Summer School
Mathematical Physics

Combinatorics
and
Statistical Mechanics
(III)

Frutillar
7-11 December

Xavier Viennot
CNRS, LaBRI, Bordeaux

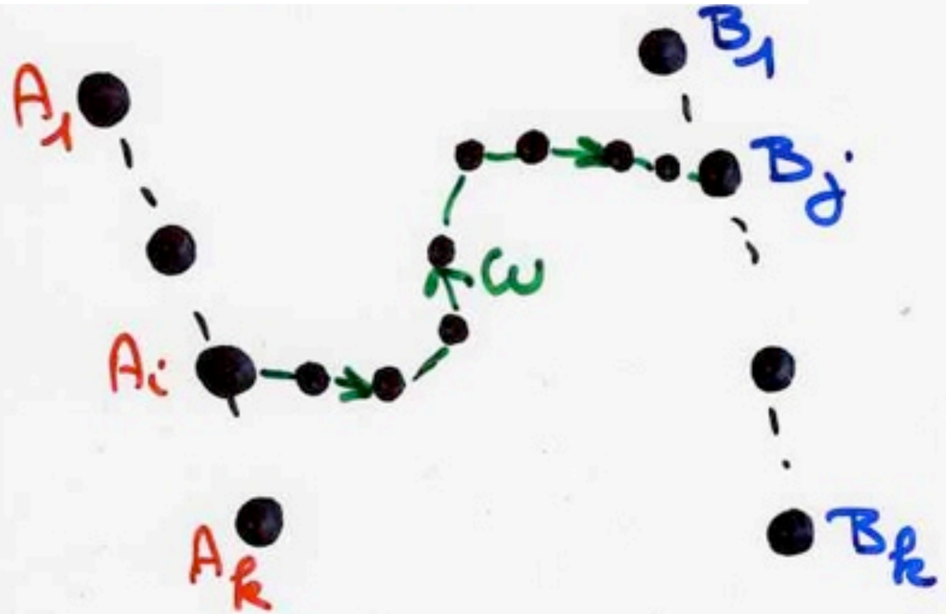
Paths, determinants and plane partitions

§1 The LGV Lemma

I. Gessel and X.G. Viennot, *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

LGV methodology



path

$\omega = (s_0, \ldots, s_n) \quad s_i \in \Pi$

valuation

$v : \Pi \times \Pi \to \mathbb{K}$ ring

$v(\omega) = v(s_0, s_1) \cdots v(s_{n-1}, s_n)$

$A_1, \ldots, A_k$

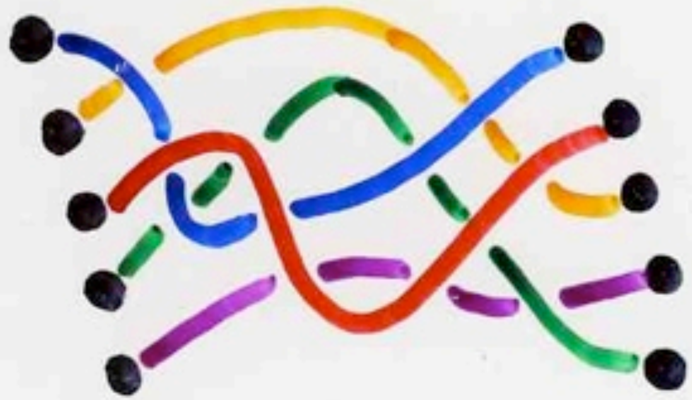
$B_1, \ldots, B_k$

$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

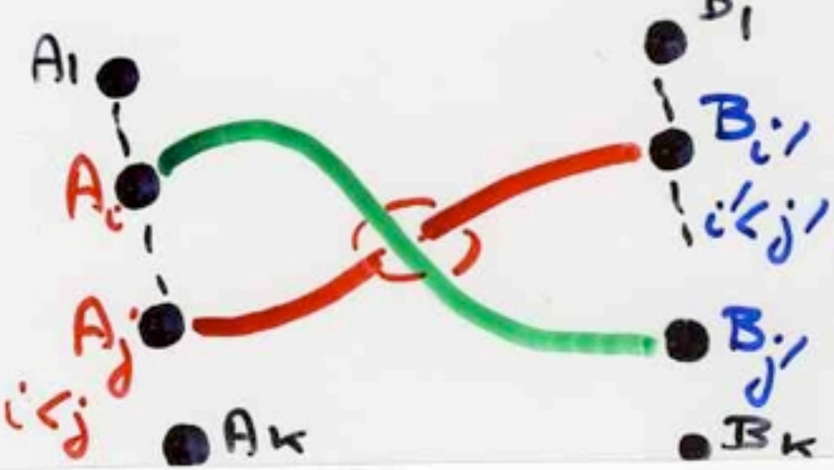
suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{Inv}(\sigma)}\, v(\omega_1) \cdots v(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

