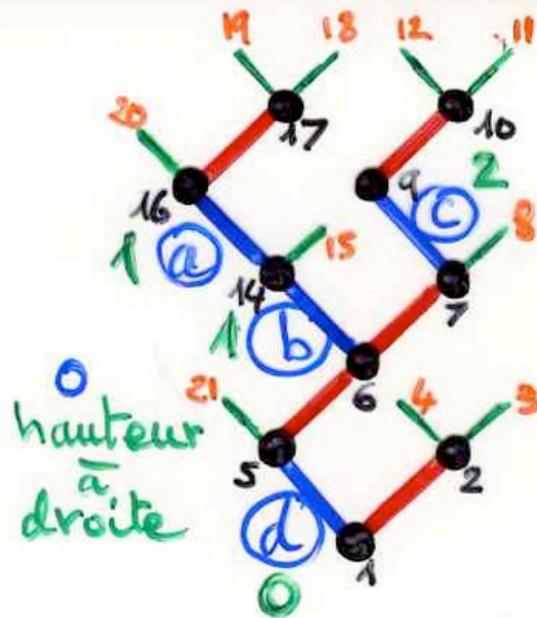
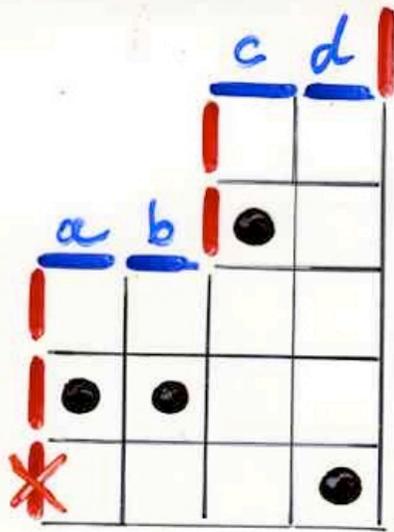












examples:

TASEP

$$\left\{ \begin{array}{l} DE = D + E \\ D|V\rangle = \bar{\beta}|V\rangle \\ \langle W|E = \bar{\alpha}\langle W| \end{array} \right.$$

examples:

$$D = \begin{bmatrix} 0 & \bar{\beta} & 0 & \dots \\ & 0 & \bar{\beta} & \dots \\ & & 0 & \dots \\ & & & \ddots \end{bmatrix}$$

$$E = \begin{bmatrix} \alpha & 1 & 0 & \dots \\ \alpha \beta & \alpha & 1 & \dots \\ \alpha \beta^2 & \alpha \beta & \alpha & \dots \\ \alpha \beta^3 & \alpha \beta^2 & \alpha \beta & \dots \\ & & & \ddots \end{bmatrix}$$

(infinite matrices)

$$\bar{\beta} = \frac{1}{\beta}, \quad \bar{\alpha} = \frac{1}{\alpha}$$

$$\langle w | = (1, 0, \dots)$$

$$|v\rangle = (1, 1, \dots)^T$$

TASEP

$$D = \begin{bmatrix} 0 & 1 & 0 & \dots \\ & 0 & 1 & \dots \\ & & 0 & \dots \\ & & & \ddots \end{bmatrix}$$

$$E = \begin{bmatrix} \bar{\beta} & 1 & 0 & \dots \\ \bar{\beta} & \bar{\beta} & 1 & \dots \\ \bar{\beta} & \bar{\beta} & \bar{\beta} & \dots \\ & & & \ddots \end{bmatrix}$$

(infinite matrices)

$$\bar{\beta} = \frac{1}{\beta}, \quad \bar{\alpha} = \frac{1}{\alpha}$$

$$\langle w | = (1, 0, \dots)$$

$$|v\rangle = (1, \bar{\alpha}, \bar{\alpha}^2, \dots)^T$$

examples:

TASEP

$$D = \begin{bmatrix} \bar{\beta} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} \alpha & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

(infinite matrices)

$$\langle w | = (1, 0, \dots, 0)$$

$$| v \rangle = (1, 0, \dots, 0)$$

$$\alpha = \frac{1}{2}$$

$$\bar{\beta} = \frac{1}{\beta}$$

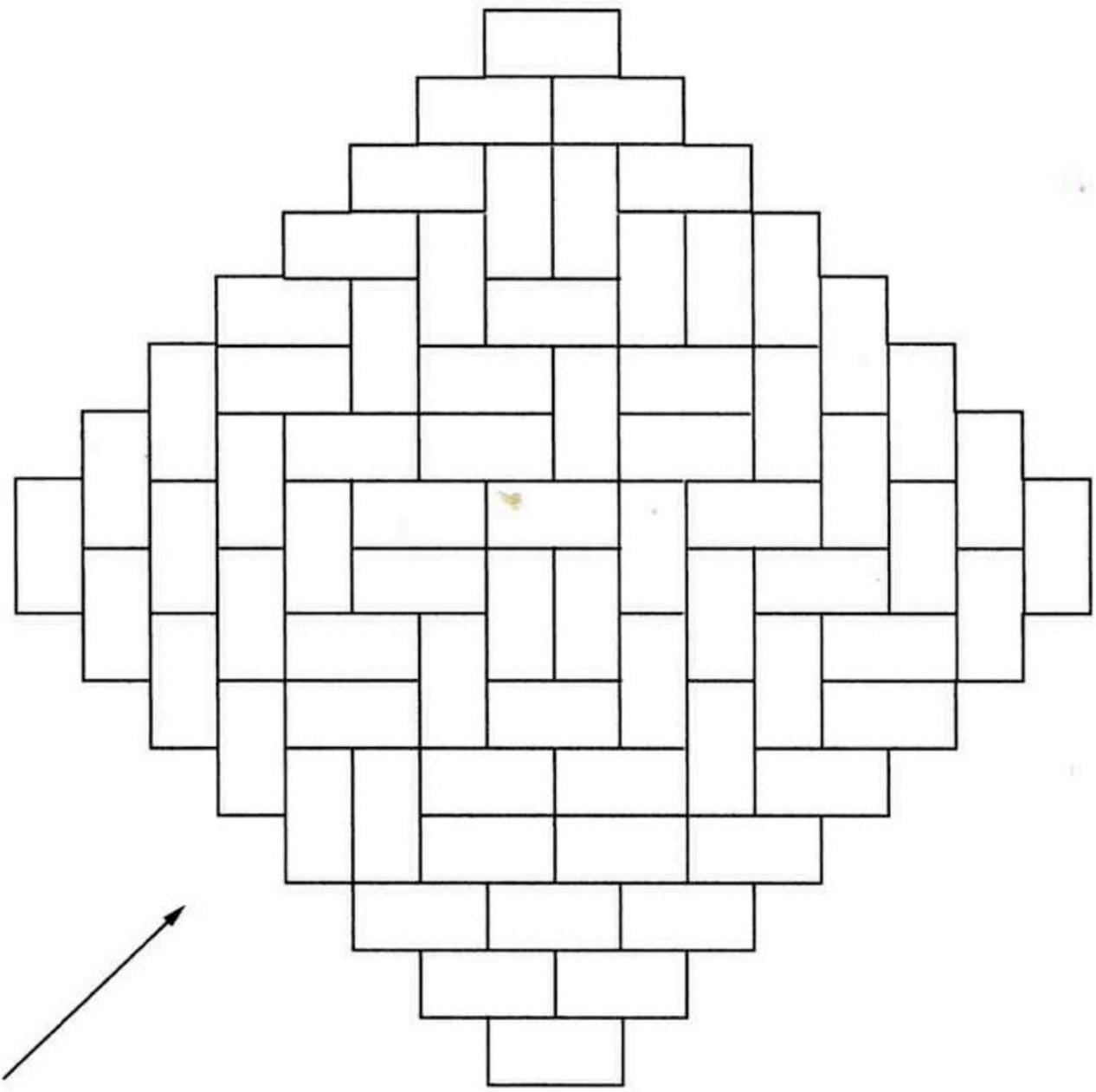
$$K^2 = \alpha + \bar{\beta} - 2\alpha\bar{\beta}$$

## 5. Pavages Aztec

Construire une algèbre quadratique  $Q$  (ou un automate planaire) permettant de "reconnaître" un pavage Aztec comme un  $Q$ -tableau. En déduire que le nombre de pavages Aztec de taille  $n$  est égal à  $A_n(2)$ , dans lequel  $A_n(x)$  désigne le polynôme énumérant les matrices à signes alternants de taille  $n$  selon le nombre de  $(-1)$ . (tourner d'abord de 45 degrés le diagramme Aztec et son pavage).

$$2^{n(n-1)/2}$$

$$A_n(2)$$



Elkies,  
Kuperberg,  
Larsen,  
Propp  
(1992)



# The quadratic algebra $Z$

4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A \cdot B \\ B \cdot A = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} A B \\ B \cdot A = q_{\cdot\cdot} A B + t_{\cdot\cdot} A \cdot B \\ BA = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} A B \end{array} \right.$$

$$t_{\bullet\bullet} = t_{\bullet\bullet} = 0$$

The quadratic algebra  $\mathbb{Z}$

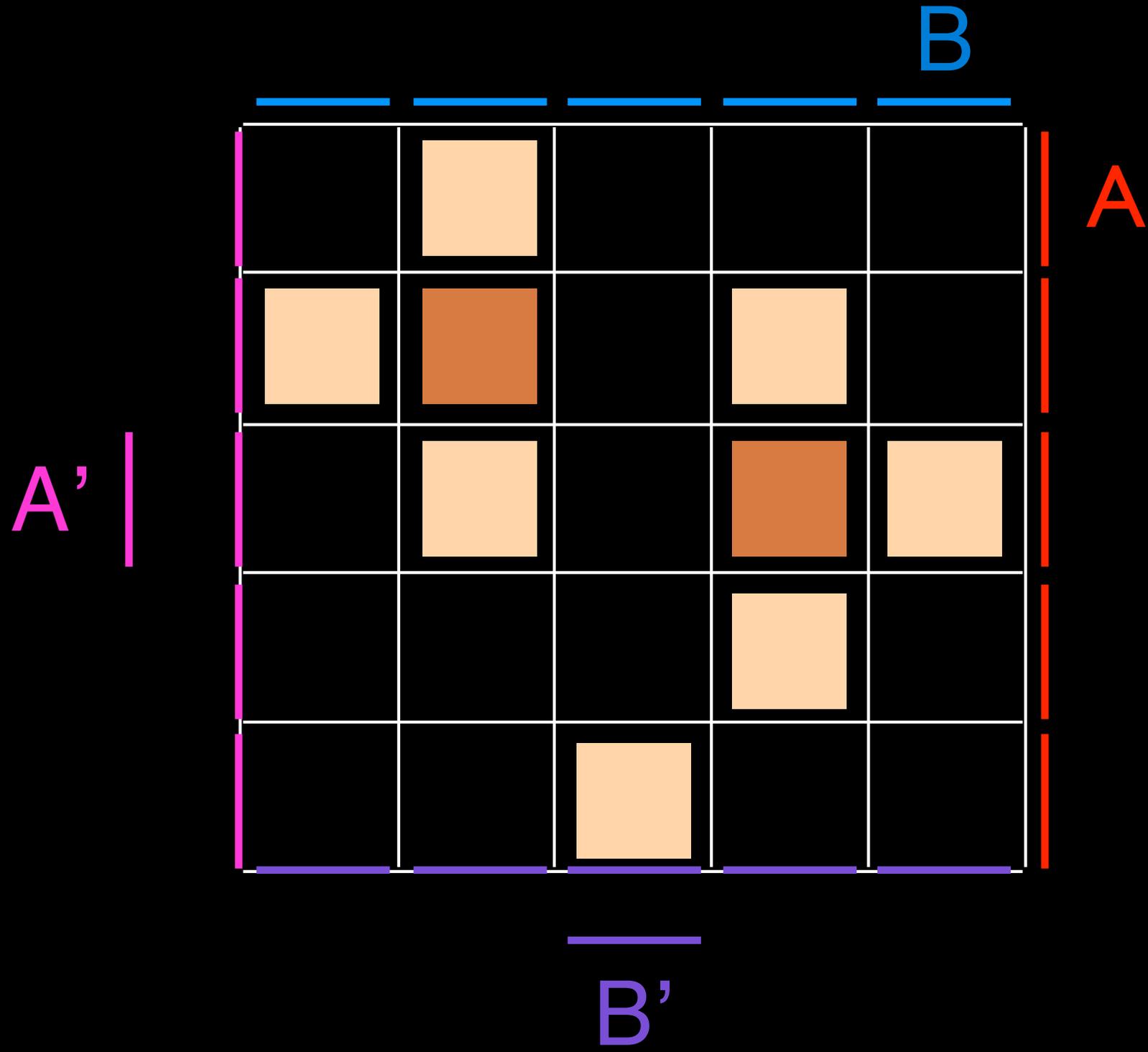
4 generators  $B, A, BA$

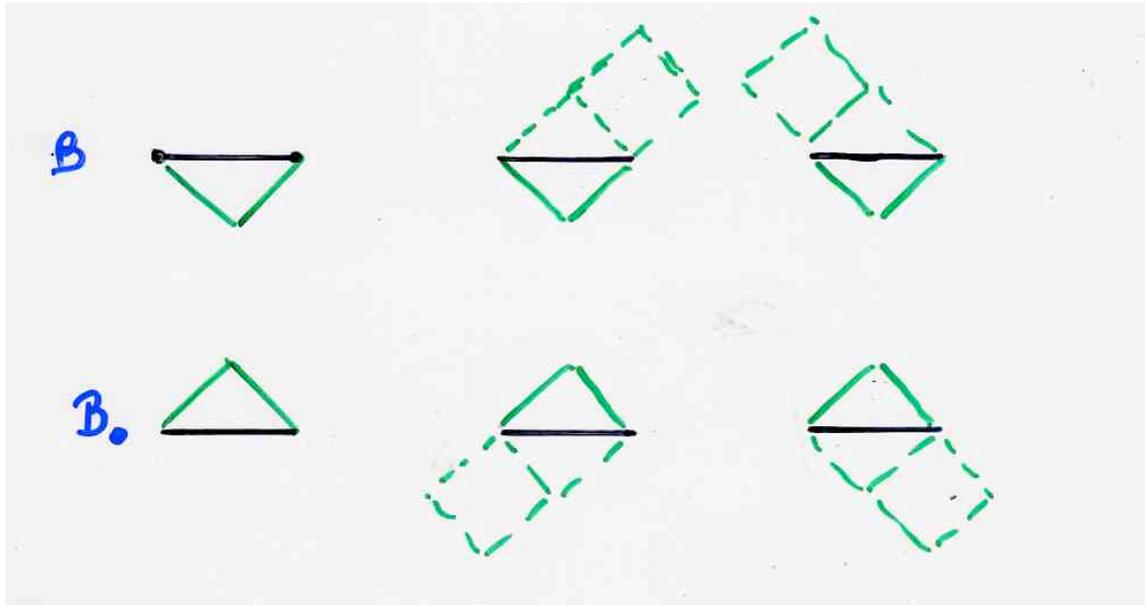
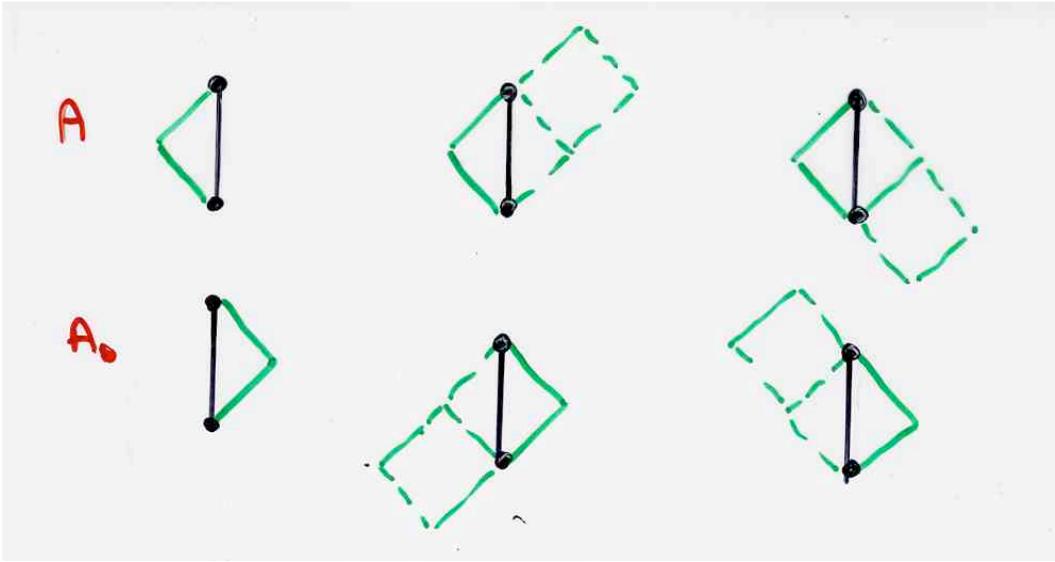
8 parameters  $q, \dots, t, \dots$

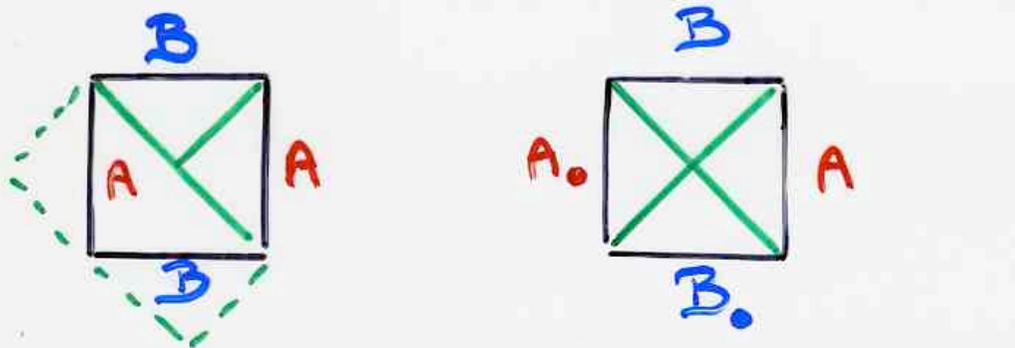
$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A \cdot B \\ B \cdot A = q_{\bullet\bullet} A \cdot B + t_{\bullet\bullet} A B \\ B \cdot A = q_{\bullet\bullet} A B + \bigcirc A \cdot B \\ BA = q_{\bullet\bullet} A \cdot B + \bigcirc A B \end{array} \right.$$

$$w = B^n A^n \quad uv = A^n B^n$$

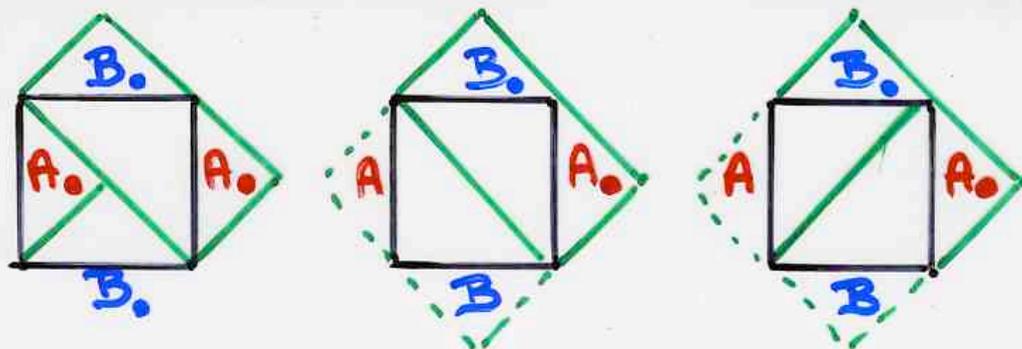
$$e(u, v; w) = \text{nb of ASM } n \times n$$







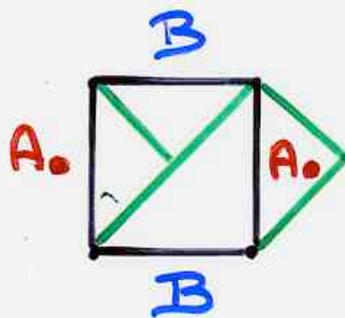
$$BA = AB + A \cdot B$$



$$B \cdot A = A \cdot B + 2AB$$



$$B \cdot A = AB$$



$$BA = A \cdot B$$

## Aztec tilings

$$t_{\bullet\bullet} = t_{\bullet\bullet} = 0 \quad (\text{ASM})$$

$$t_{\bullet\bullet} = 2$$

(nb of  $-1$  in ASM)

The quadratic algebra  $\mathbb{Z}$

4 generators  $B, A, BA$   
8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{\bullet\bullet} AB + t_{\bullet\bullet} A \cdot B \\ B \cdot A = q_{\bullet\bullet} A \cdot B + 2 AB \\ B \cdot A = q_{\bullet\bullet} A \cdot B + \bigcirc A \cdot B \\ BA = q_{\bullet\bullet} A \cdot B + \bigcirc A \cdot B \end{array} \right.$$

## 6. ASM et FPL

Pour toute matrice à signe alternants (ASM), on associe une "configuration B.A.BA" en identifiant les entrées  $1$  et  $-1$ . Caractériser de telles configurations. Caractériser leur complémentaire. Montrer que cette bijection "passage au complémentaire" est sous une "forme déguisée" la bijection classique entre ASM et FPL ("fully packed loops").


■		■	■	■
		■		■
■		■		
■	■	■		■
■	■		■	■

configuration

B.A. BA

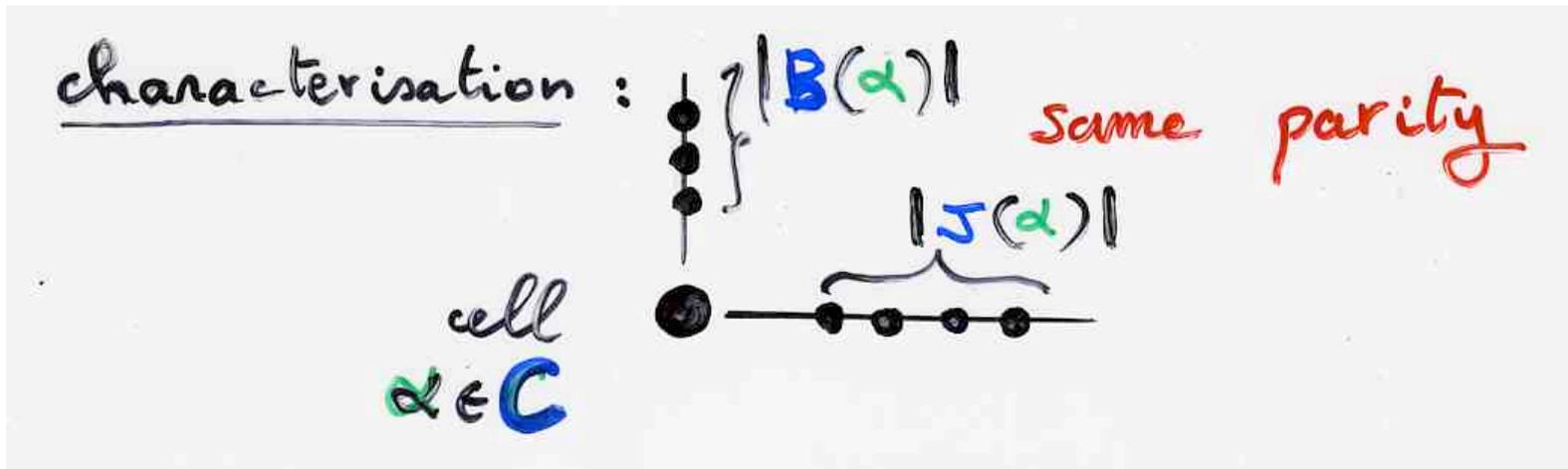
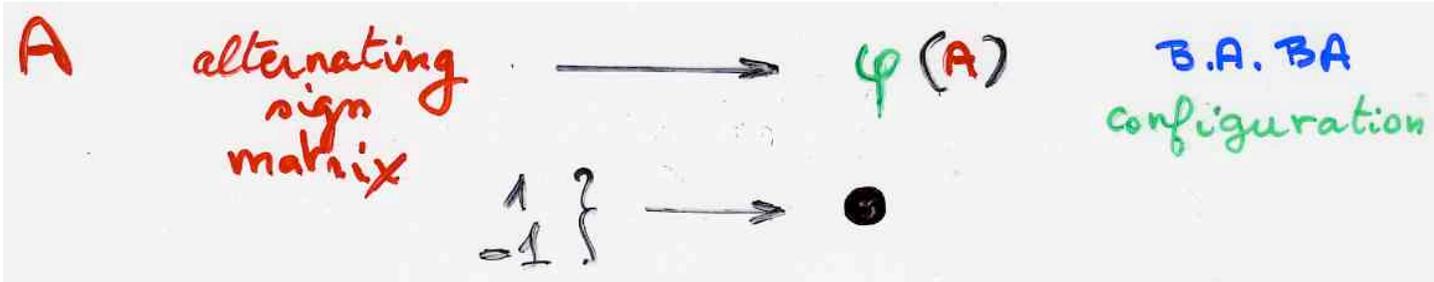
configuration  
B.A. BA

Prop. The number of configuration B.A. BA  
on  $n \times n$  is  $2^{(n^2)}$

A alternating sign matrix  $\longrightarrow \varphi(A)$  B.A. BA configuration  
 $\left. \begin{matrix} 1 \\ -1 \end{matrix} \right\} \longrightarrow \bullet$

	Light Orange			
Light Orange	Dark Orange		Light Orange	
	Light Orange		Dark Orange	Light Orange
			Light Orange	
		Light Orange		

	■			
■	■		■	
	■		■	■
			■	
		■		



+ odd in each row and column number of cells in C

passage au complémentaire

# The quadratic algebra $Z$

4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A \cdot B \\ B \cdot A = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} A B \\ B \cdot A = q_{\cdot\cdot} A B + t_{\cdot\cdot} A \cdot B \\ BA = q_{\cdot\cdot} A \cdot B + t_{\cdot\cdot} A B \end{array} \right.$$

$$t_{\bullet\bullet} = t_{\bullet\bullet} = 0$$

The quadratic algebra  $\mathbb{Z}$

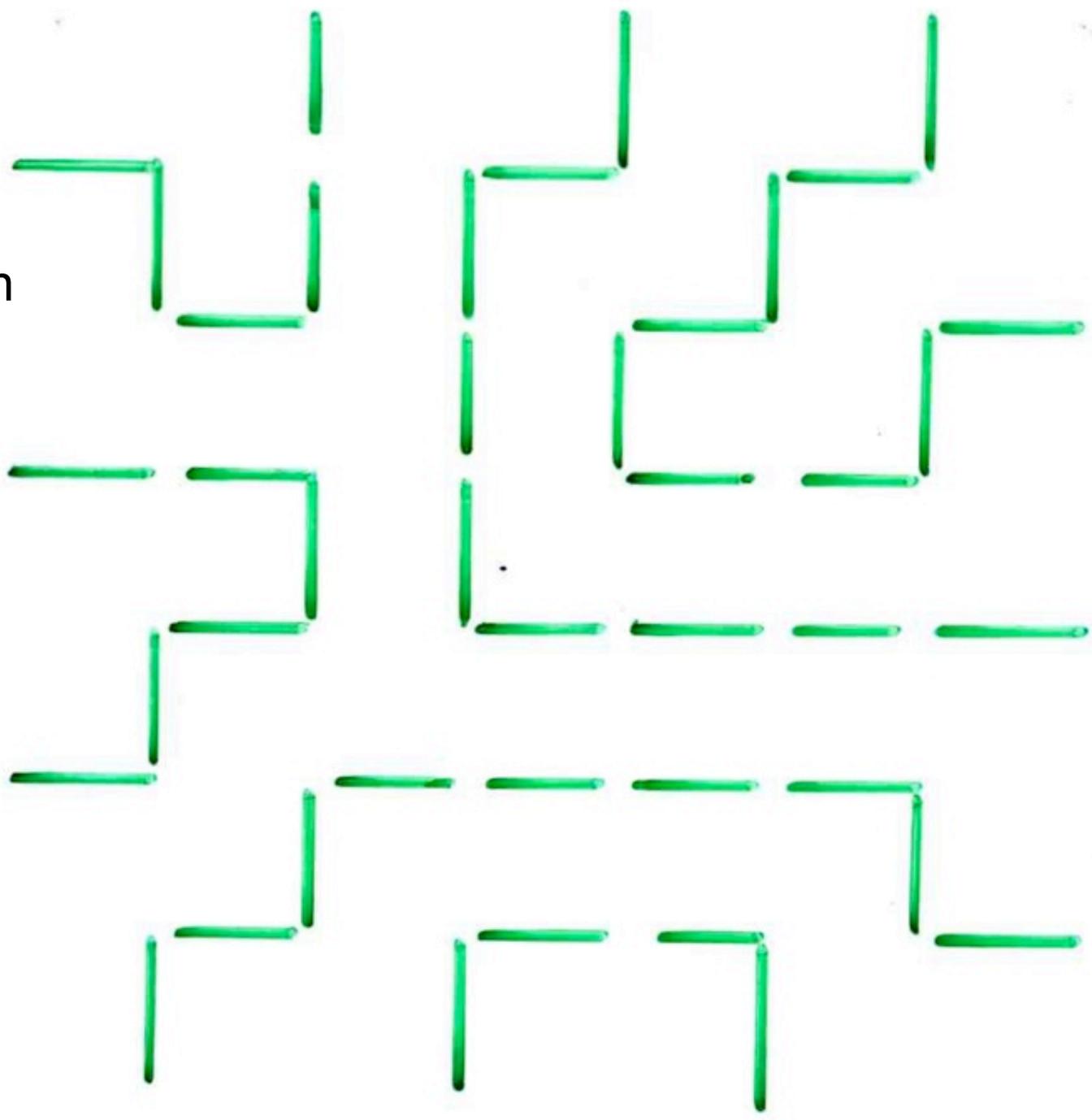
4 generators  $B, A, BA$   
 8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = q_{00} AB + t_{00} A \cdot B \\ B \cdot A = q_{\bullet\bullet} A \cdot B + t_{\bullet\bullet} A B \\ B \cdot A = q_{\bullet\bullet} A B + \bigcirc A \cdot B \\ BA = q_{\bullet\bullet} A \cdot B + \bigcirc A B \end{array} \right.$$

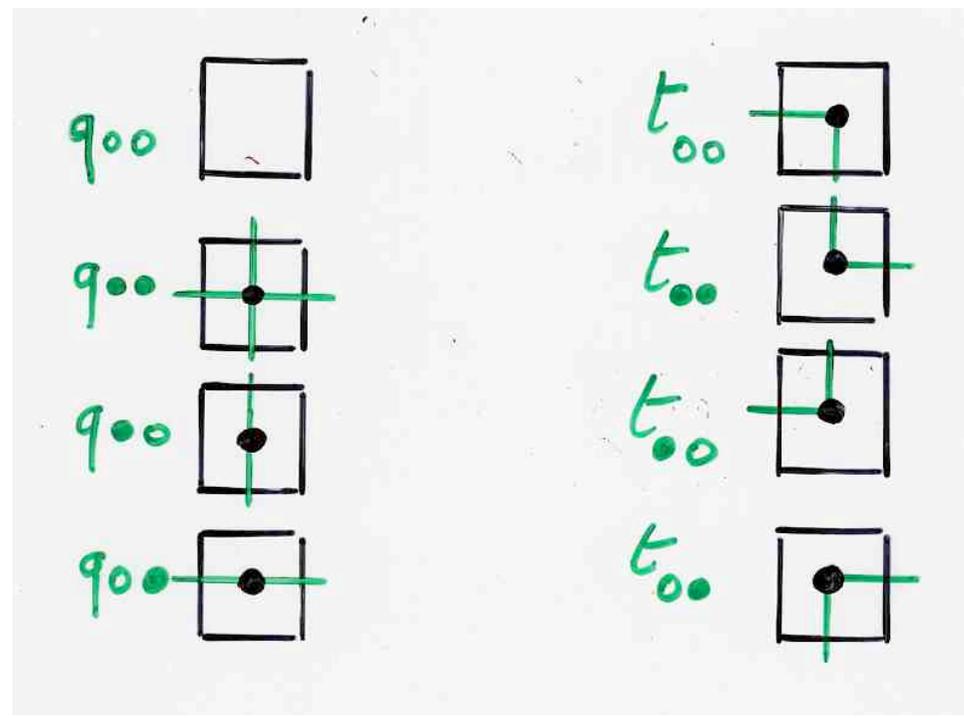
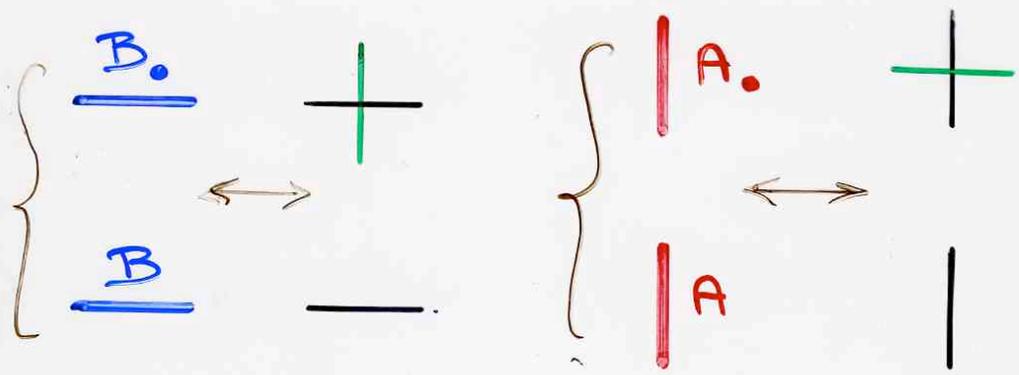
$$w = B^n A^n \quad uv = A^n B^n$$