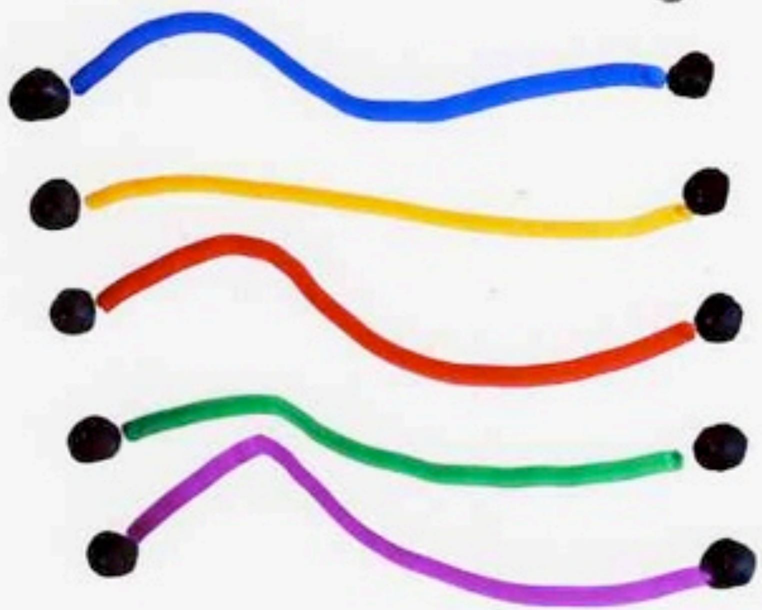


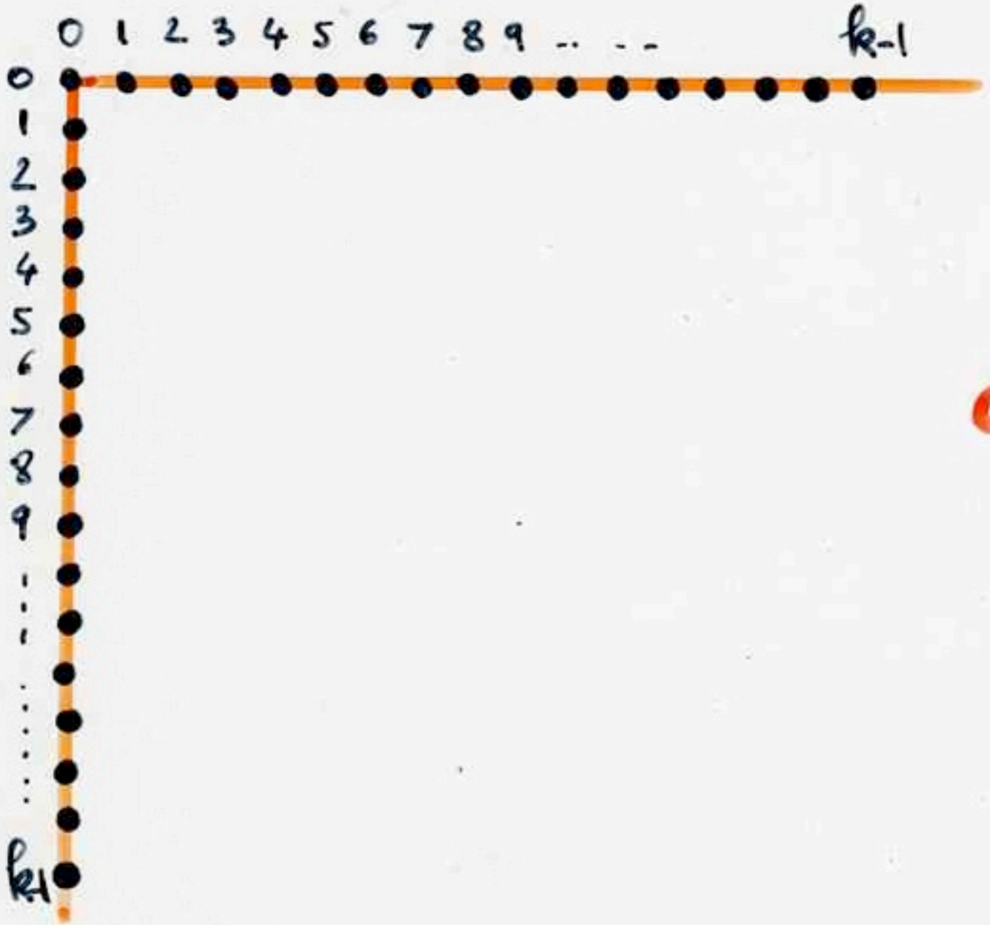
(C) crossing condition



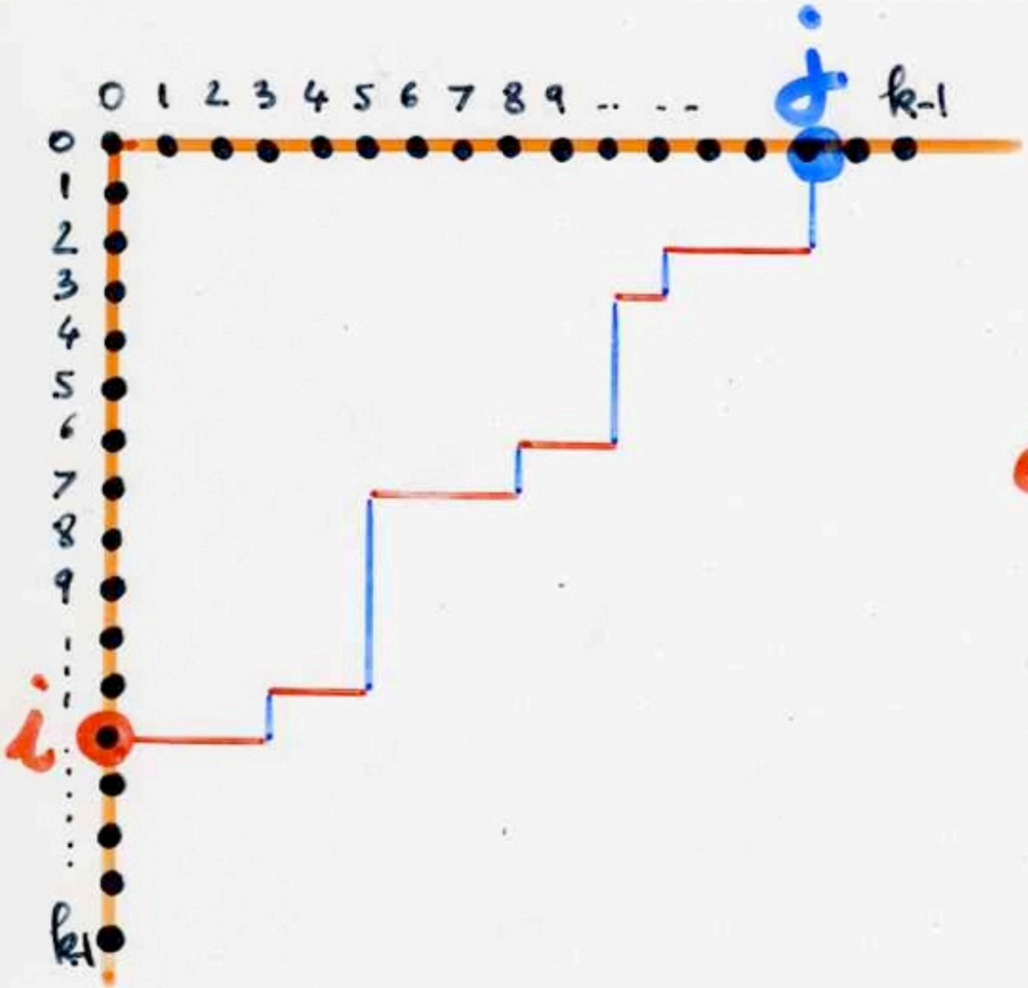
Prop. (C)

$$\det(a_{ij}) = \sum_{\substack{\Omega=(\omega_1,\dots,\omega_k) \\ \omega_i : A_i \rightsquigarrow B_i \\ \text{2 by 2 disjoints}}} v(\omega_1)\dots v(\omega_k)$$

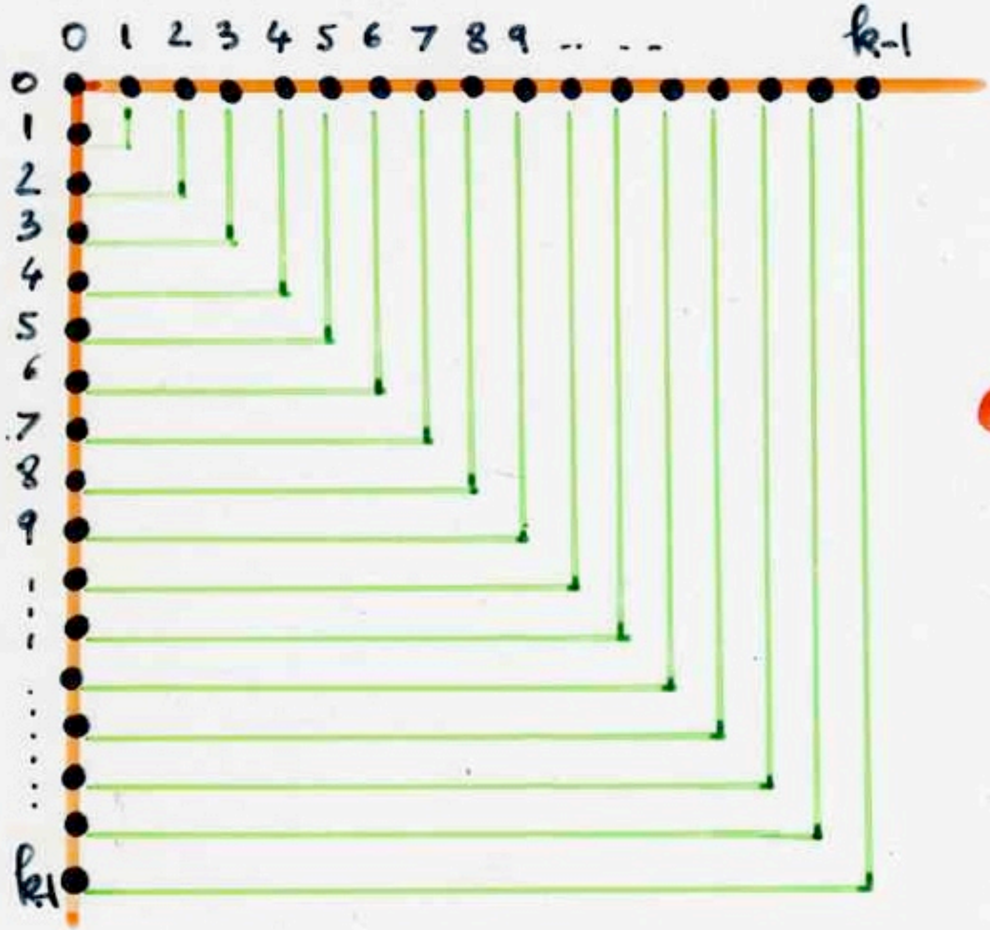




$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k \times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdot \\ 1 & 2 & 3 & 4 & 5 & \cdot & \\ 1 & 3 & 6 & 10 & \cdot & & \\ 1 & 4 & 10 & \cdot & & & \\ 1 & 5 & \cdot & & \binom{i+j}{i} & & \\ 1 & \cdot & & & & & \\ & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