$$e(u, v; w) = \text{nb of ASM } n \times n$$

FPL  
“Fully  
Packed  
Loop”  
configuration



geometric interpretations of  $Z$ -tableaux

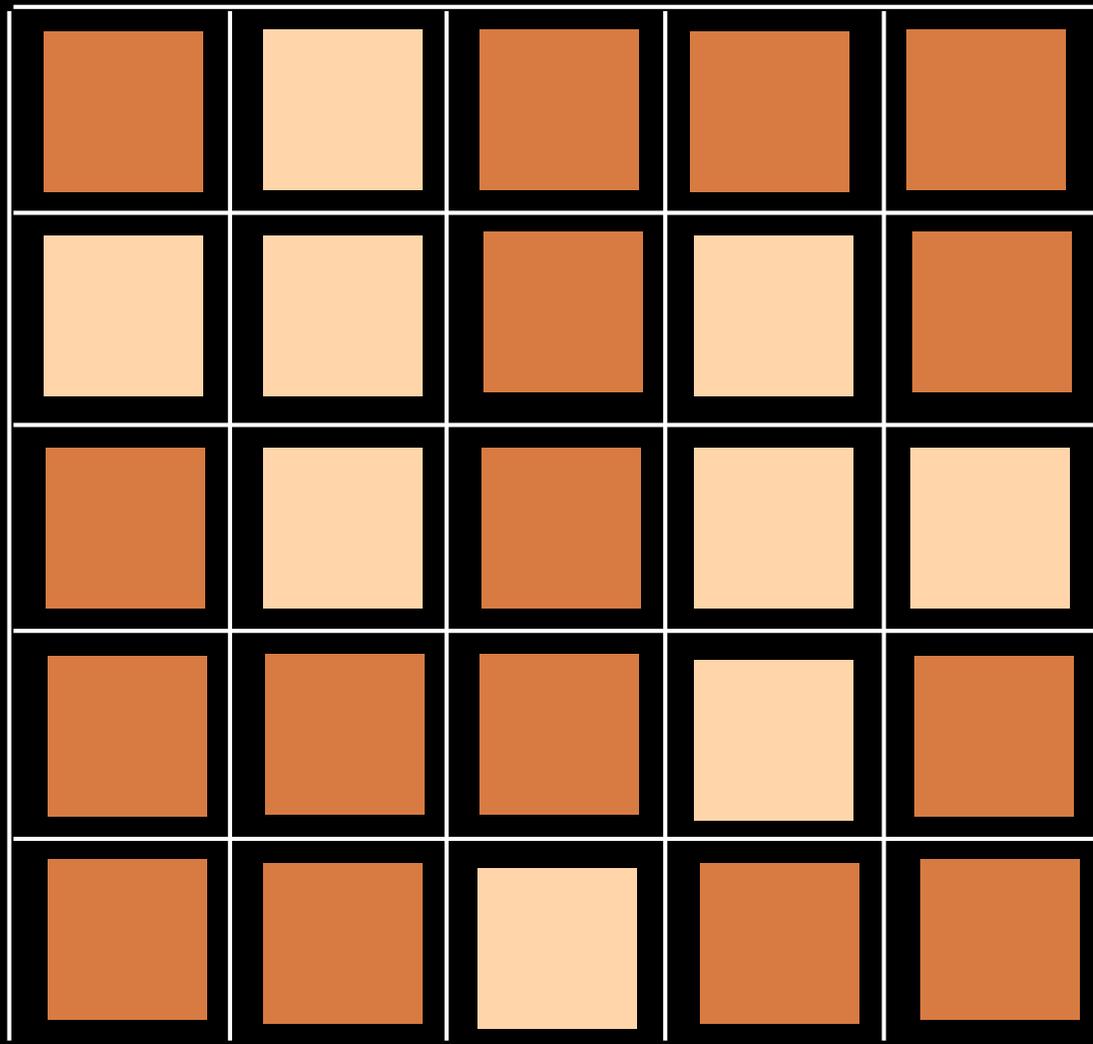


8-vertex model

The quadratic algebra  $\mathcal{Z}$

4 generators  $B, A, B, A$   
8 parameters  $q, \dots, t, \dots$

$$\left\{ \begin{array}{l} BA = \bigcirc AB + t_{00} A \cdot B \\ B \cdot A = \bigcirc A \cdot B + t_{\cdot\cdot} A B \\ B \cdot A = q_{00} A B + t_{\cdot\cdot} A \cdot B \\ BA = q_{00} A \cdot B + t_{00} A B \end{array} \right.$$



ex: complement of a configuration  $C$

$C$  rules on  $\alpha$ : always take  $t_{xy}$   
 if  $\alpha \in C$   
 if  $w = B^n A^n \rightarrow T$   $\Sigma$ -tollean

The label of  $\alpha$  in  $T$  is :

- if  $\alpha \in C$ ,  $t_{xy}$  with  $\begin{cases} x = \bullet \text{ or } 0 \\ y = \bullet \text{ or } 0 \end{cases}$  if  $\begin{cases} J(\alpha) \text{ odd} \\ B(\alpha) \text{ even} \end{cases}$

-  $\alpha \notin C$   $q_{xy}$  with  $\begin{cases} \square \\ \vdots \\ \square \dots \end{cases}$  " "

Change the rule (complement)  
 if  $x \in C$  always take  $\bar{x}$

Coloring  $[n] \times [n]$  in four colors

-	-	PP	PI	PP
-	.	IP	II	IP
-	.	PP	PI	PP
		,	,	,

$\boxed{PP}$

$\boxed{II}$

$\boxed{PI}$

$\boxed{IP}$

$$q_{xy} \leftrightarrow t_{xy}$$

$$q_{xy} \leftrightarrow t_{\bar{x}\bar{y}}$$

$$q_{xy} \leftrightarrow t_{x\bar{y}}$$

$$q_{xy} \leftrightarrow t_{\bar{x}y}$$

$$x = \begin{pmatrix} \bullet \\ 0 \end{pmatrix} \quad \bar{x} = \begin{pmatrix} 0 \\ \bullet \end{pmatrix}$$

ASM

$$t_{00} = t_{00} = 0 \quad w = B^n A^n$$

complement

$\Rightarrow$   $(q_{00}$  and  $q_{00}$  forbidden on cells  $\{PP, II\}$ )

and  $(q_{00}$  and  $q_{00}$  forbidden on cells  $\{PI, IP\}$ )

if

$$w = \underbrace{B_x \dots B \cdot B \cdot B}_n \cdot \underbrace{A \cdot A \cdot A \cdot \dots}_n$$

then

$q_{00}$  and  $q_{00}$  forbidden for every cell

$\rightarrow$  bijection  $ASM \leftrightarrow FPL$

same with  $w = \dots B \cdot B \cdot B \cdot A \cdot A \cdot A \cdot A \cdot \dots$

## 7. Automate planaire

Est-ce qu'un tableau de permutation est reconnaissable par un automate planaire ?  
En d'autres termes, les tableaux de permutations sont-ils des  $Q$ -tableaux ?

# Def. planar automaton $\mathcal{P}$

- 3 finite sets  $\left\{ \begin{array}{l} \cdot \mathcal{B} \\ \cdot \mathcal{A} \\ \cdot \mathcal{S} \end{array} \right.$  horizontal vertical alphabet  
planar labels (state)

- $\theta$  (partial) transition function

$$(\mathcal{A}, \mathcal{B}, \mathcal{A}) \xrightarrow{\theta} (\mathcal{B}', \mathcal{A}') \quad \text{or } \emptyset$$

$\mathcal{A} \in \mathcal{S}; \quad \mathcal{B}, \mathcal{B}' \in \mathcal{B}; \quad \mathcal{A}, \mathcal{A}' \in \mathcal{A}$

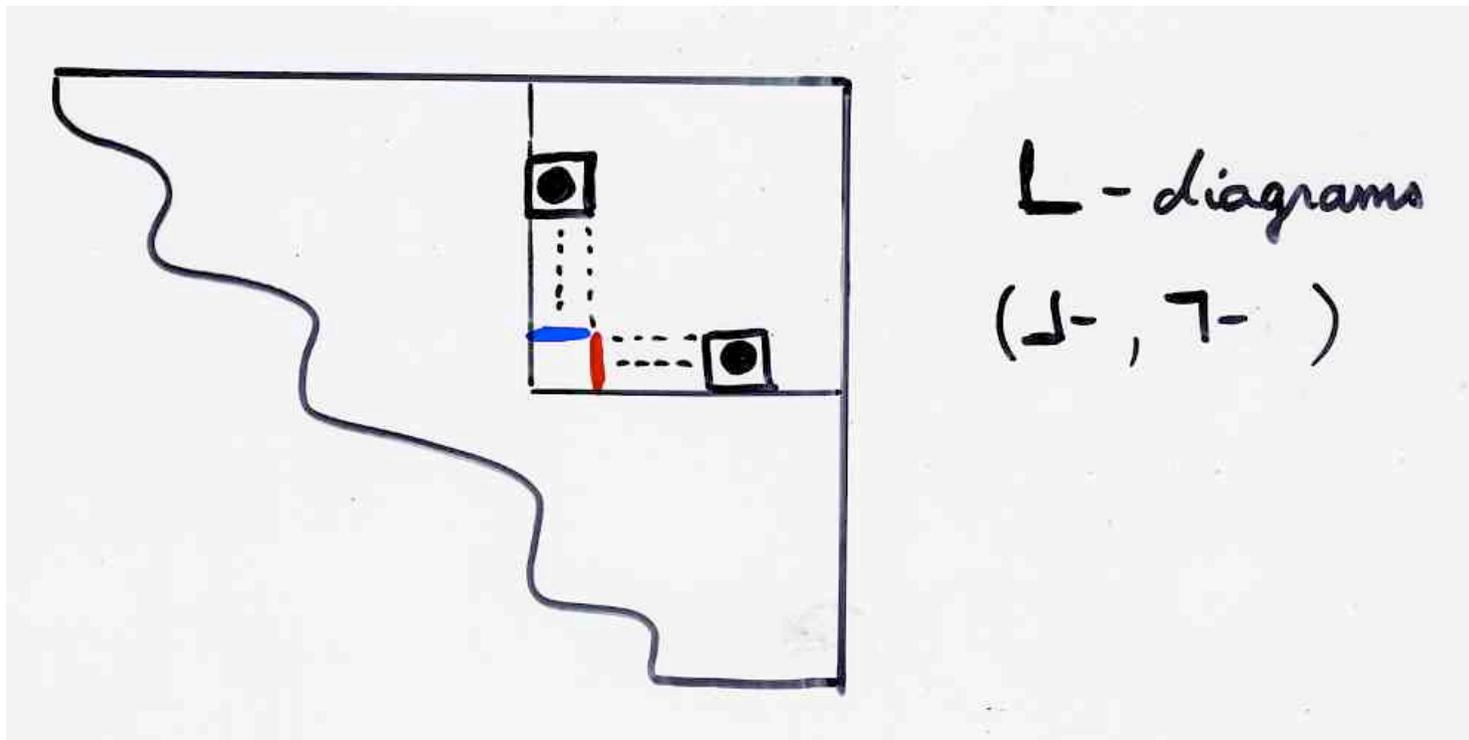
- $w \in (\mathcal{A} \cup \mathcal{B})^*$  initial word
- $uv, \quad u \in \mathcal{A}^*, \quad v \in \mathcal{B}^*$  final word





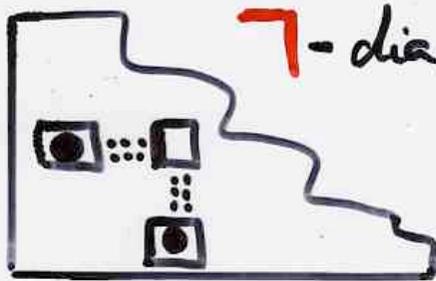
«Figures»  
accepted by planar automata ?

- **surjectivity**  
in diagrams

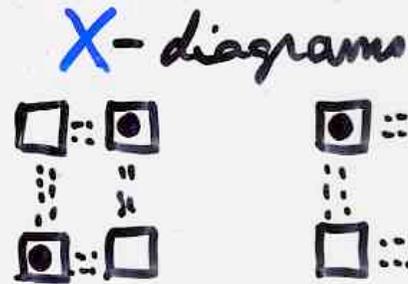


Bijections between **pattern-avoiding** fillings of Young diagrams

Josuat-Vergès (2008)



**7**-diagrams



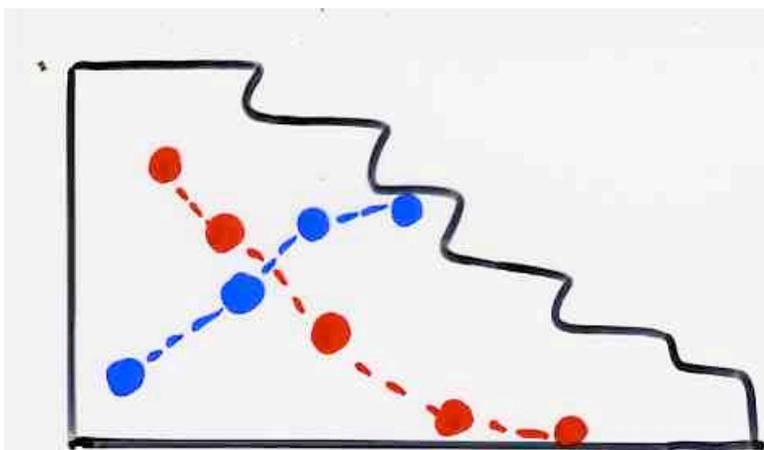
**X**-diagrams



increasing  
decreasing chains in fillings of Ferrers shapes

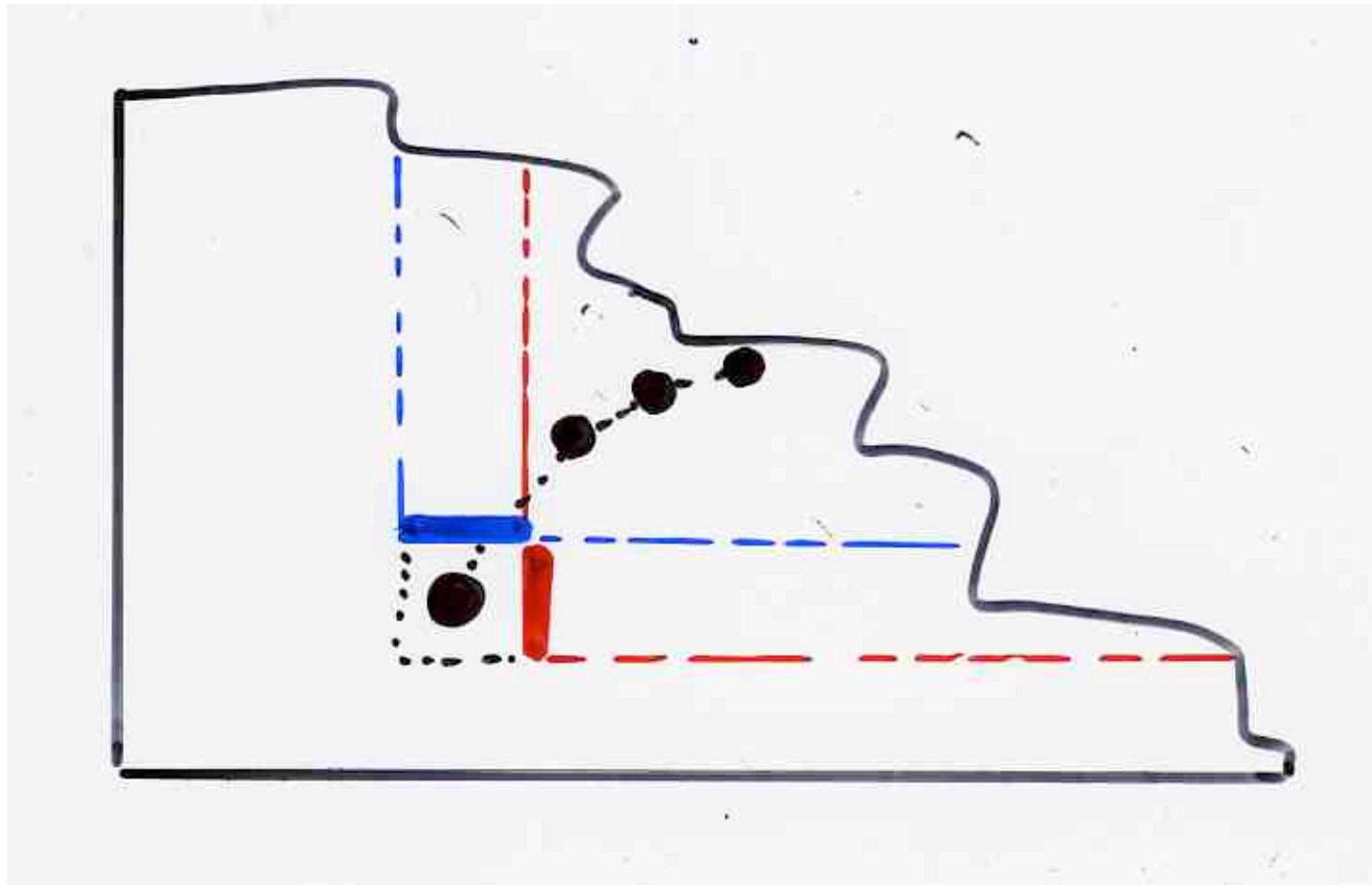
(Jonsson, 2005) (Knattenthaler, 2006)

(Bachelin, West, Xin, 2005) (Bousquet-Mélou, Steingrimsón, 2005) ...



increasing  
decreasing

subsequences  
(chains)



## 8. q-Laguerre

(cours II)

Définir la bijection inverse de la correspondance "histoires de Laguerre" -- permutations. En déduire que le paramètre "q-Laguerre" défini sur les "histoires de Laguerre" est en fait le nombre de "motifs 31-2" sur la permutation correspondante, c'est-à-dire les triplets tels que:  $i < j = (i + 1) < k, \sigma(j) < \sigma(k) < \sigma(i)$

Bijection

Permutations

$n+1$

Histoires

de Laguerre

$(\gamma_c, \mathcal{f})$

$n$

Bijection

Permutations

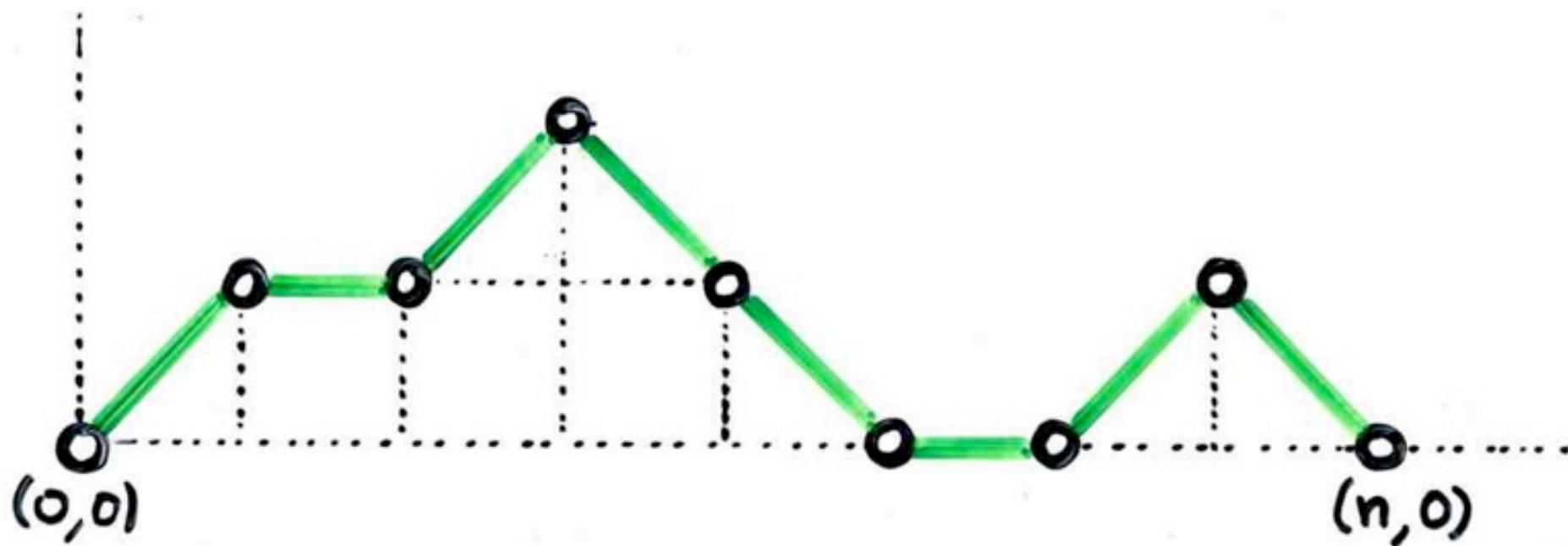
$n+1$

Histoires de Laguerre

$(\gamma_c, f)$

$n$

Chemin de Motzkin  
 $n$



Bijection

Permutations

$n+1$

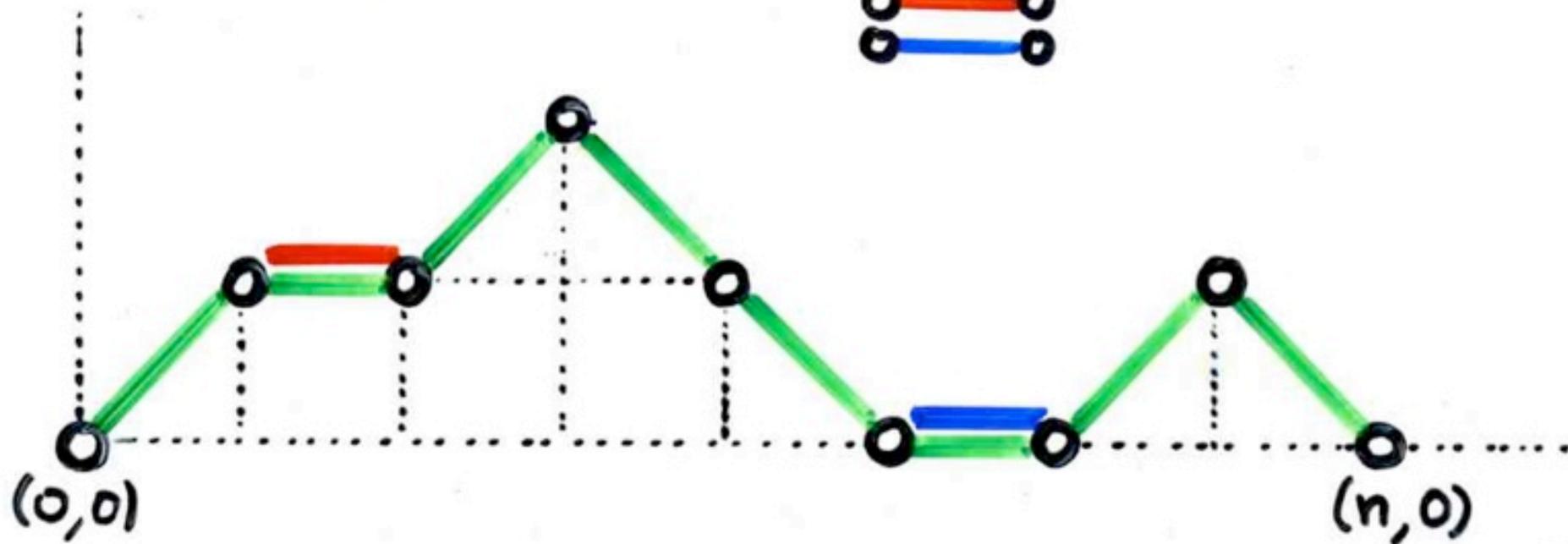
Histoires de Laguerre

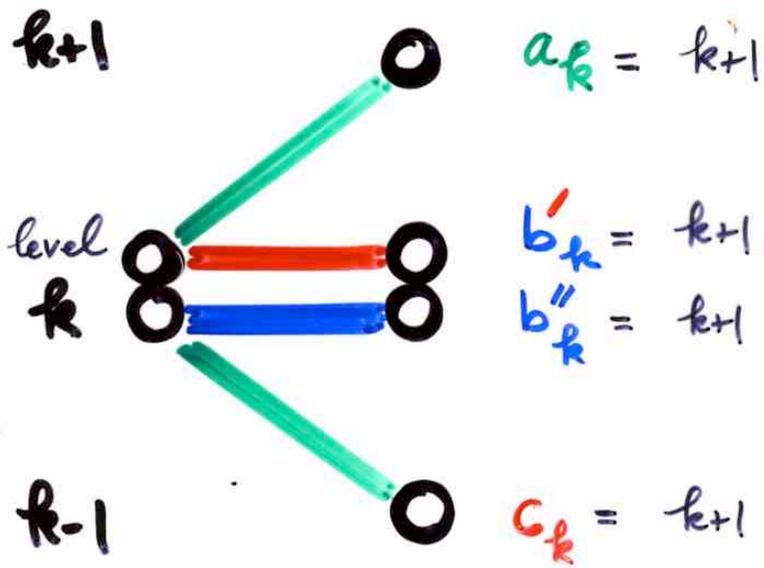
$(\gamma, c, \delta)$

$n$

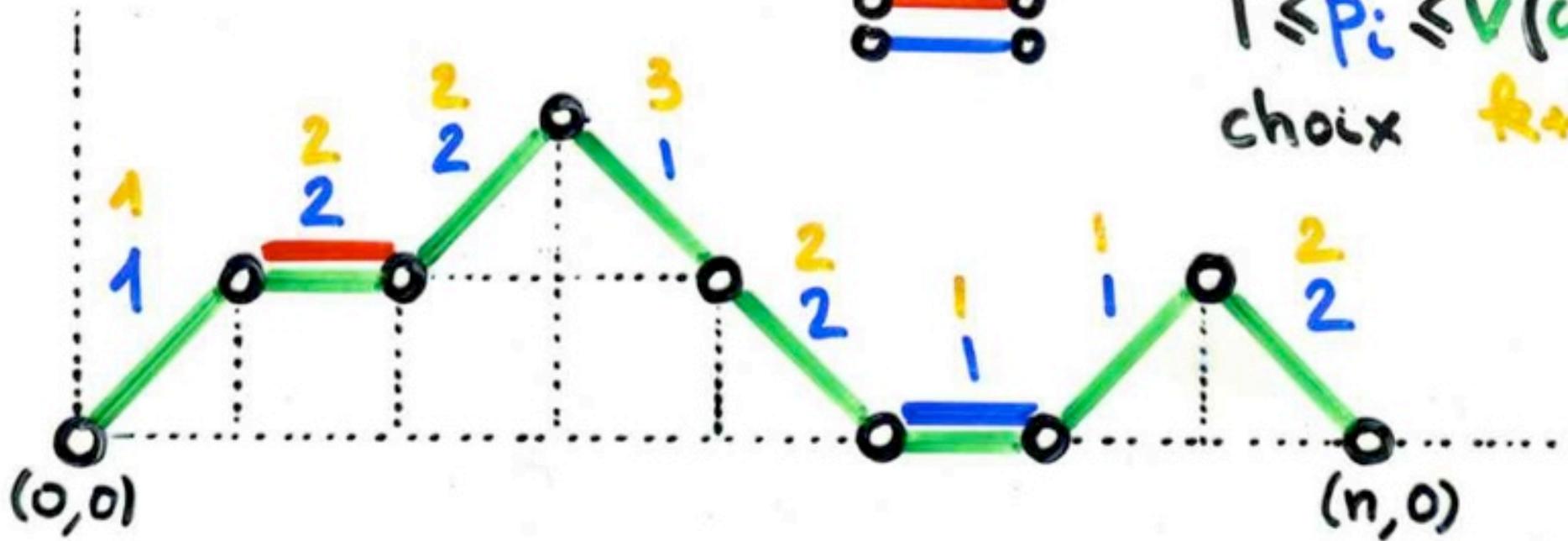
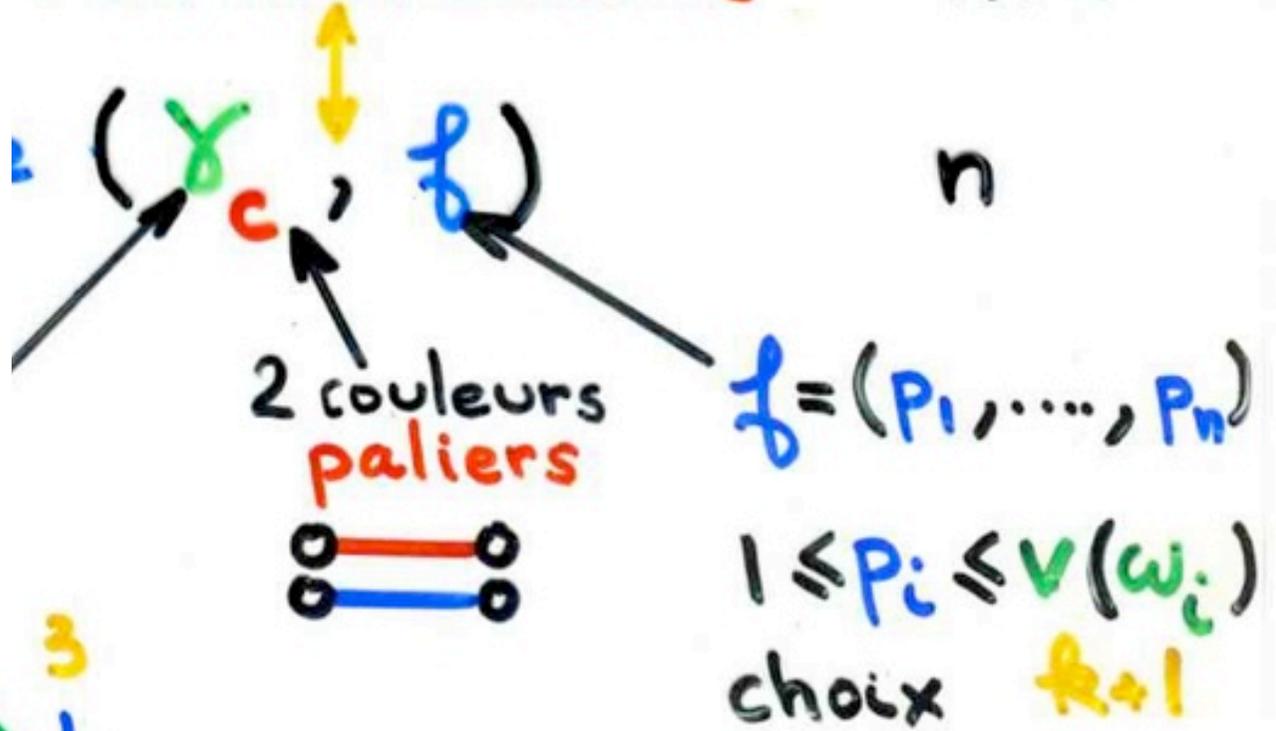
Chemin de Motzkin  
 $n$