§3 Binomial determinants

$$0 \leq a_1 < \cdots < a_k$$

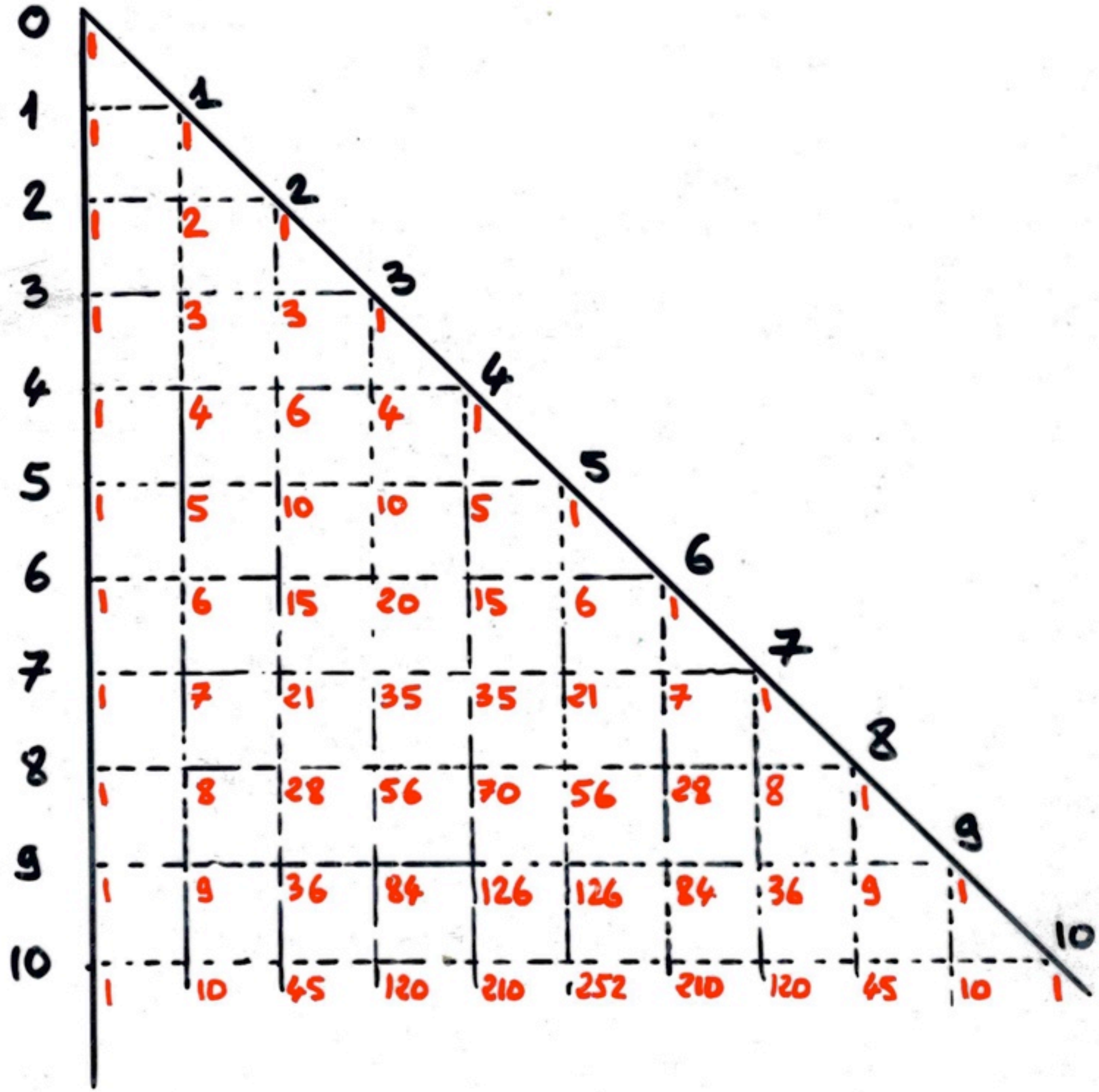
$$0 \leq b_1 < \cdots < b_k$$

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k}$$

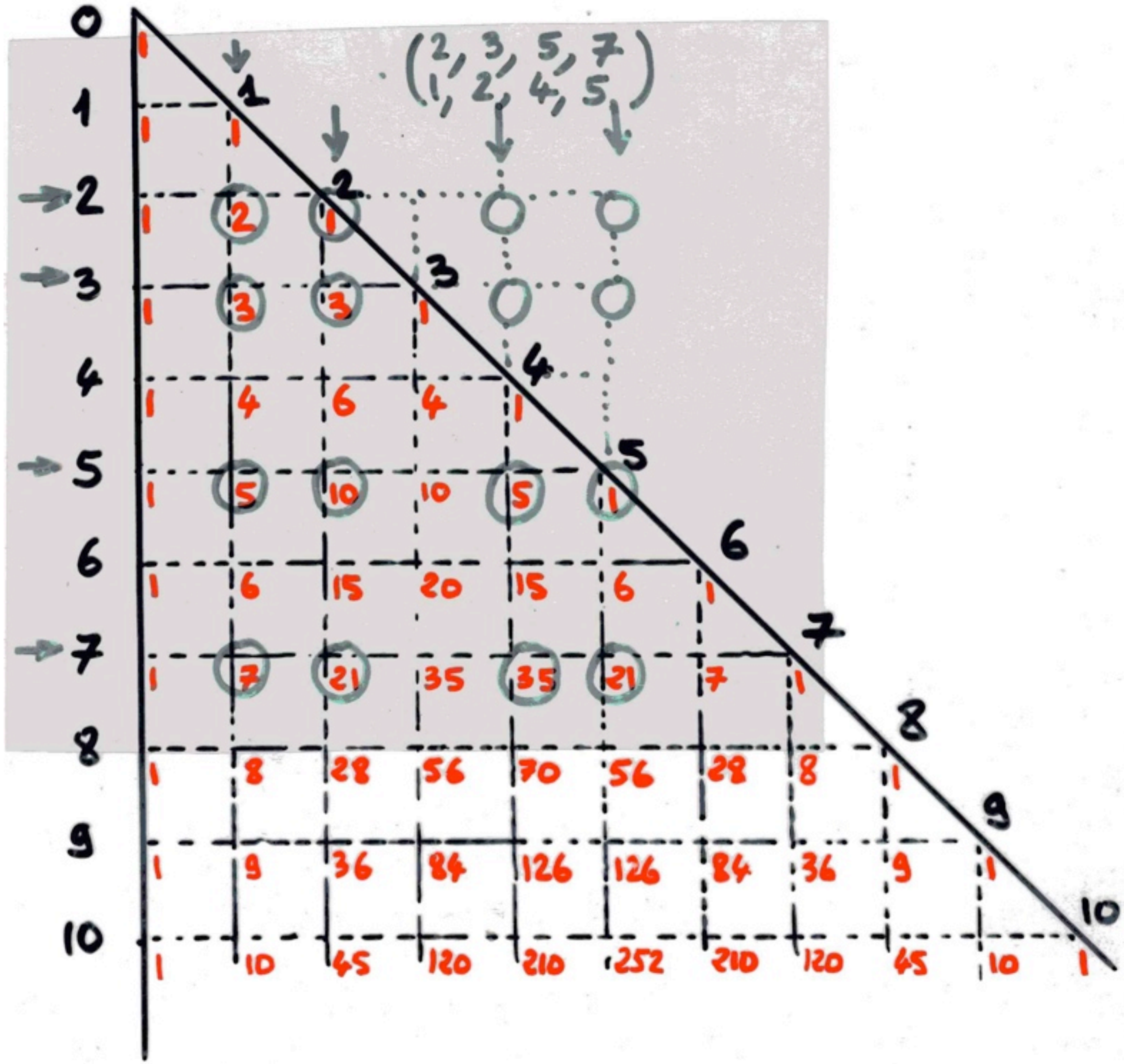
$$= \text{dét}\left(\binom{a_i}{b_j}\right)_{1 \leq i, j \leq k}$$

I. Gessel, X.G. Viennot
(Adv. in Maths
58 (1985) 300-321)

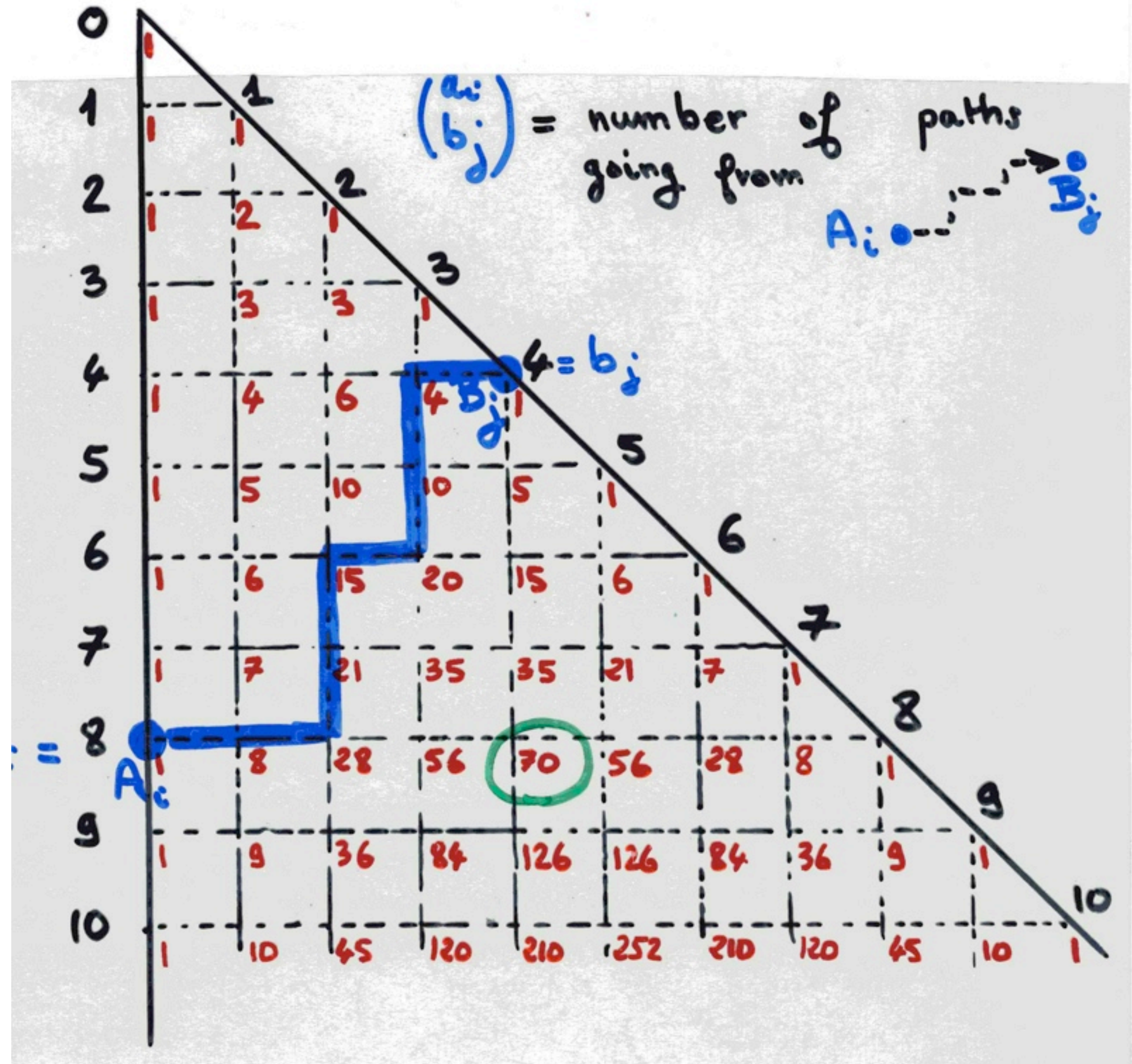
Binomial Determinant



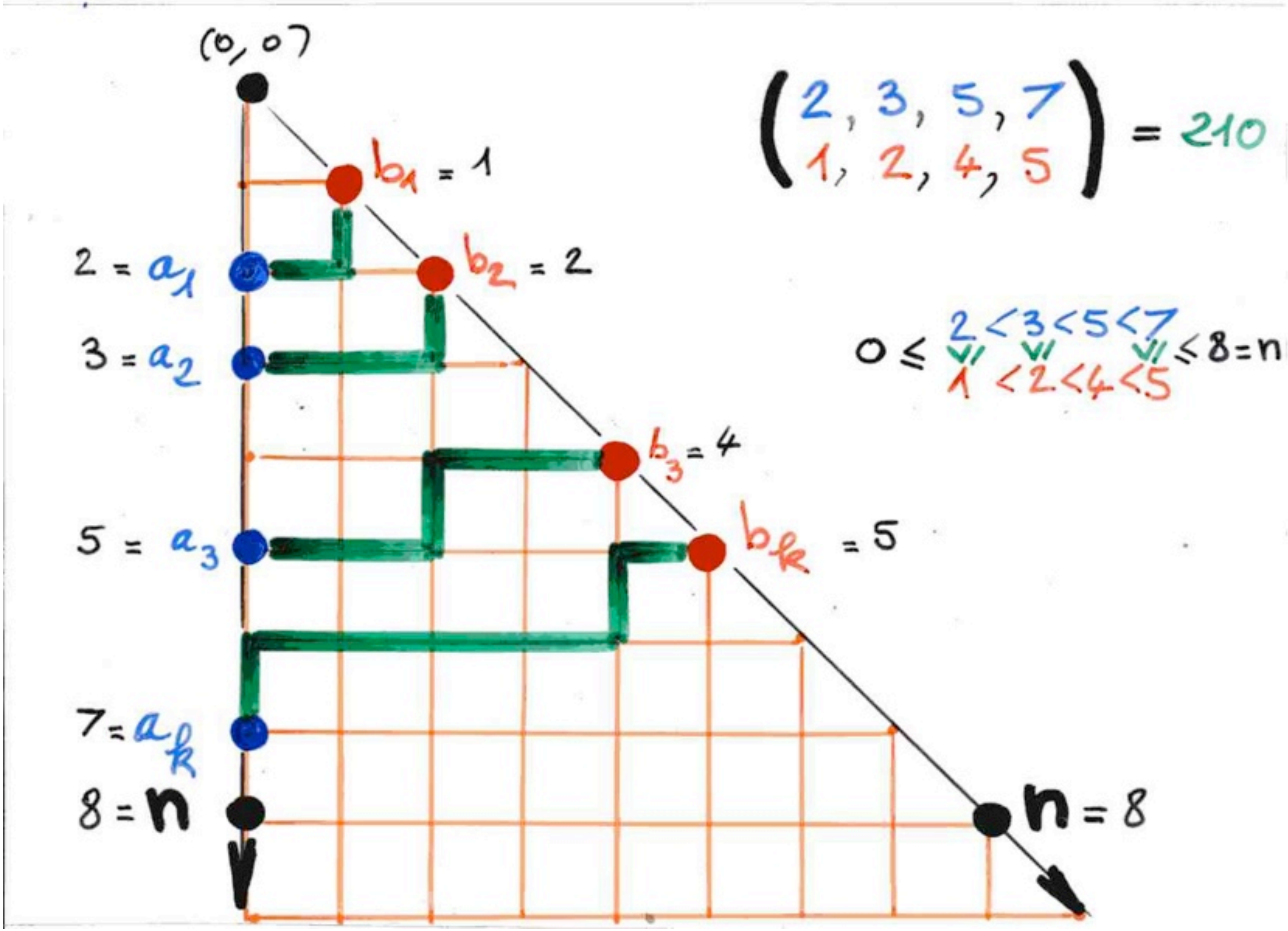
0
1
1
1 1
2
1 2 1
3
1 3 3 1
4
1 4 6 4 1
5
1 5 10 10 5 1
6
1 6 15 20 15 6 1
7
1 7 21 35 35 21 7 1
8
1 8 28 56 70 56 28 8 1
9
1 9 36 84 126 126 84 36 9 1
10
1 10 45 120 210 252 210 120 45 10 1