2 couleurs  
paliers

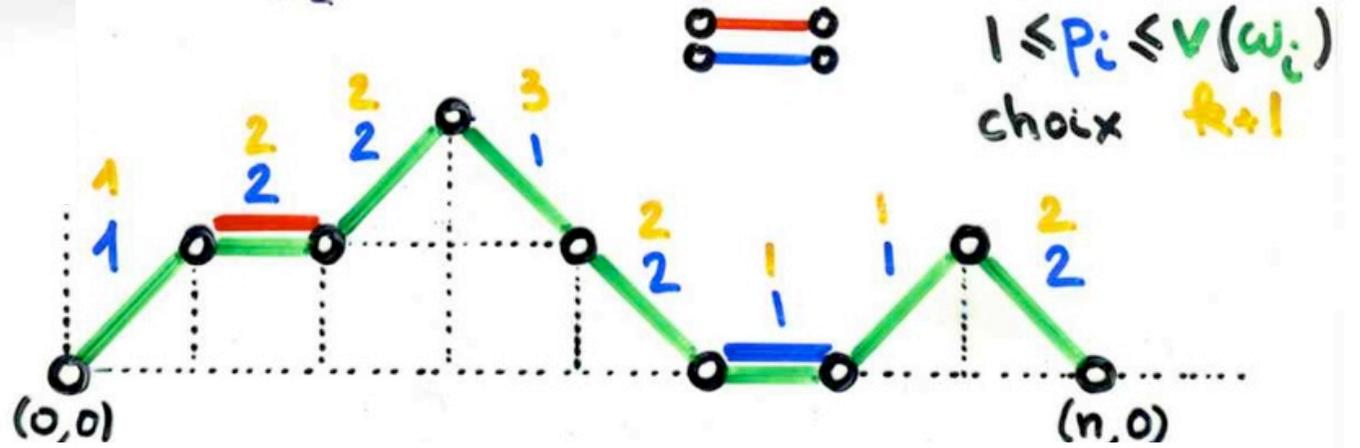




# Permutations



$$h = (\omega_c; (p_1, \dots, p_n))$$

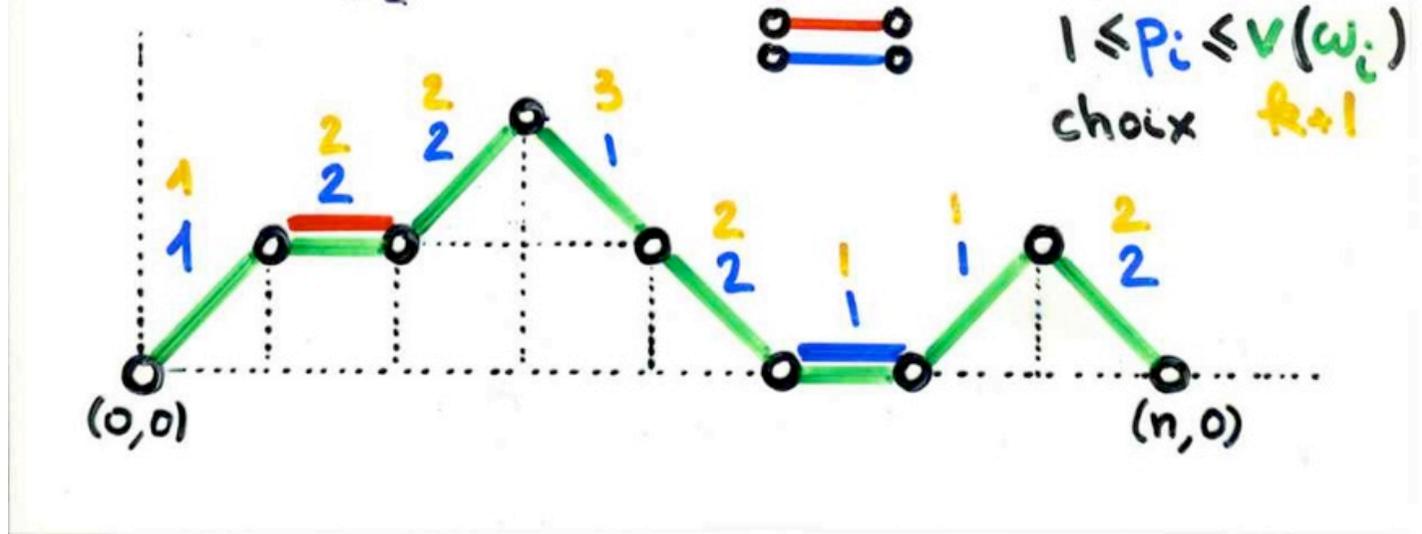


$x$	$\omega_c$	pos	$v$
1		1	1
2		2	2
3		2	2
4		1	3
5		2	2
6		1	1
7		1	1
$n=$ 8		2	2
9			

1 
  
 1 2
   
 1 3 2
   
 4 1 3 2
   
 4 1 3 5 2
   
 4 1 6 3 5 2
   
 4 1 6 7 3 5 2
   
 4 1 6 7 8 3 5 2
   
 4 1 6 9 7 8 3 5 2

$=$ 
  
 $\in$   $G_{n+1}$

“q-analogue”  
of Laguerre  
histories

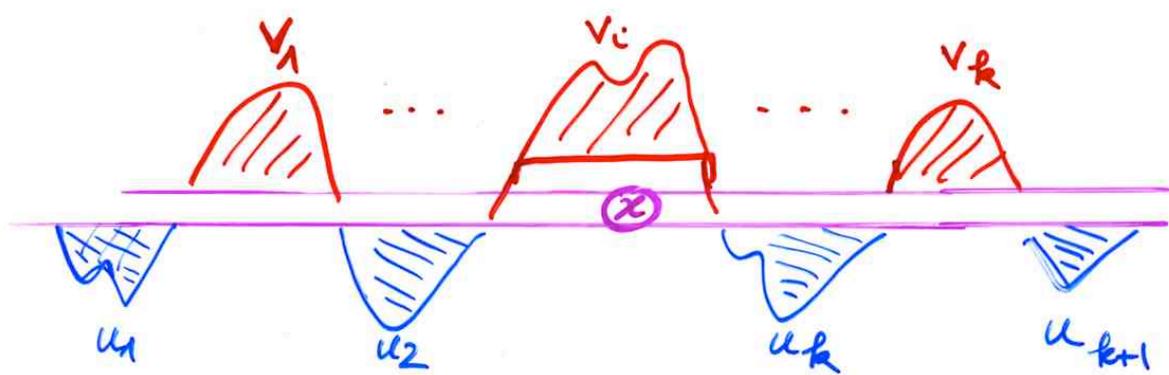


choices function

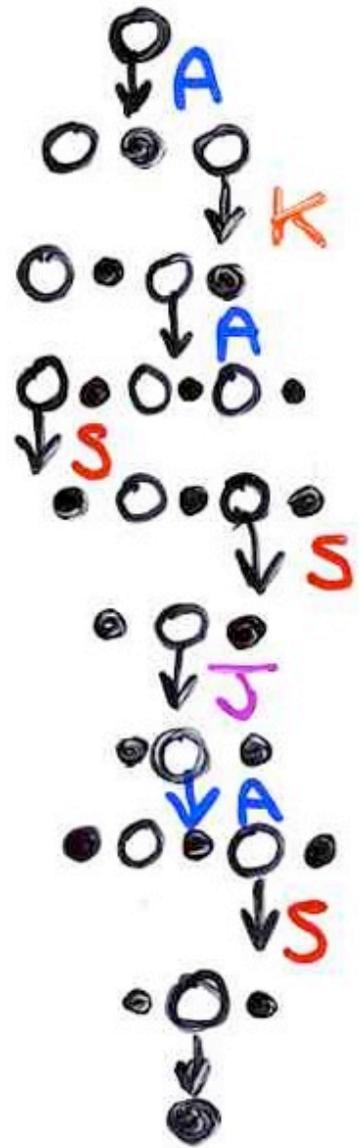
1 2 3 4 5 6 7 8  
 1 2 2 1 2 1 1 2  
 0 1 1 0 1 0 0 1

q-Laguerre :  $q^4$

$\lrcorner$   
 $\lrcorner$  1  $\lrcorner$   
 $\lrcorner$  1  $\lrcorner$  2  
 $\lrcorner$  1  $\lrcorner$  3  $\lrcorner$  2  
 4 1  $\lrcorner$  3  $\lrcorner$  2  
 4 1  $\lrcorner$  3 5 2  
 4 1 6  $\lrcorner$  3 5 2  
 4 1 6  $\lrcorner$  7  $\lrcorner$  3 5 2  
 4 1 6  $\lrcorner$  7 8 3 5 2  
 4 1 6 9 7 8 3 5 2 =  $\frac{6!}{n+1}$



1  
2  
3  
4  
5  
6  
7  
8  
9



U  
 U 1 U  
 U 1 U 2  
 U 1 U 3 U 2  
 4 1 U 3 U 2  
 4 1 U 3 5 2  
 4 1 6 U 3 5 2  
 4 1 6 U 7 U 3 5 2  
 4 1 6 U 7 8 3 5 2  
 4 1 6 9 7 8 3 5 2

# Laguerre history

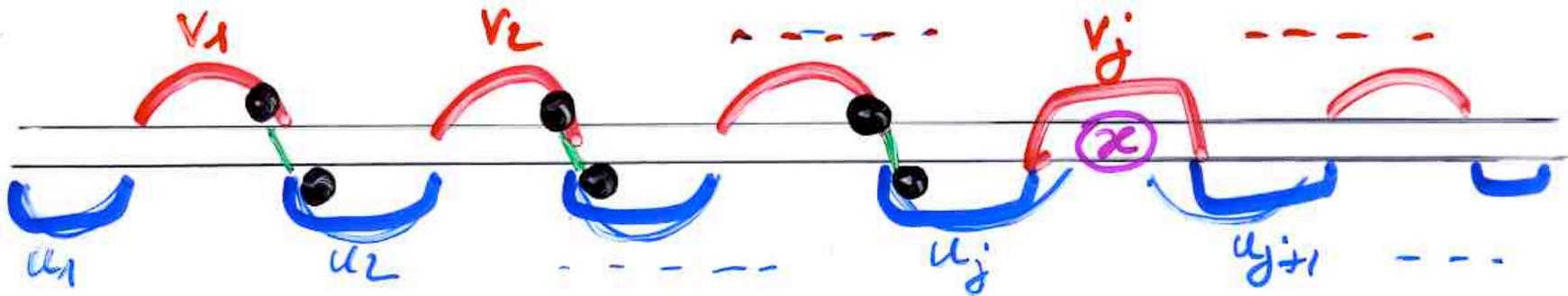
Lemme  $\pi \circ \theta \circ \pi^{-1} \circ \sigma \circ \pi = \sigma$   
 $h = (\omega_c ; (p_1, \dots, p_n)) \in \mathcal{L}_n$   
 $\downarrow$   
 $\sigma \in \mathcal{S}_{n+1}$   
 permutation

$P_x = j$  est aussi :

$j = 1 + \text{nb}$  de triplets  $(a, b, x)$   
 ayant le "motif"  $(31-2)$  c.à.d. :

$$a = \sigma(i), \quad b = \sigma(i+1), \quad x = \sigma(l)$$

$$i < i+1 < l \quad b < x < a$$



“q-analogue” of Laguerre histories

complements:  
deux  $q$ -Laguerre

## q-Laguerre

$$b_k = 2k+1$$

$$\lambda_k = k^2$$

$$\mu_n = n! \quad (\text{restricted})$$

Laguerre  
histories

$$b_k = 2k+2$$

$$\lambda_k = k(k+1)$$

$$\mu_n = (n+1)!$$

(large)

## q-analogue

A. de Médicis, XGV

"Moments des q-Polynômes de Laguerre  
et la bijection de Foata-Zeilberger"  
Adv. in Applied Maths 15, 262-304 (1994)

$$\text{If } \mu_n = [n]_q !$$

$$= 1(1+q)(1+q+q^2) \dots (1+q+\dots+q^{n-1})$$

$$\text{then } \begin{cases} b_k = q^k ( [k]_q + [k+1]_q ) \\ \lambda_k = q^{2k-1} [k]_q \times [k]_q \end{cases}$$

IP

$$b_k = [k]_q + [k+1]_q$$

$$\lambda_k = [k]_q \times [k]_q$$

then  $\mu_n \leftrightarrow$   $q$ -Laguerre histories  
(restricted)

$$b_k = [k+1]_q + [k+1]_q$$

$$\lambda_k = [k]_q \times [k+1]_q$$

$\mu_n \leftrightarrow$   $q$ -Laguerre histories  
(large)

$\rightarrow$  PASEP

$$L_n^{(\alpha)}(x)$$

$$\mu_n^{(\alpha)} = (\alpha+1)_n = (\alpha+1)(\alpha+2) \dots (\alpha+n)$$

$$\beta = (\alpha+1)$$

$$[n; \beta] = \beta + q + q^2 + \dots + q^{n-1}$$

$$\left\{ \begin{array}{l} b_{k,q}^{(\beta)} = q^k ( [k]_q + [k+1; \beta]_q ) \\ \cancel{b_{k,q}^{(\beta)}} = q^{2k-1} [k]_q \times [k; \beta]_q \end{array} \right.$$

9.  $q$ -Hermite  
et  $UD = DU + Id$

(cours I et II)

Donner une bijection entre les "placements de tours" sur un diagramme de Ferrers et les involutions dont les points fixes sont colorés en deux couleurs. (penser aux "histoires"). Interpréter, en terme d'involution sans point fixe, le terme constant (qui est un polynôme en  $q$ ) dans le "normal ordering" d'un mot  $w$  de l'algèbre quadratique définie par  $UD = qDU + I$ . (idée: définir le paramètre nombre de "croisement").

$$UD = DU + I$$

Lemme - Tout mot  $w \in \{U, D\}^*$   
s'écrit

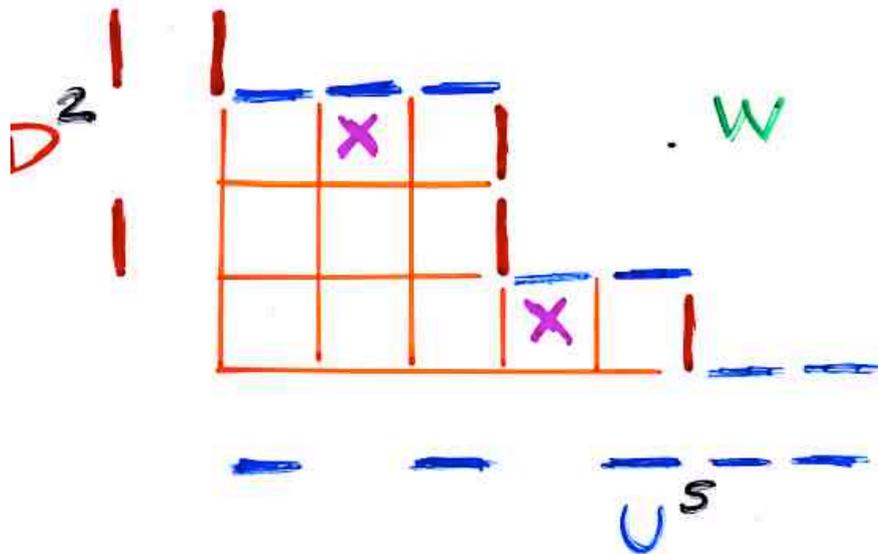
$$w = \sum_{i, j \geq 0}$$

$$c_{i,j}(w) D^i U^j$$

notation

$$w \rightarrow F_w$$

Diagram  
Ferrers



Prop.  $c_{i,j}^k(w) =$  nb de "placement"  
de  $k$  sur  $F$   
tours

avec 
$$e = |w|_D - k$$
$$d = |w|_U - k$$

$$UD = DU + I$$

Prop-  $w$  mot de Dyck

$$c_{0,0}(w) = v_H(w) \quad \text{valuation Hermite}$$

$$= \text{nb d'histoires d'Hermite associées à } w$$

$\lambda_k = k$

polynôme d'Hermites  $H_n(x)$

$$\lambda_k = k ; \quad b_k = 0$$

$(k \geq 1)$                        $(k \geq 0)$

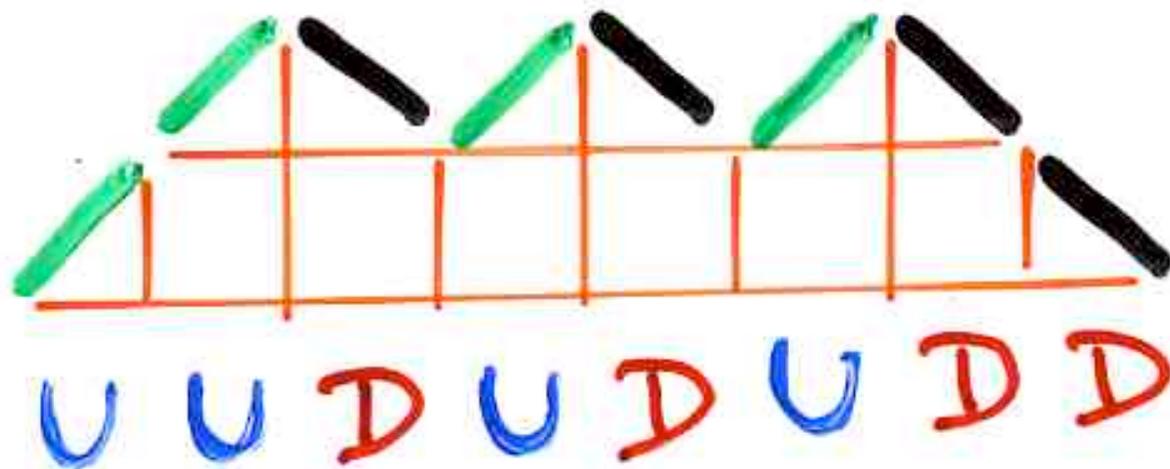
$$a_k = 1$$

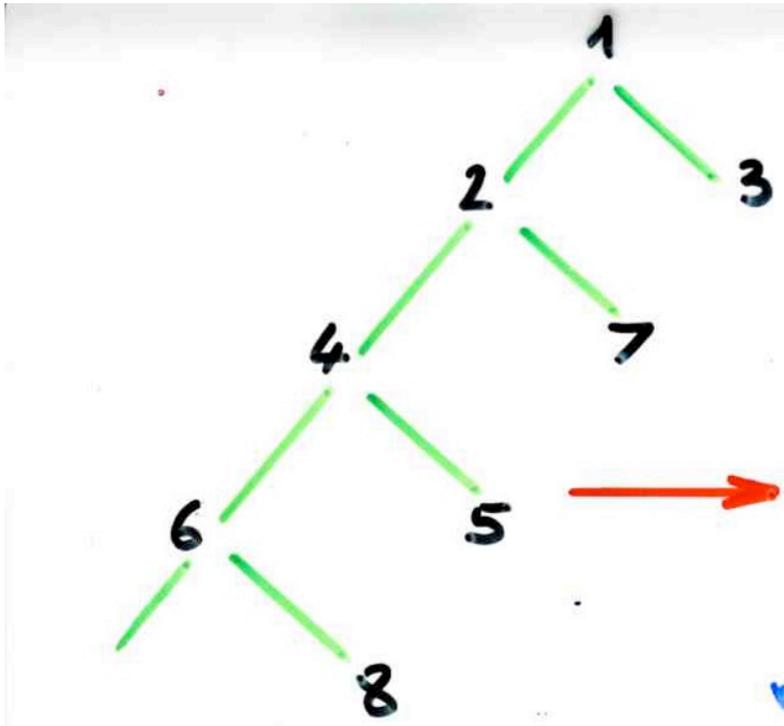
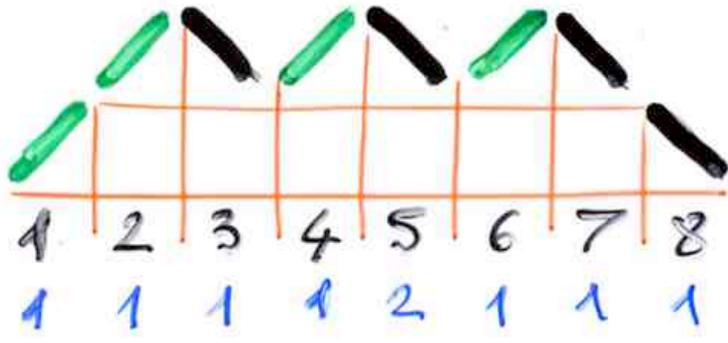
$$\begin{cases} b'_k = 0 \\ b''_k = 0 \end{cases}$$

$$c_k = k$$

Histoires d'Hermites

ex :





Involution

$$\sigma = (13)(27)(45)(68)$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 5 & 4 & 8 & 2 & 6 \end{pmatrix}$$

no fixed points

$q$ -Hermite

$$b_k = 0$$

$$\lambda_k = [k]_q = (1 + q + \dots + q^{k-1})$$

$$\mu_{2n} = \sum_{\alpha} q^{\text{cr}(\alpha)}$$

$\alpha$   
involutions  
no fixed point

$\text{cr}(\alpha)$  number of crossings



$$UD = qDU + I$$

$\text{emb}(\alpha)$  number of pairs  
"emboîtée"

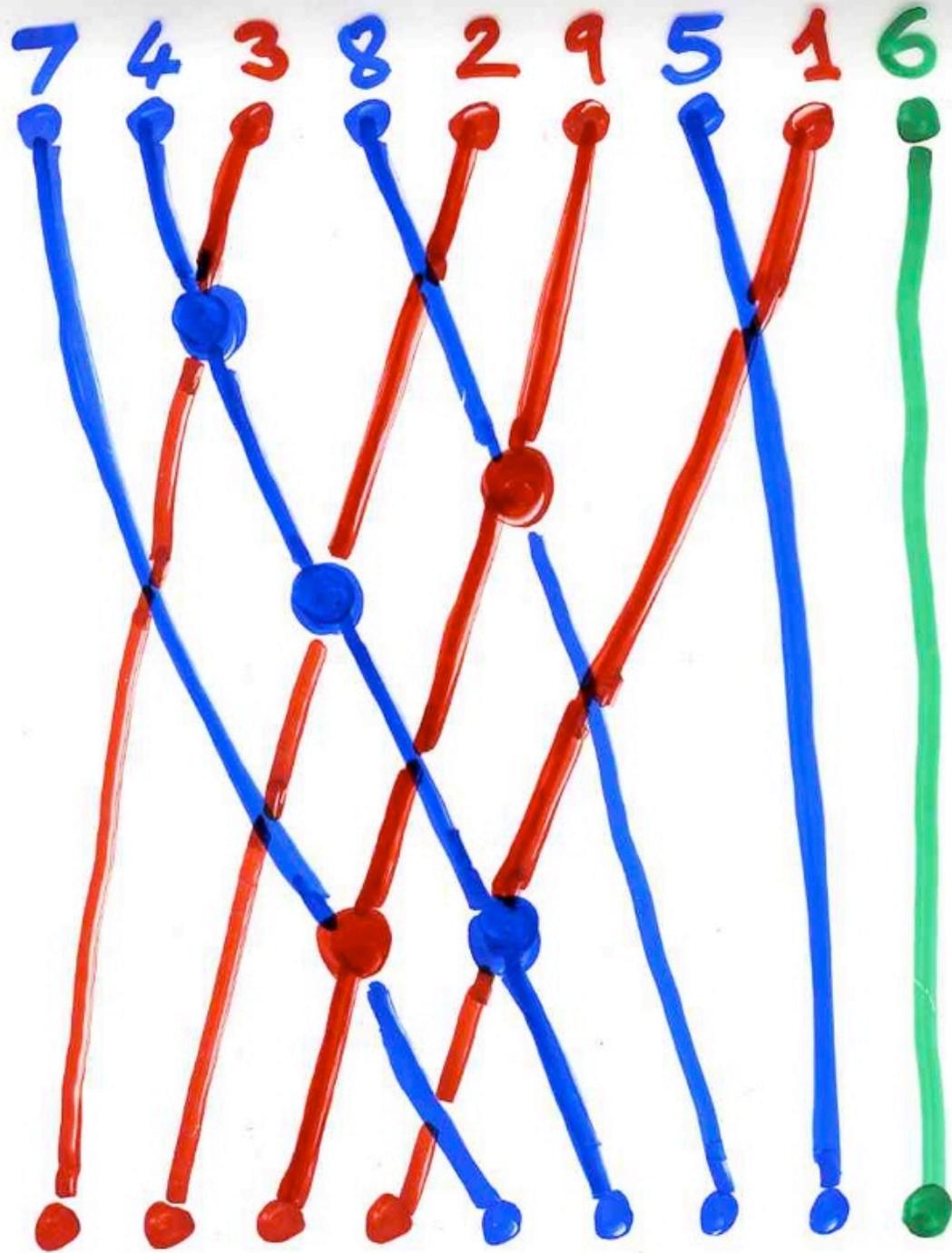


$$\mu_{2n} = \sum_{\alpha} q^{\text{emb}(\alpha)}$$

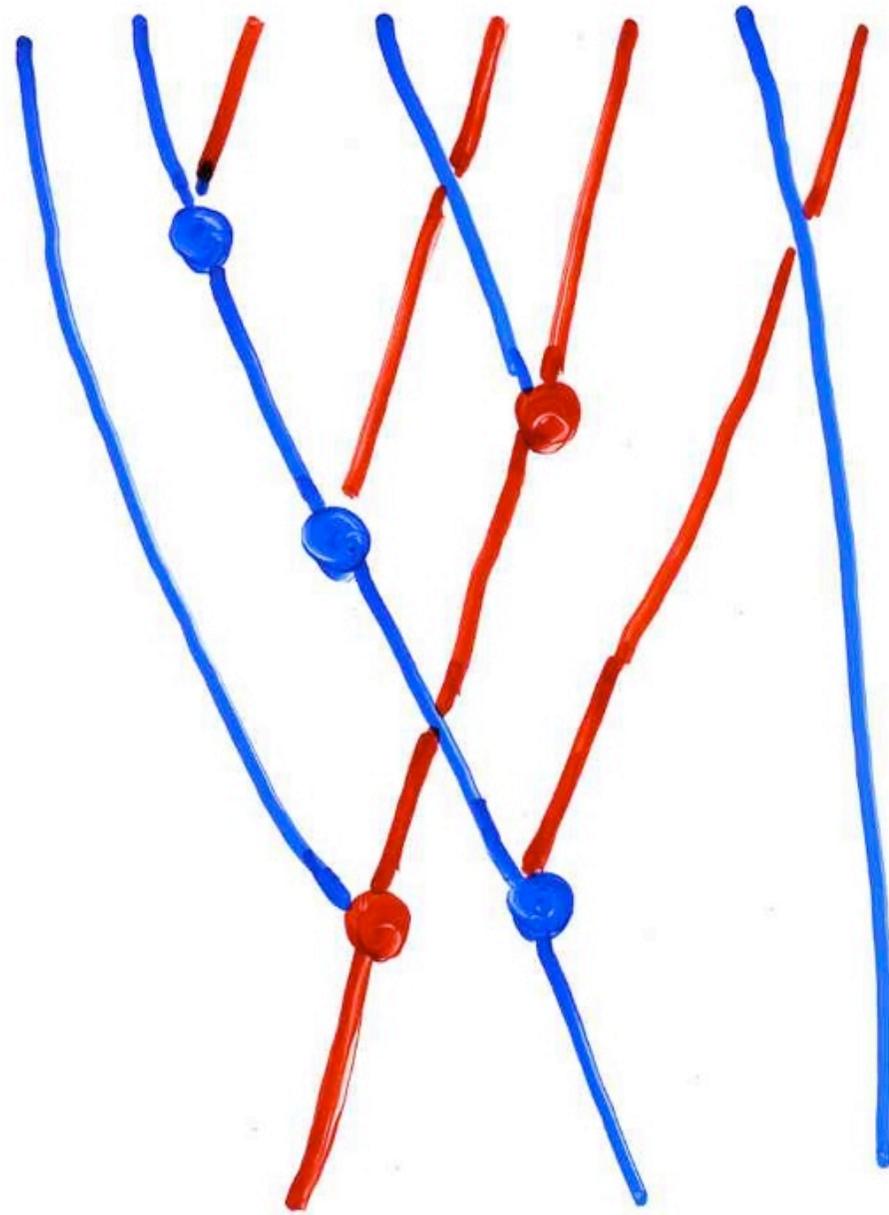
$\alpha$   
involutions  
no fixed points

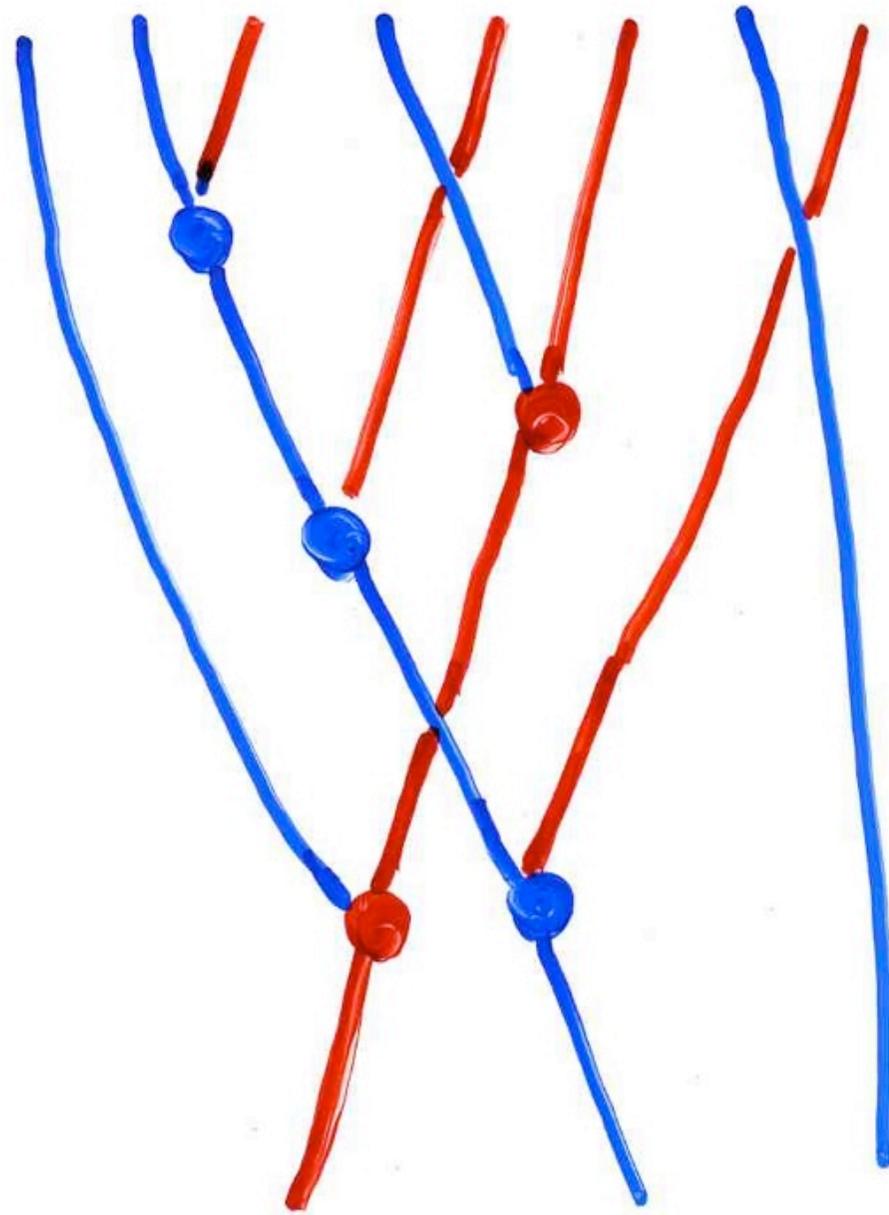
## 10. Algorithme inverse "échange-fusion" (cours II)

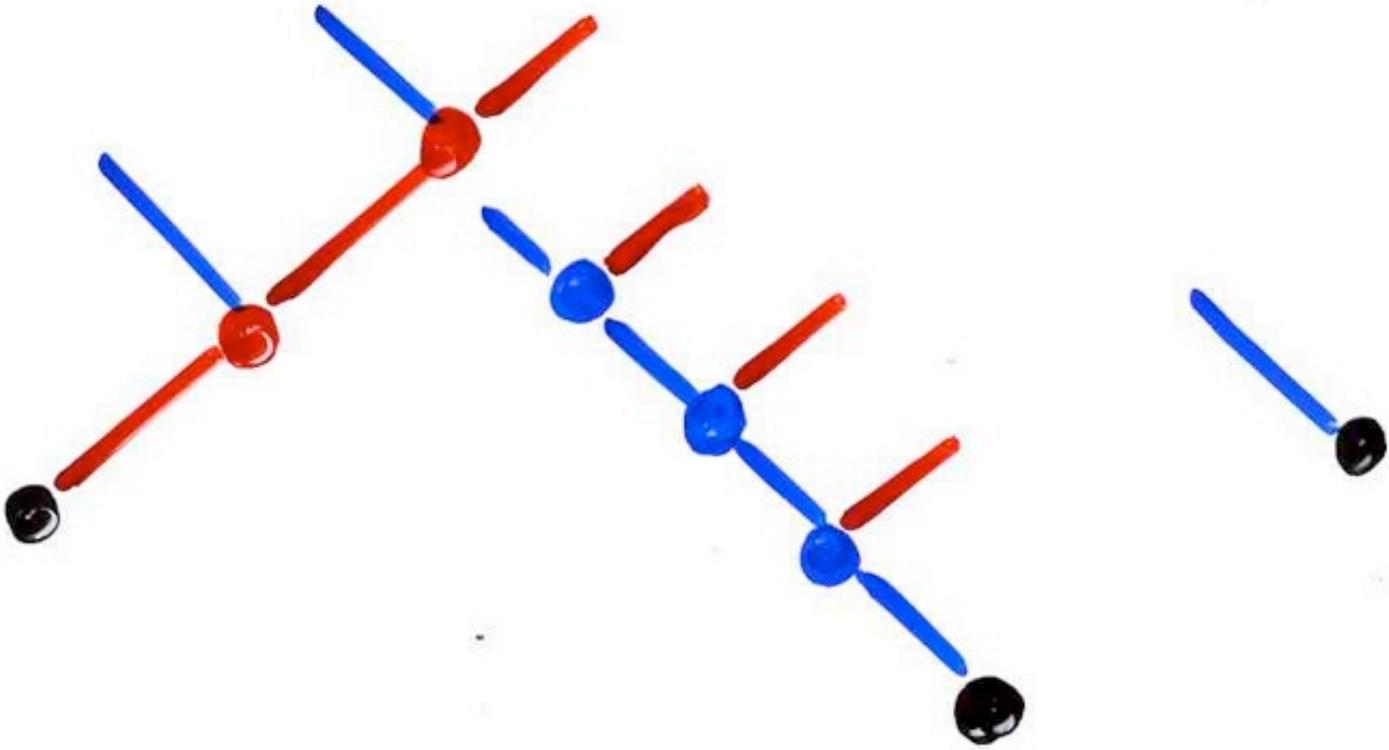
Construire la bijection inverse de l'algorithme "échange-fusion" associant un tableau alternatif à une permutation. (idée: dégager des arbres dans le dessin formé des fils rouges et bleus avec leurs croisements).