(2, 3, 5, 7
1, 2, 4, 5)
0 1 2 3 4 5 6 7 8 9 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



= number of paths going from
A_i → B_j
$4 = b_j$
B_j
A_i
0 1 2 3 4 5 6 7 8 9 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1


(0, 0)
$b_1 = 1$
$2 = a_1$
$b_2 = 2$
$3 = a_2$
$b_3 = 4$
$5 = a_3$
$b_k = 5$
$7 = a_k$
$8 = n$
$n = 8$
$\begin{pmatrix} 2, 3, 5, 7 \\ 1, 2, 4, 5 \end{pmatrix} = 210$
$0 \leq \begin{matrix} 2 < 3 < 5 < 7 \\ \geq \quad \geq \quad \geq \quad \geq \\ 1 < 2 < 4 < 5 \end{matrix} \leq 8 = n$

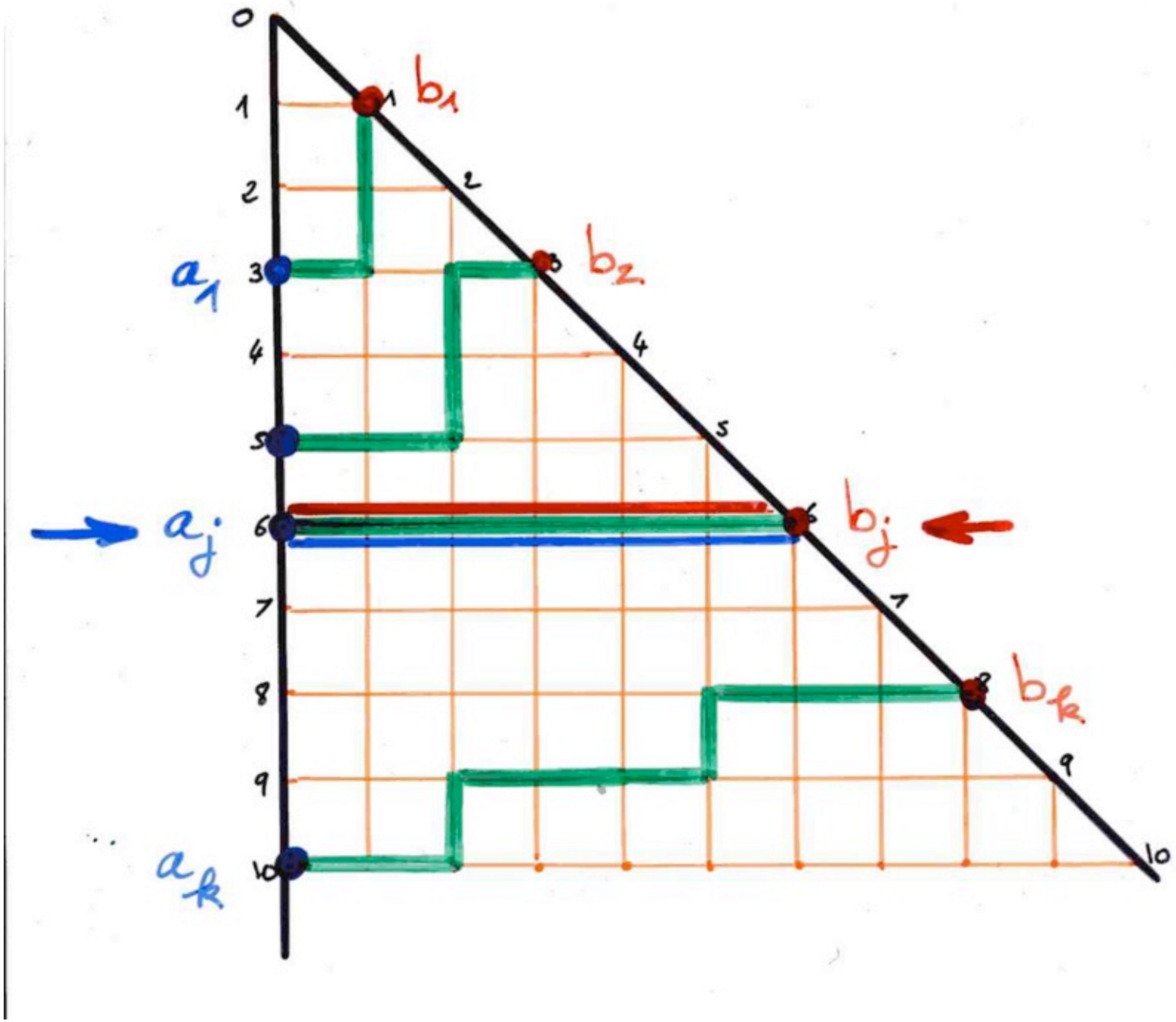
Cor 1 - $\begin{pmatrix} a_1, \ldots, a_k \\ b_1, \ldots, b_k \end{pmatrix} \geq 0$

Cor 3. If $a_j = b_j$

$$\begin{pmatrix} a_1, & \dots\dots, & a_j, & \dots\dots, & a_k \\ & & \parallel & & \\ b_1, & \dots\dots, & b_j, & \dots\dots, & b_k \end{pmatrix}$$

Cor 3. If $a_j = b_j$

$$\begin{pmatrix} a_1, \dots, a_{j-1} \\ b_1, \dots, b_{j-1} \end{pmatrix} \begin{pmatrix} a_{j+1}, \dots, a_k \\ b_{j+1}, \dots, b_k \end{pmatrix}$$



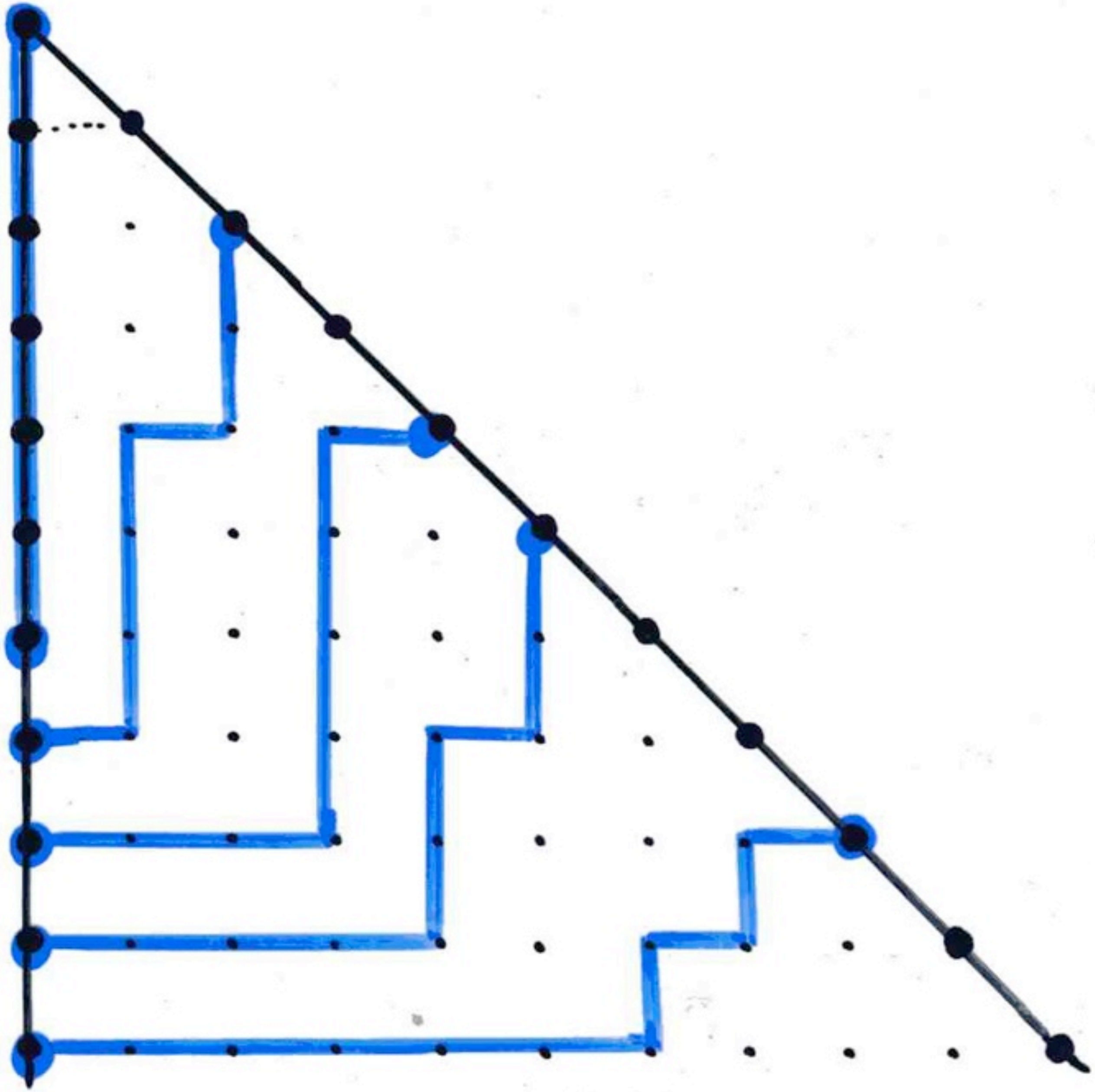
0
1
2
3
4
5
6
7
8
9
10
b_1
b_2
b_j
b_k
a_1
a_j
a_k

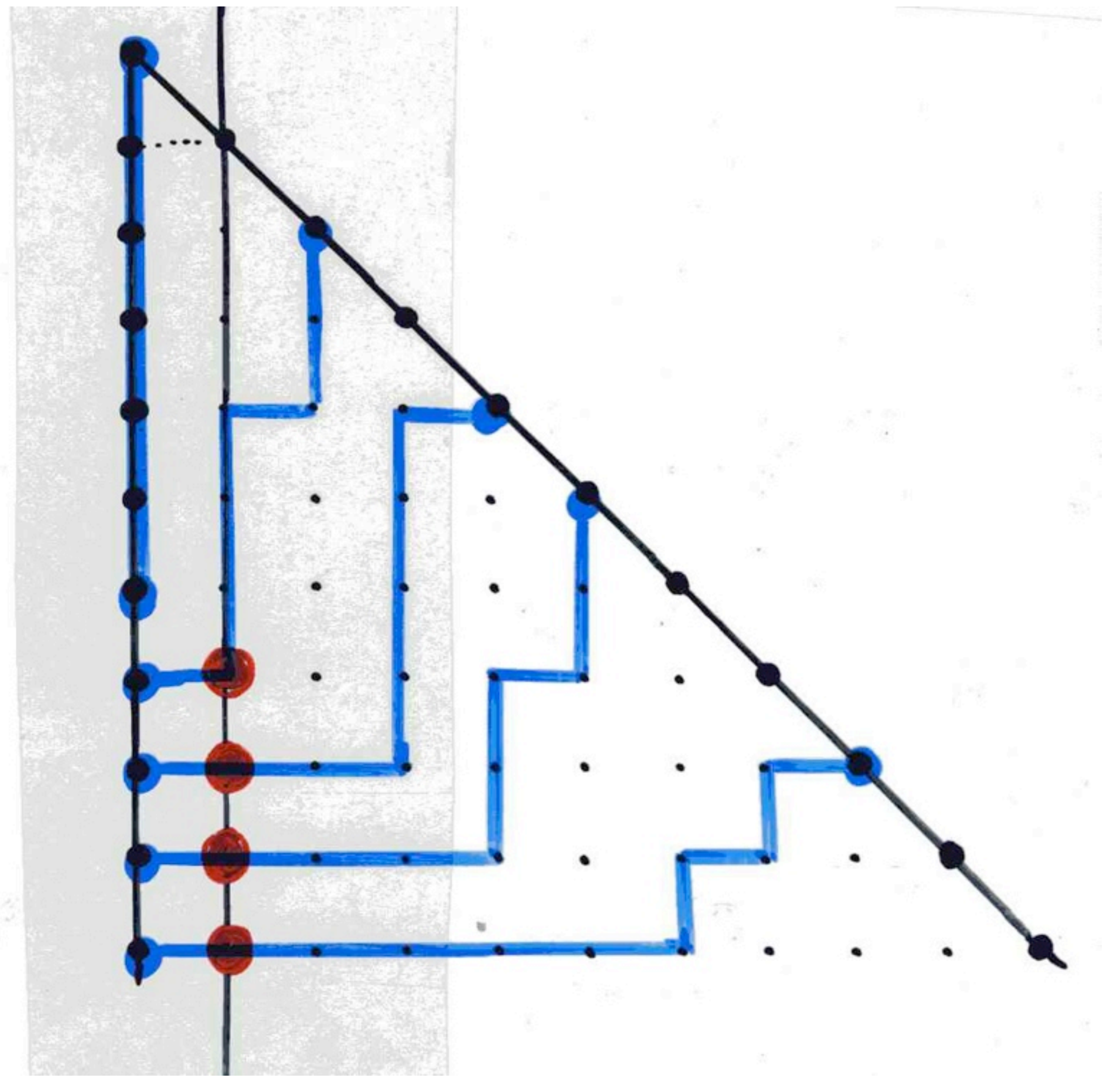
Lemma 1. If $b_1 \neq 0$, then

$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \frac{a_1 \cdots a_k}{b_1 \cdots b_k} \binom{a_1 - 1, \ldots, a_k - 1}{b_1 - 1, \ldots, b_k - 1}$$

Lemma 2.

$$\binom{a, a+1, \ldots, a+k-1}{0, b_2, \ldots, b_k} = \binom{a, a+1, \ldots, a+k-2}{b_2 - 1, b_3 - 1, \ldots, b_k - 1}$$