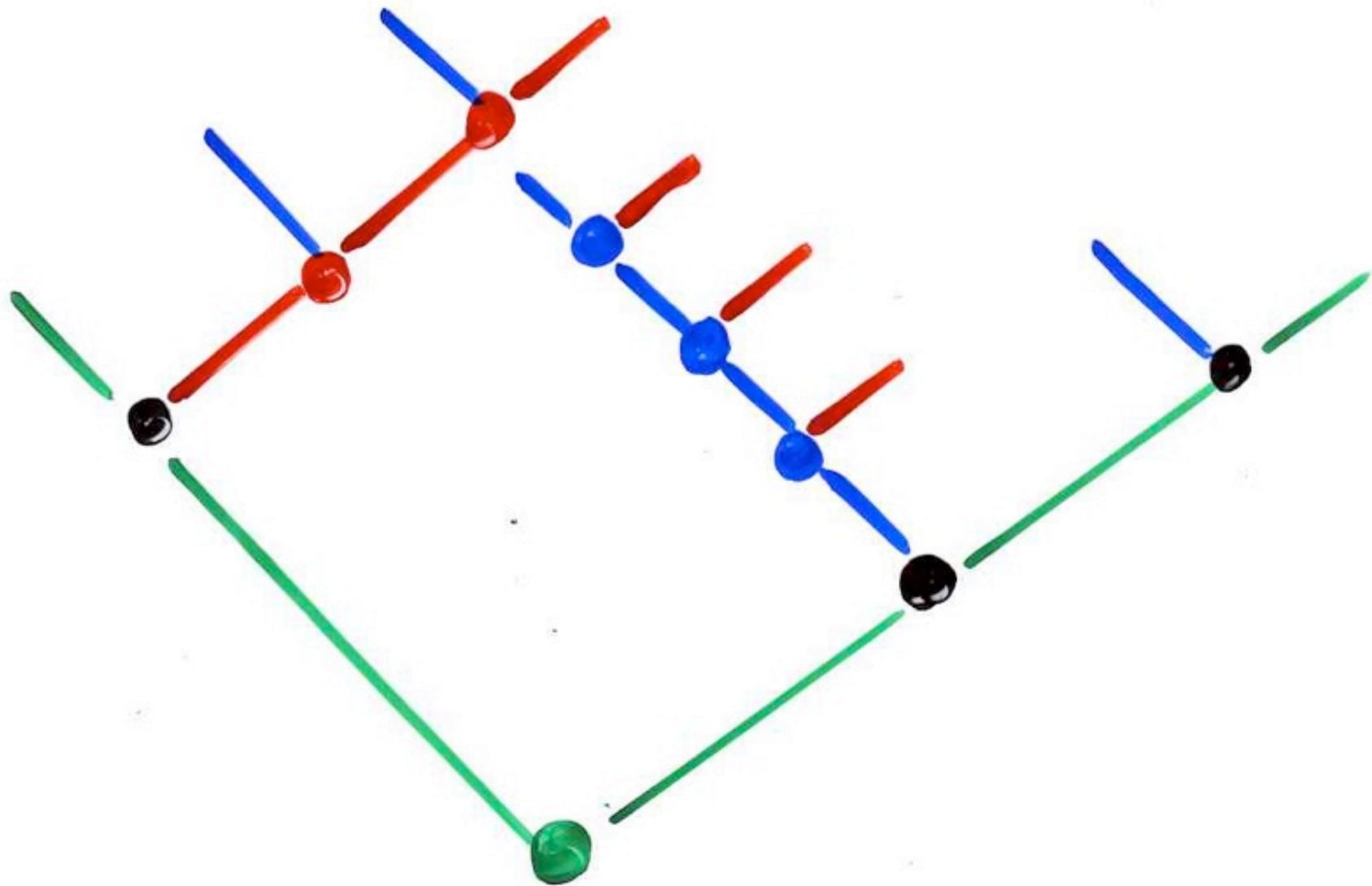


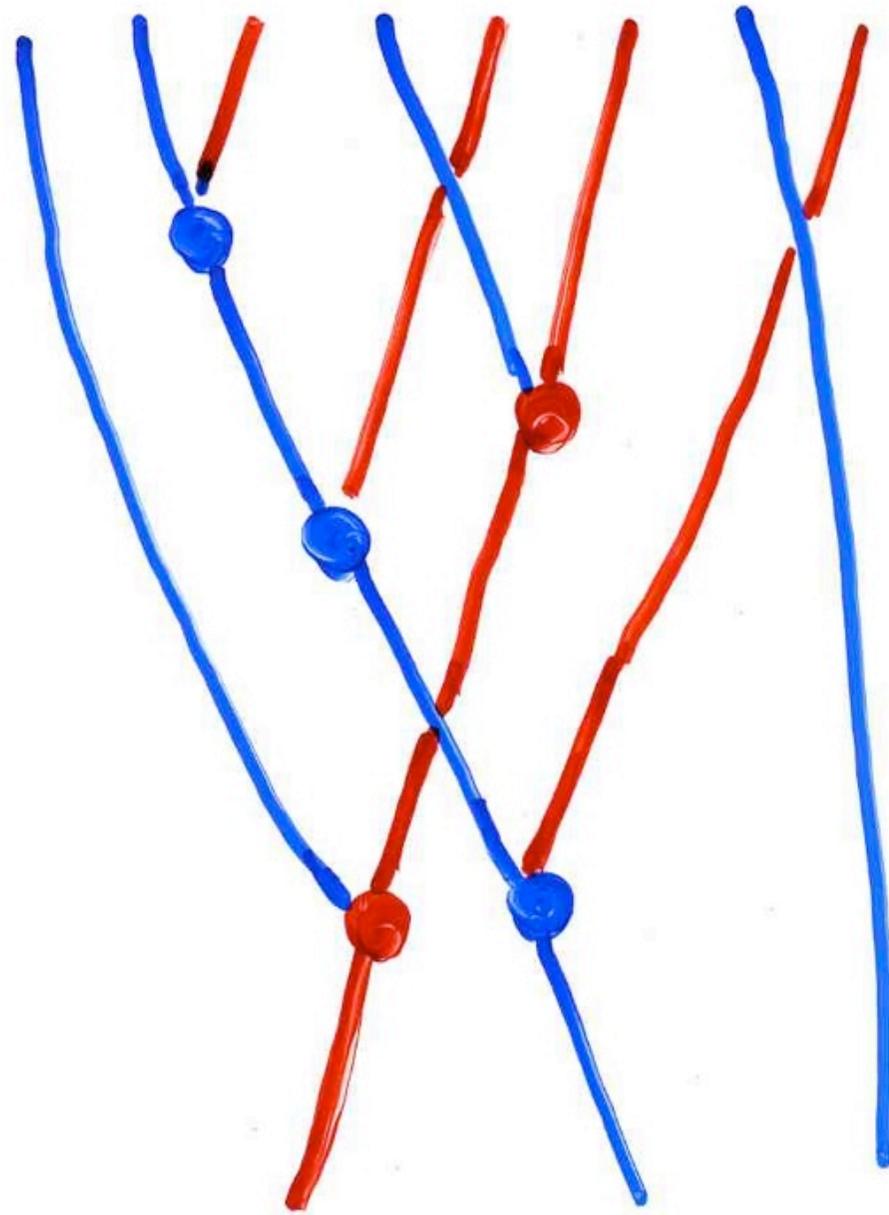


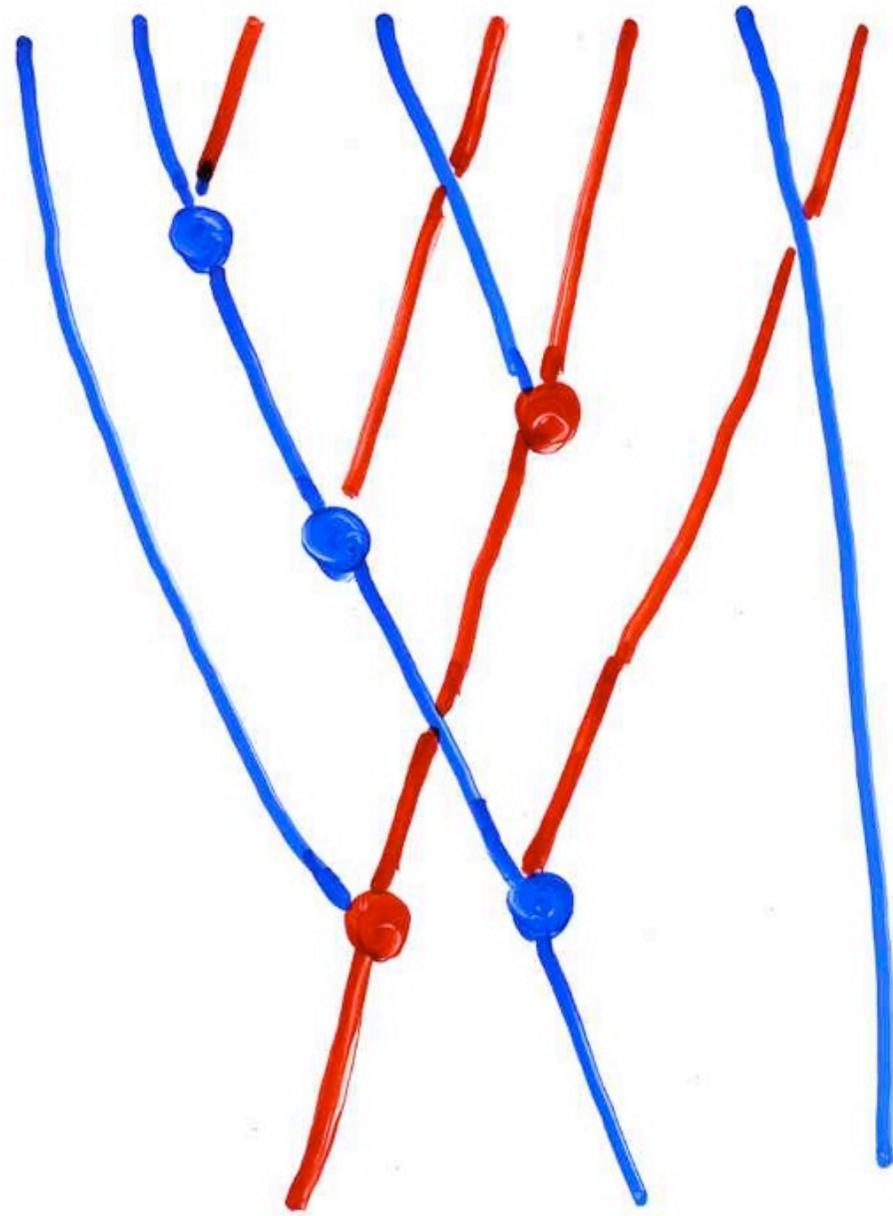










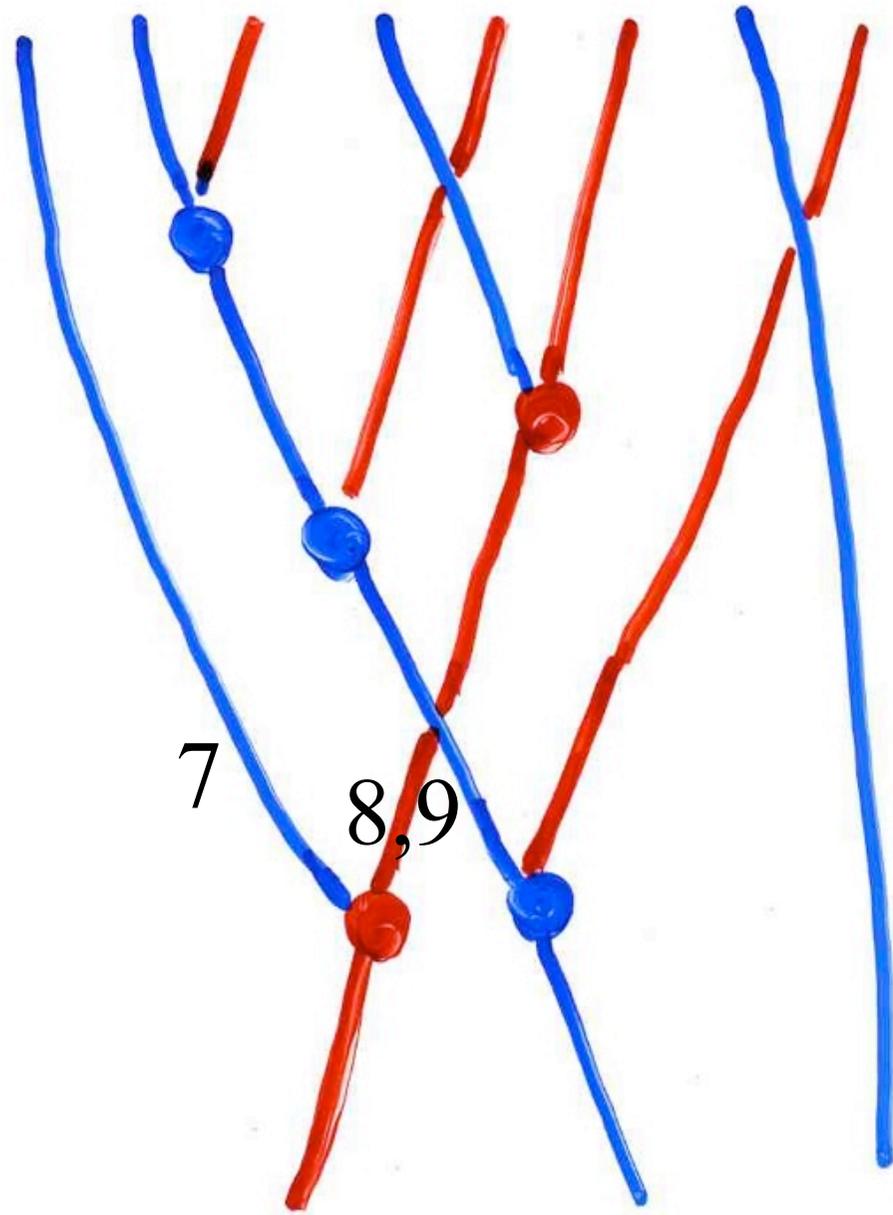


7,8,9

1,2,3,4

5

6



7

8,9

1,2,3,4

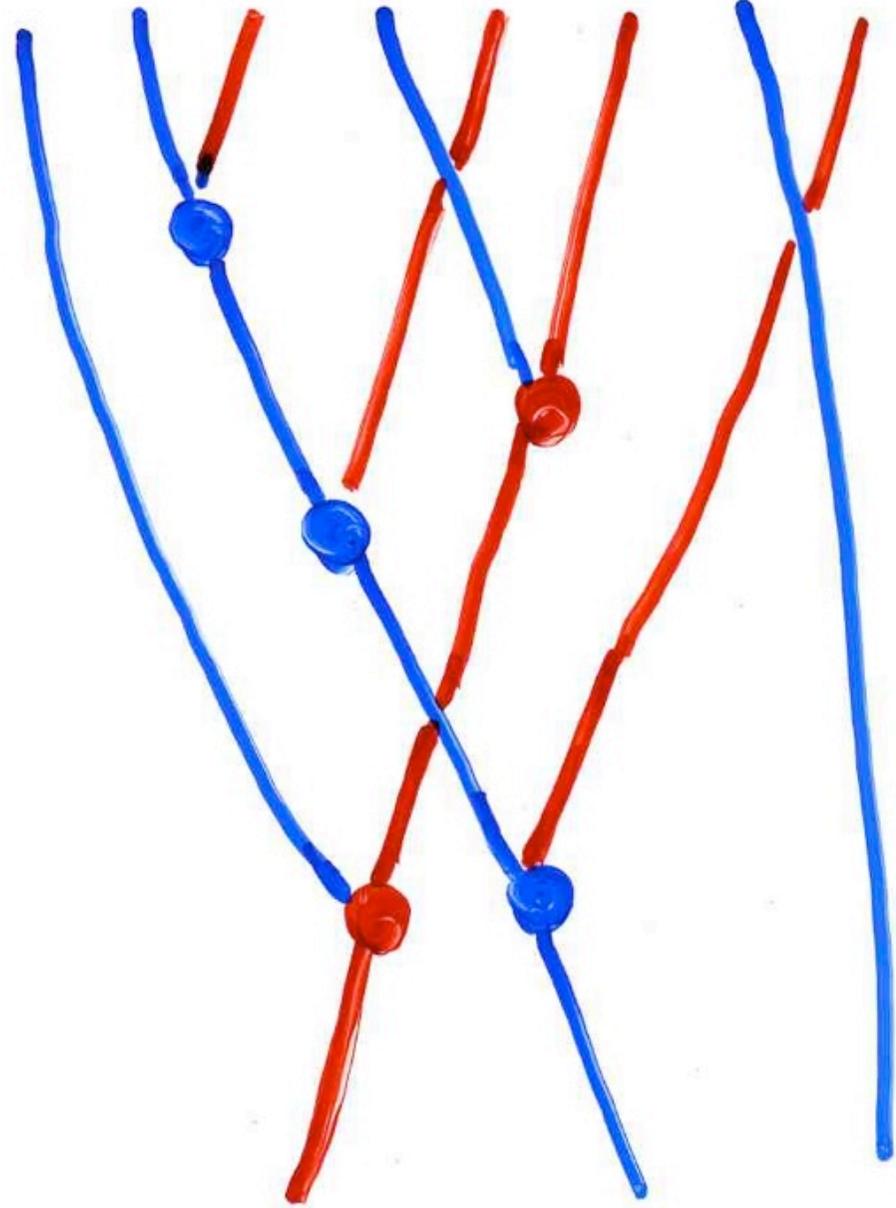
5

6

7

8

9



1,2,3,4

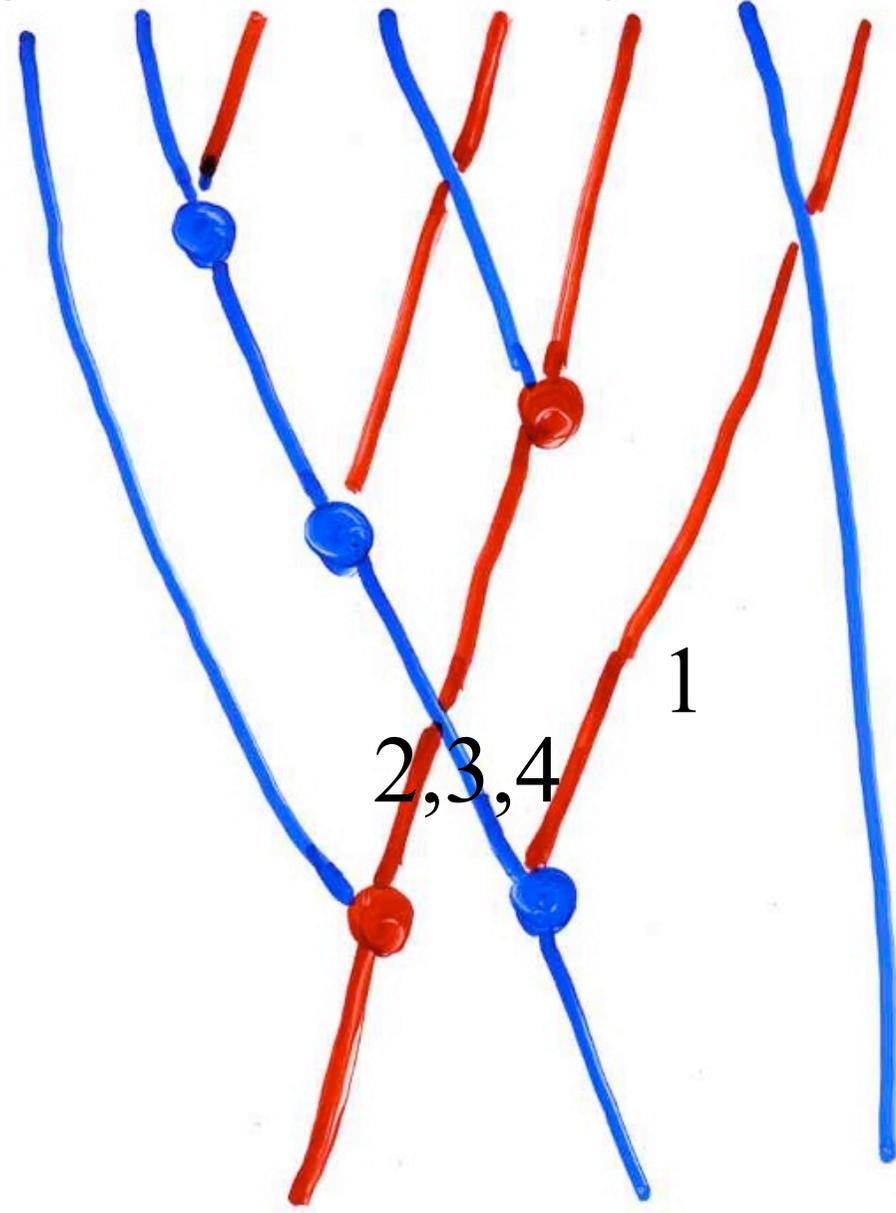
5

6

7

8

9

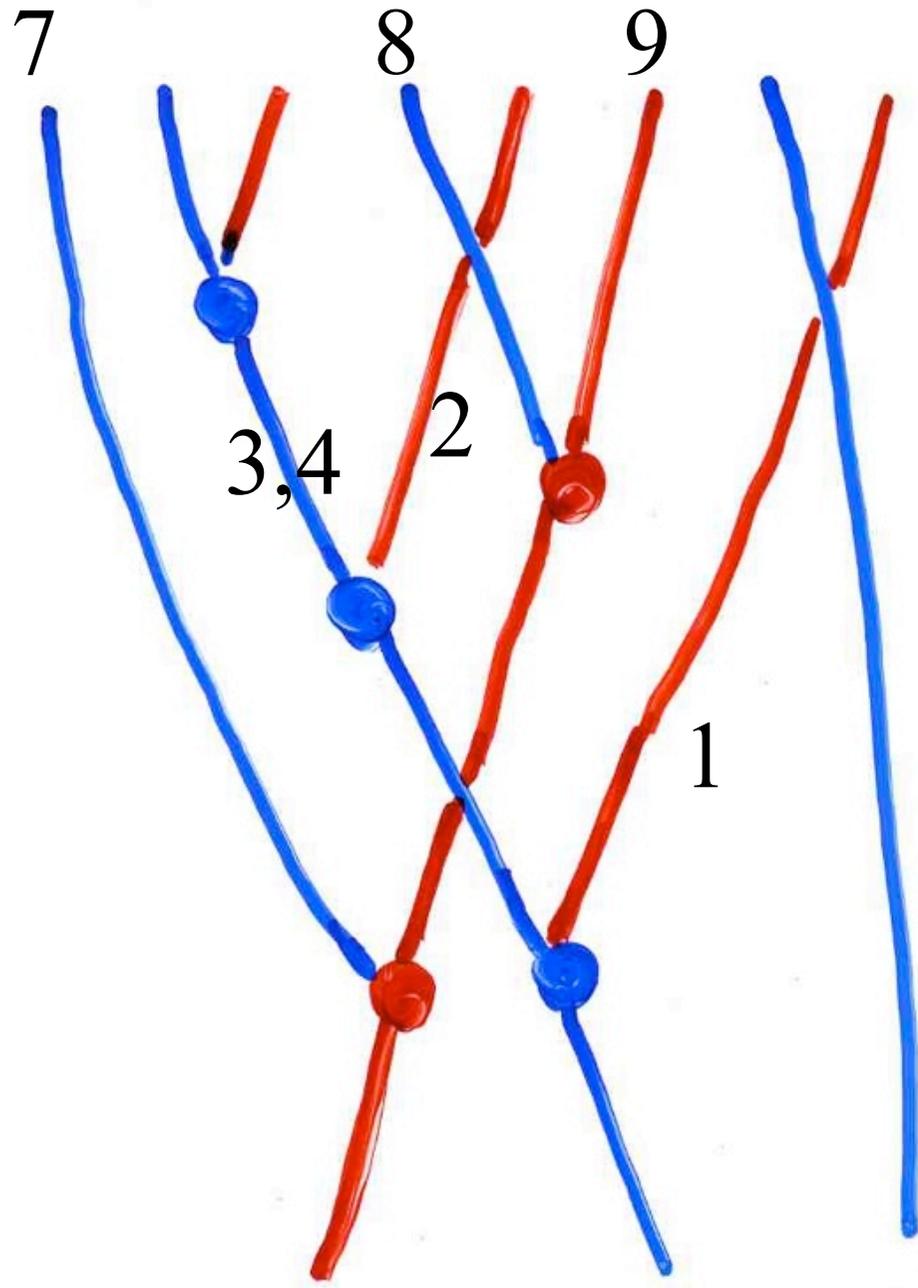


1

2,3,4

5

6



9

8

7