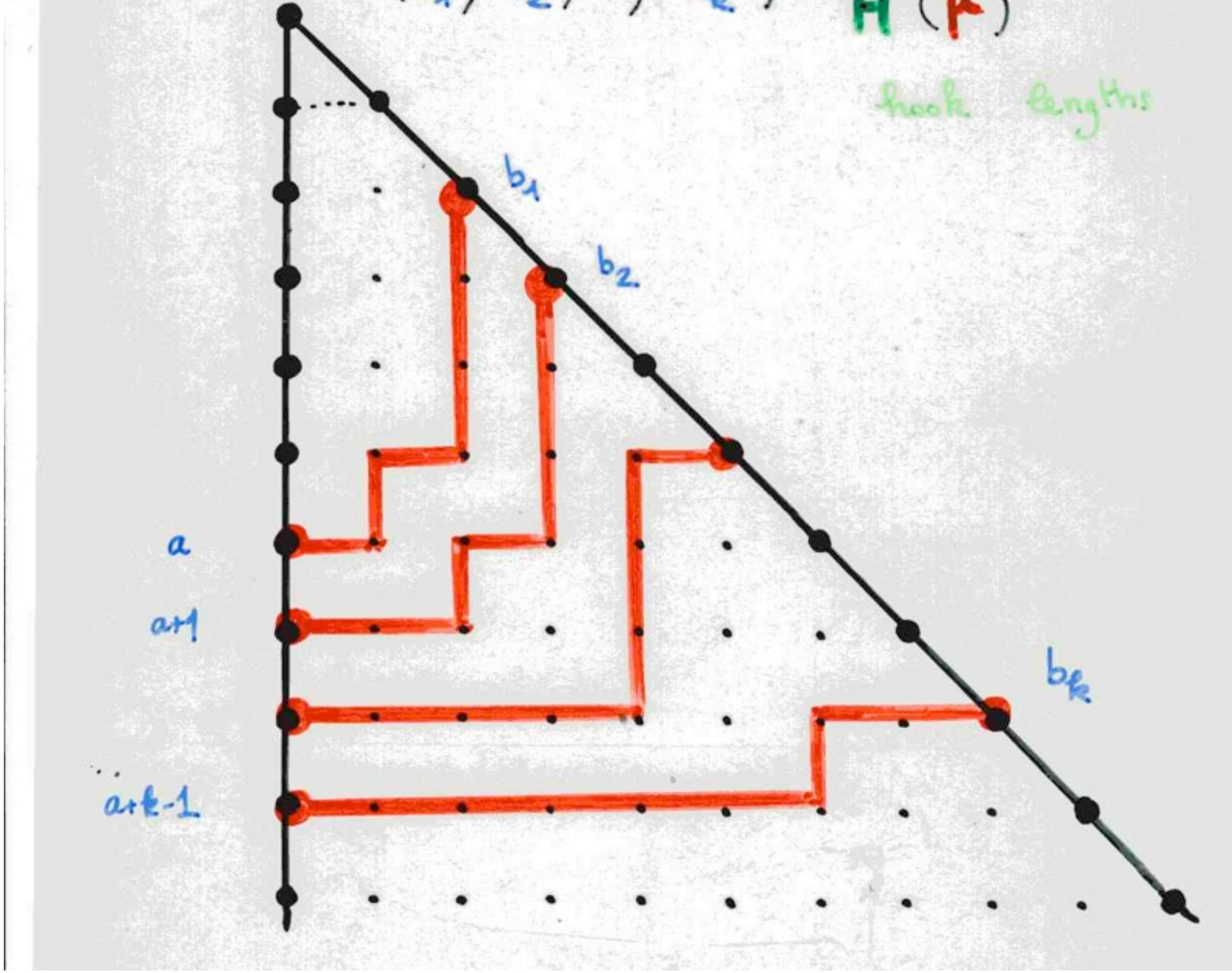




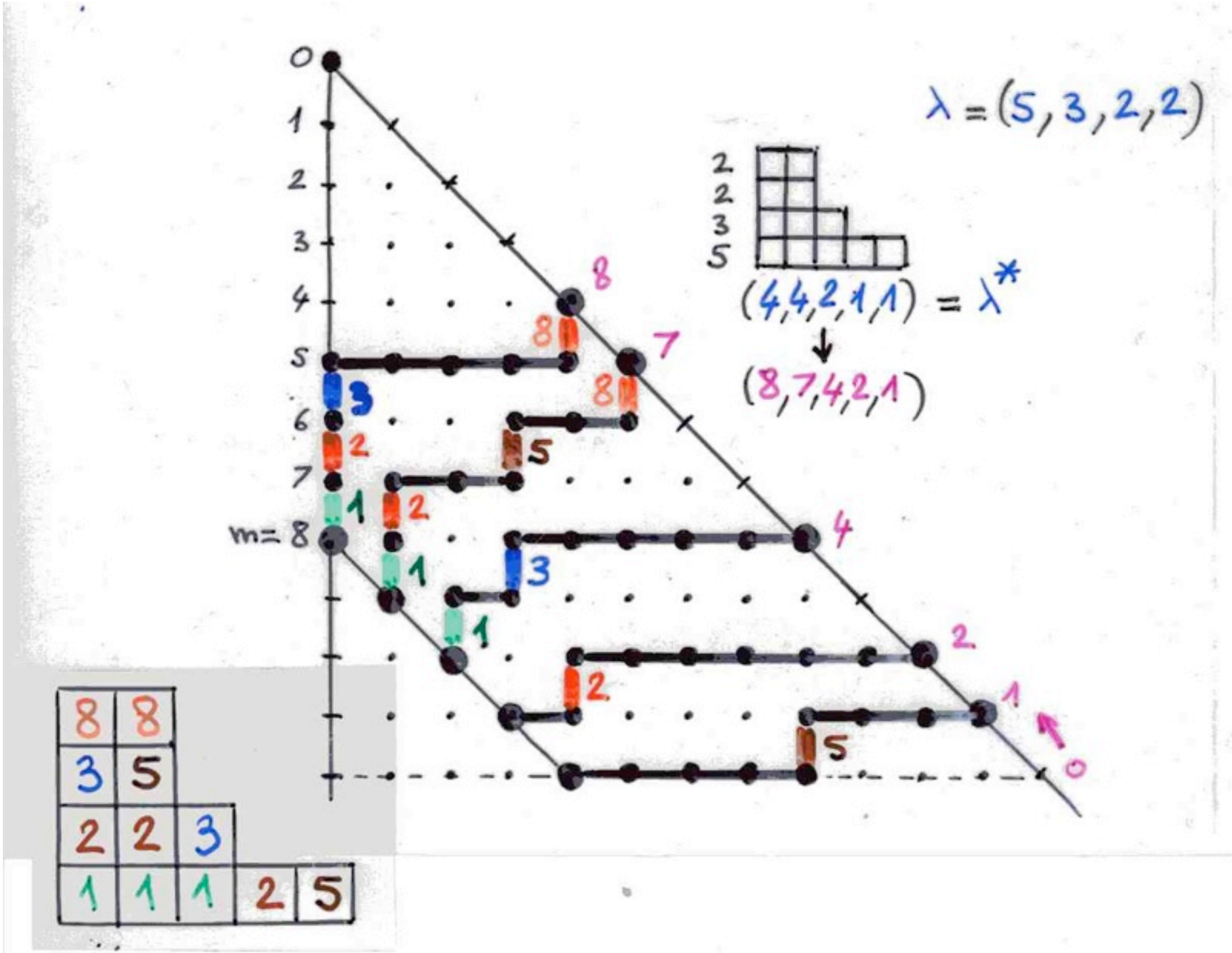
contents

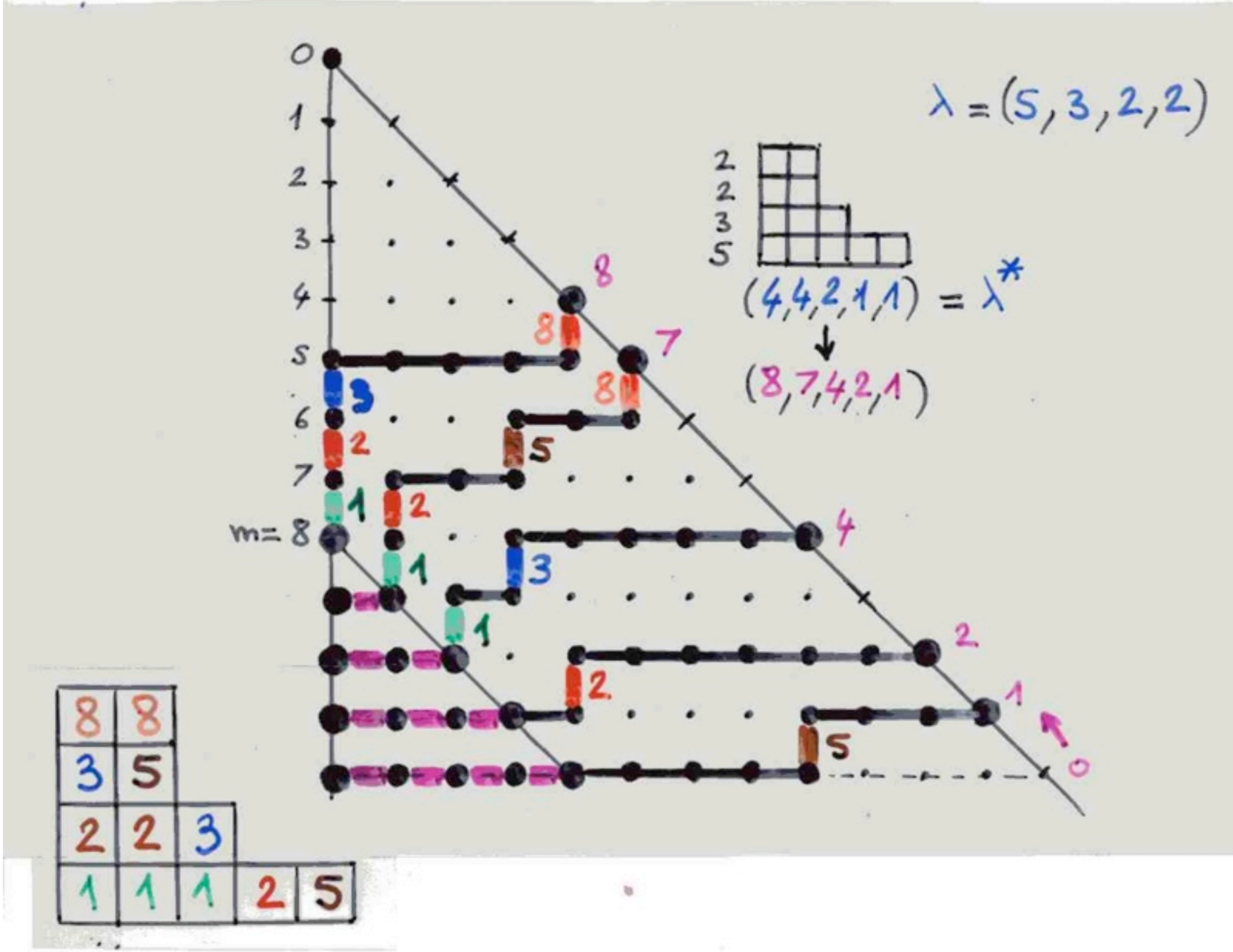
$$\begin{pmatrix} a, & a+1, & \ldots, & a+k-1 \\ b_1, & b_2, & \ldots, & b_k \end{pmatrix} = \frac{C_a(\mu)}{H(\mu)}$$

hook lengths



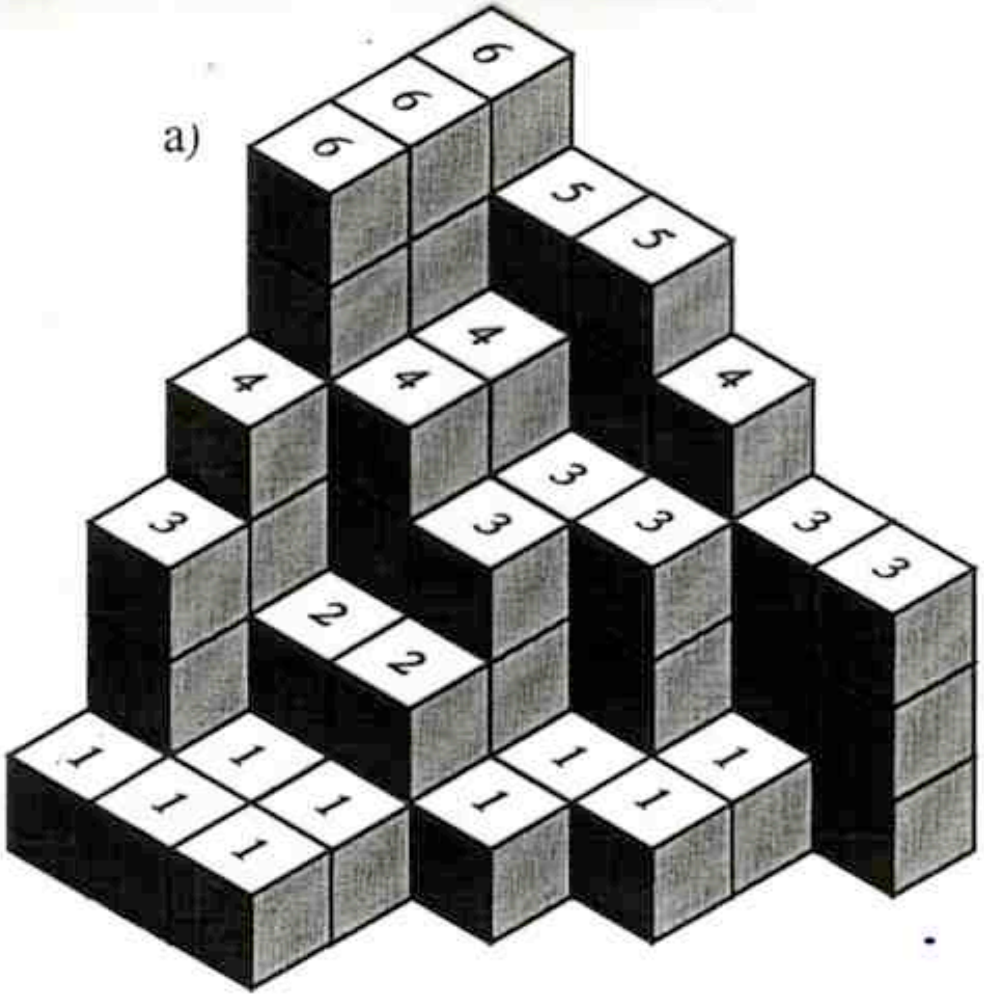
§4 Young tableaux


λ = (5, 3, 2, 2)
(4,4,2,1,1) = λ*
(8,7,4,2,1)
m= 8


λ = (5, 3, 2, 2)
(4,4,2,1,1) = λ*
(8,7,4,2,1)
m= 8
0
1
2
3
4
5
6
7
8
2
2
3
5
8 8
3 5
2 2 3
1 1 1 2 5