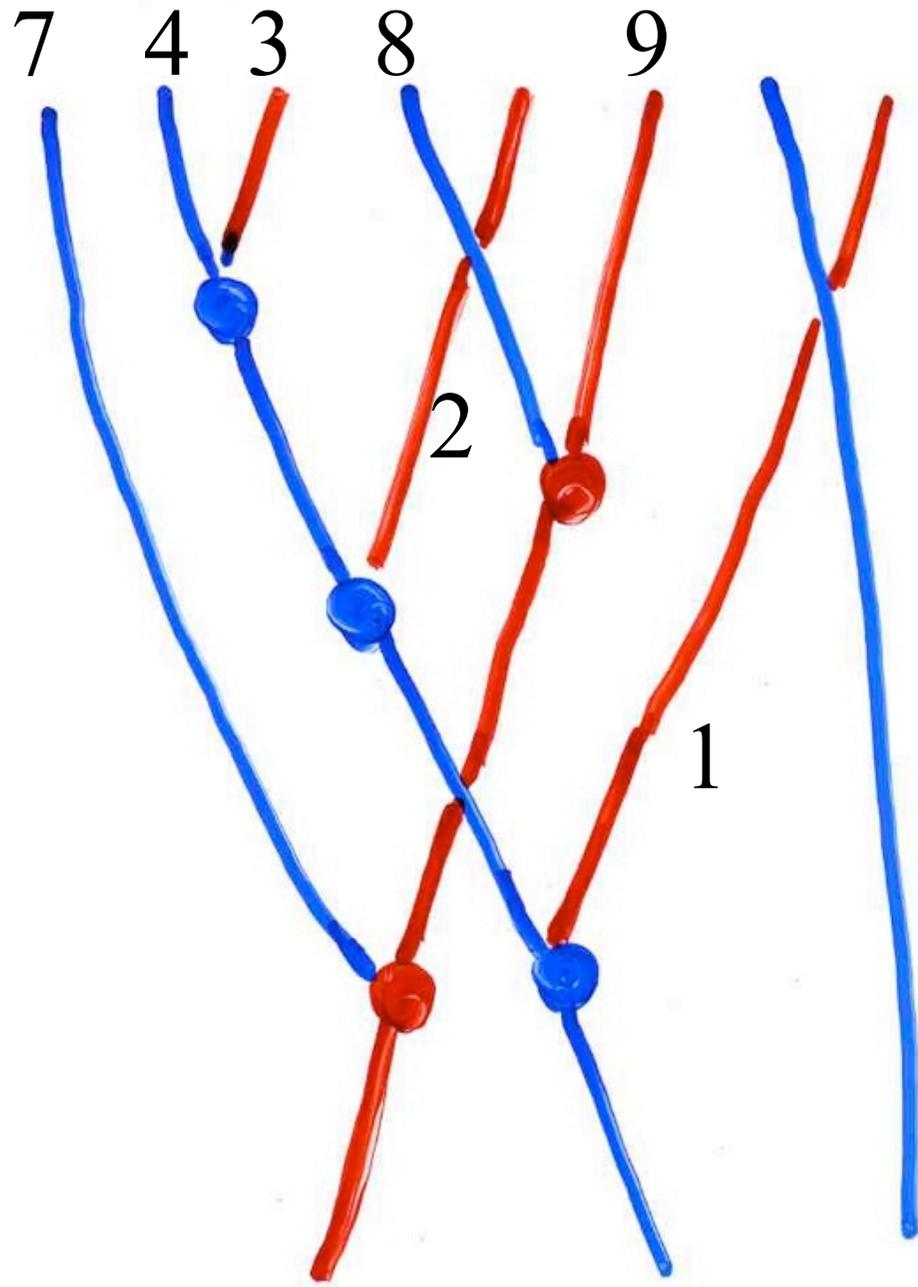
3,4

2

1

5

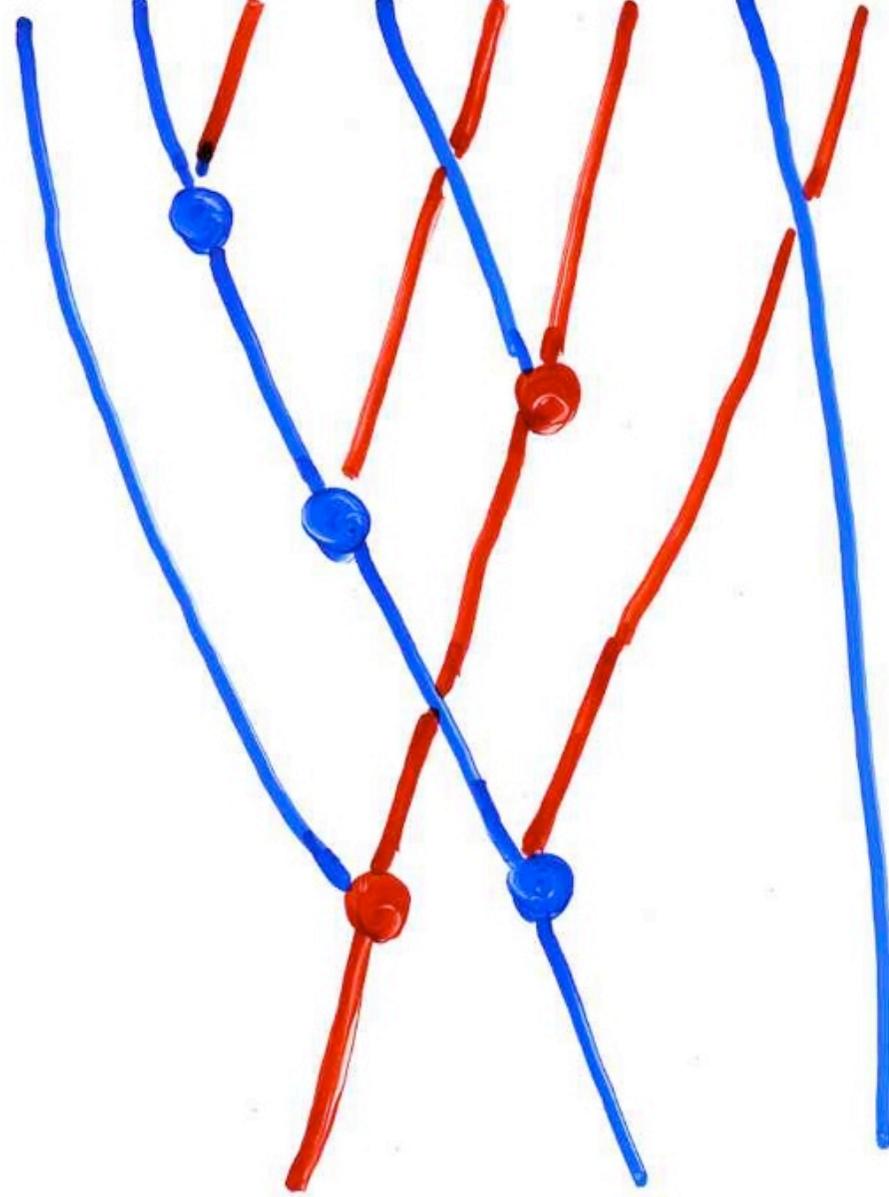
6



5

6

7 4 3 8 2 9 5 1 6



1. Tableaux alternatifs et tableaux de permutations
2. Tableaux alternatifs de Catalan: caractérisation
3. Fonction de partition du TASEP
4. Tableaux de Catalan et paires de chemins
5. Pavages Aztec
6. ASM et FPL
7. Automate planaire
8.  $q$ -Laguerre
9.  $q$ -Hermite et  $UD = DU + Id$
10. Algorithme inverse "échange-fusion"