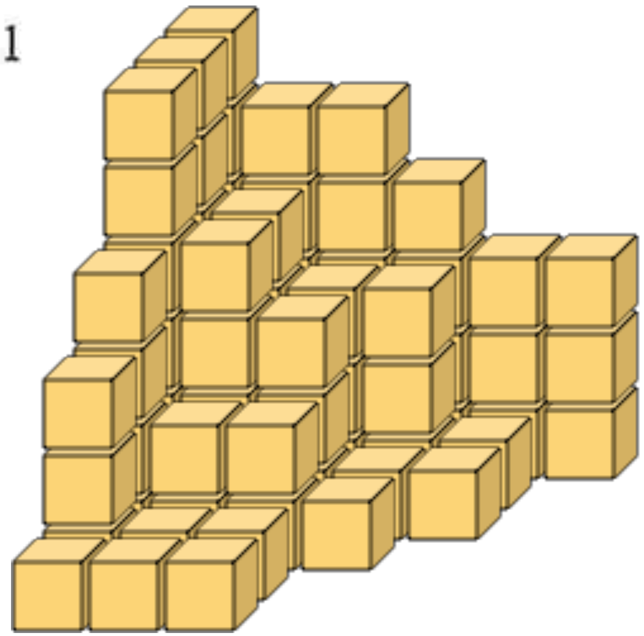
Plane partitions

a)



b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

Partitions planes bornées

diagramme 3D $\quad F \subseteq \mathcal{B}(r,s,t)$

$$\mathcal{B}(r,s,t) = \left\{ (i,j,k) \in \mathbb{N}^3, \begin{array}{l} 1 \le i \le r \\ 1 \le j \le s \\ 1 \le k \le t \end{array} \right\}$$

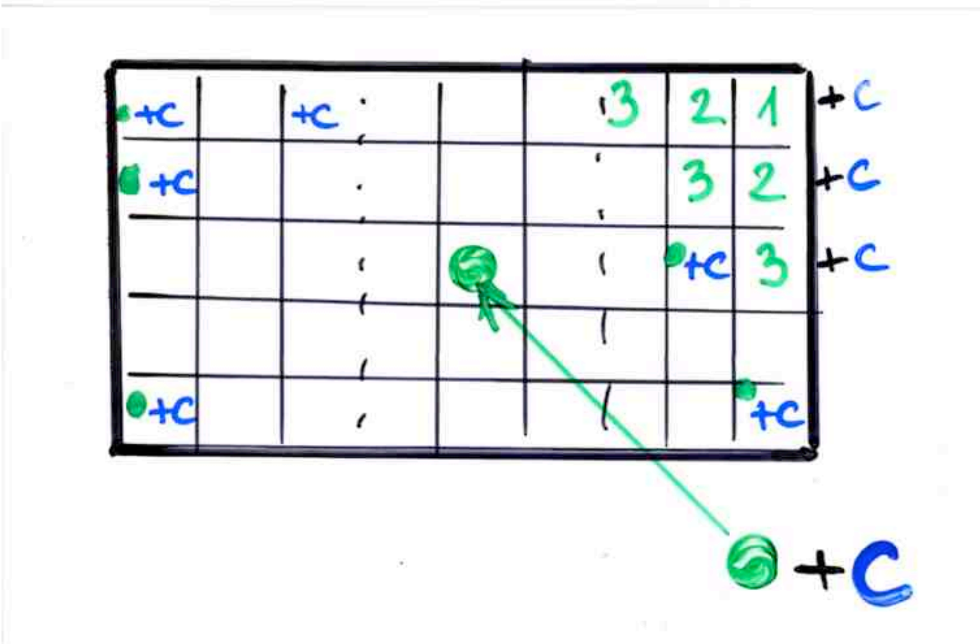
$\mathcal{B}(r,s,t)$

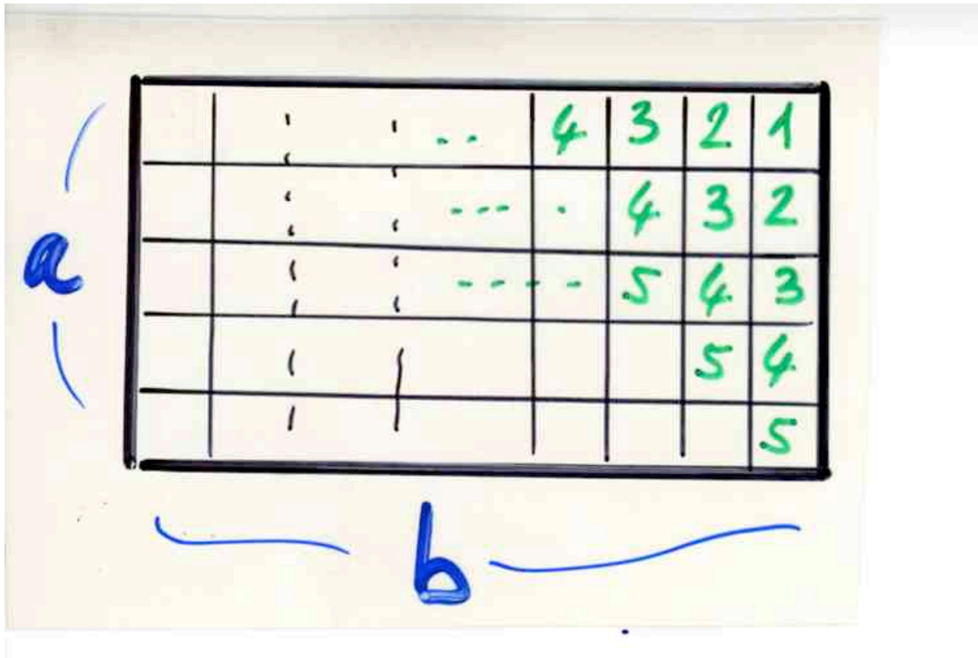
at most r rows
at most s columns
parts $\le t$

$\mathcal{B}(7,6,6)$

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

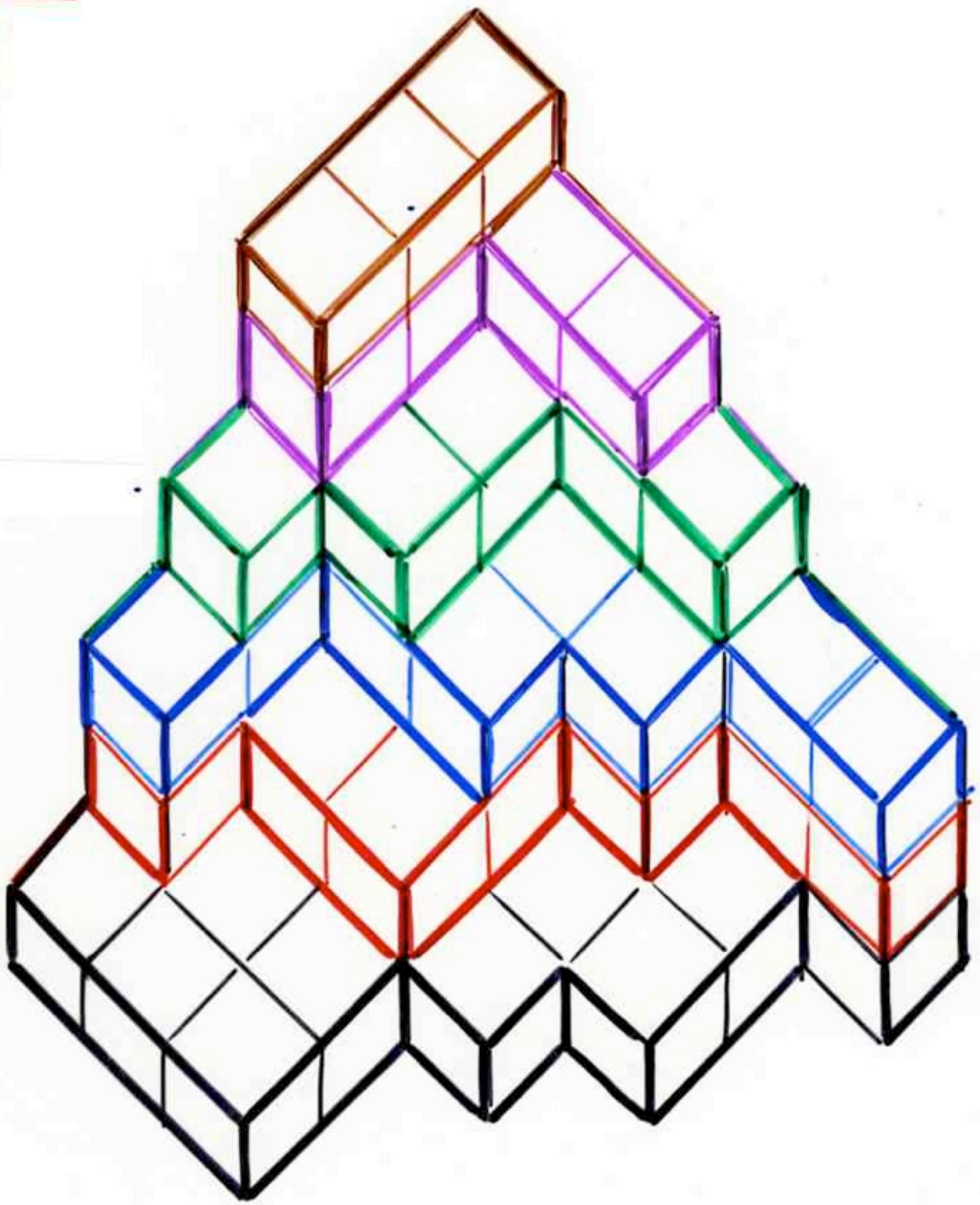
$$\prod_{\substack{1 \le i \le a \\ 1 \le j \le b \\ 1 \le k \le c}} \frac{i+j+k-1}{i+j+k-2}$$

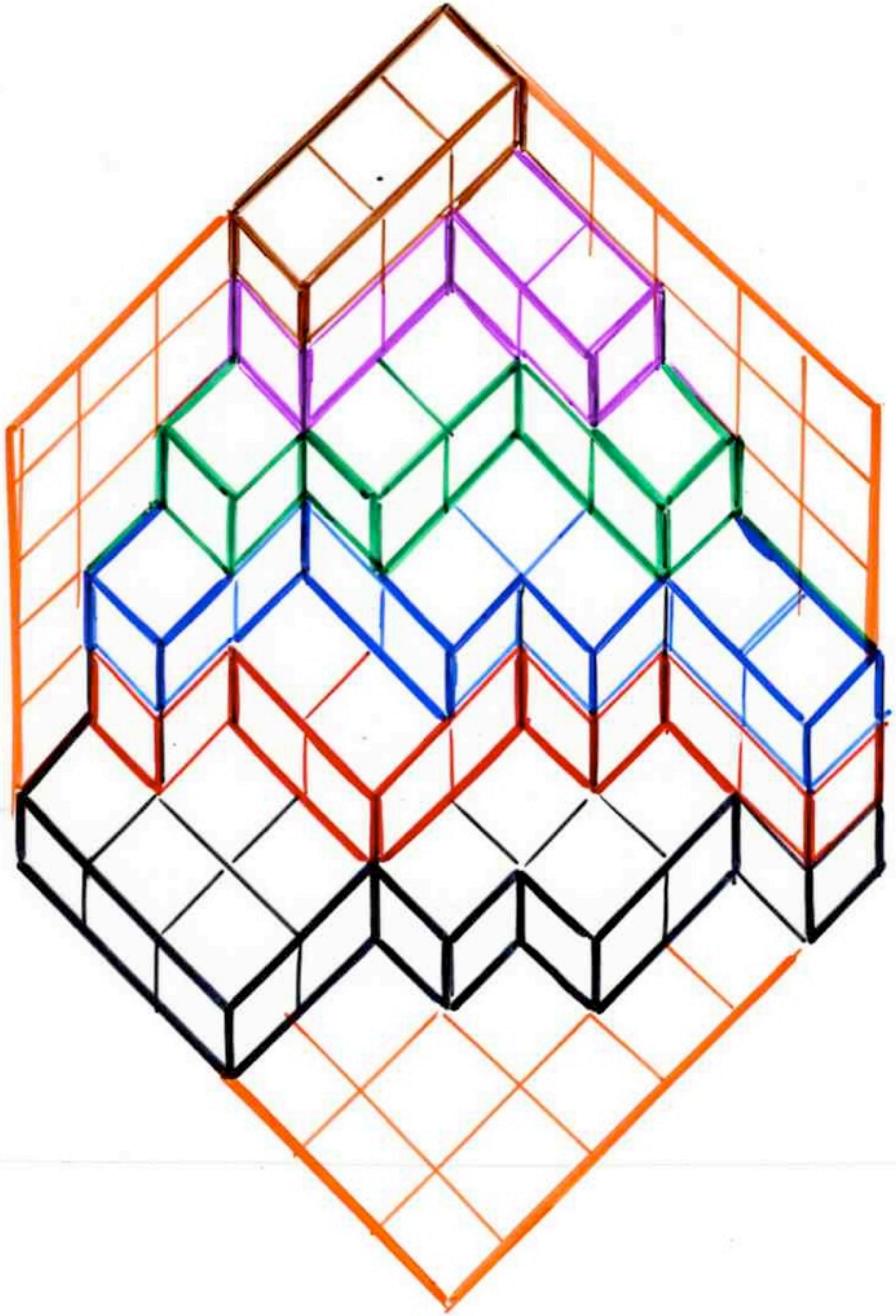

3 2 1
3 2
3
+C
+C
+C
+C
+C
+C
+C
+C
+C
+C

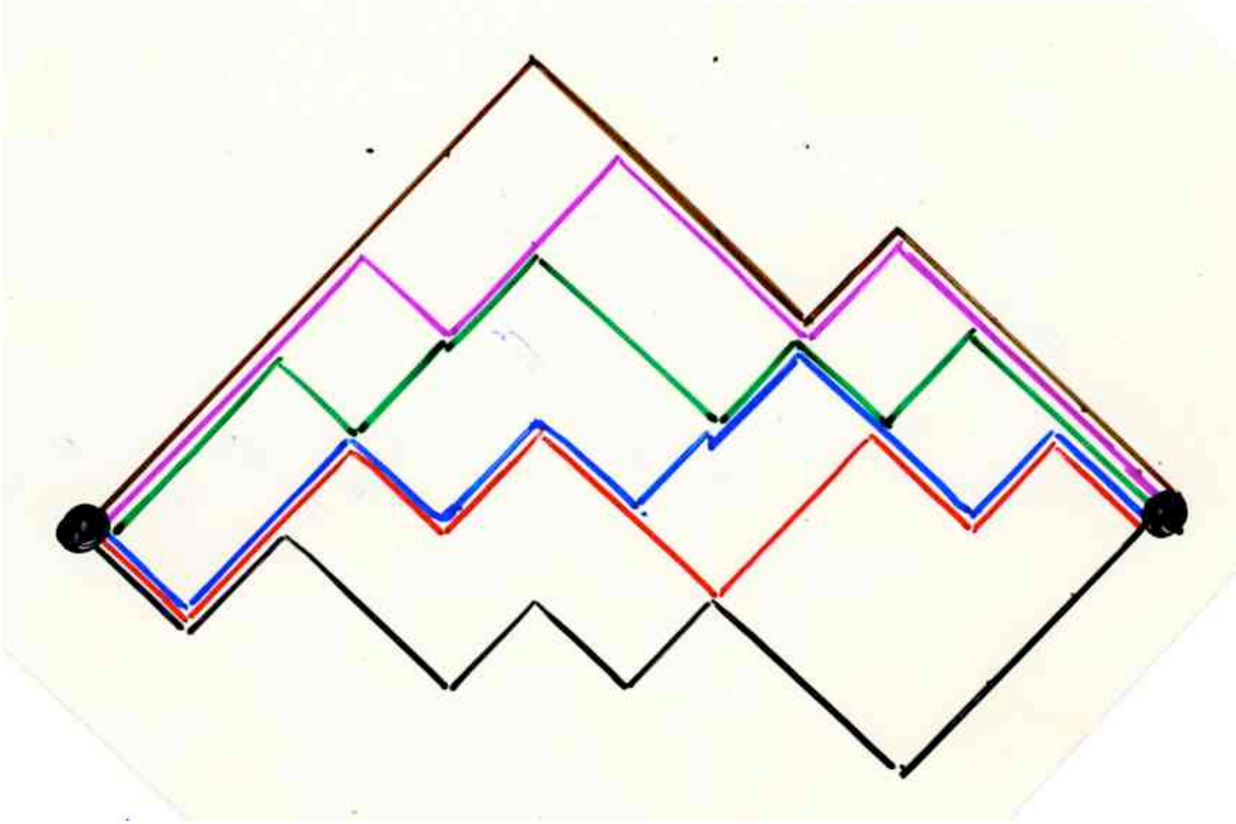

4 3 2 1
4 3 2
5 4 3
5 4
5
a
b

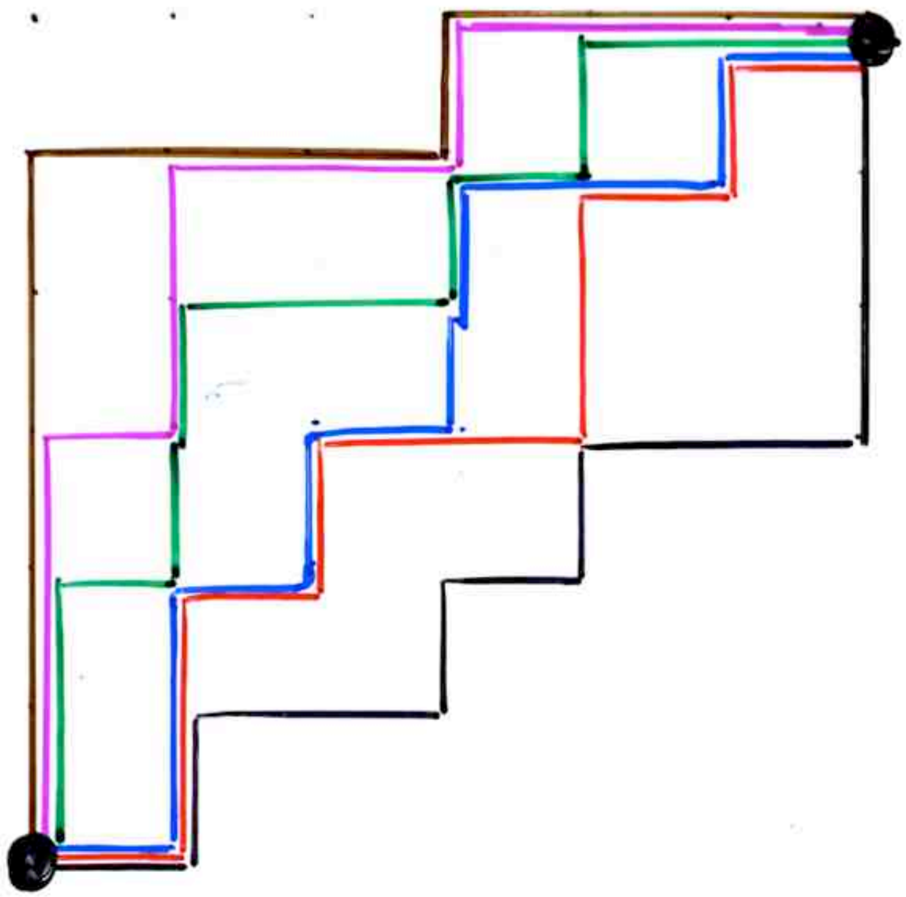
Plane partitions and paths

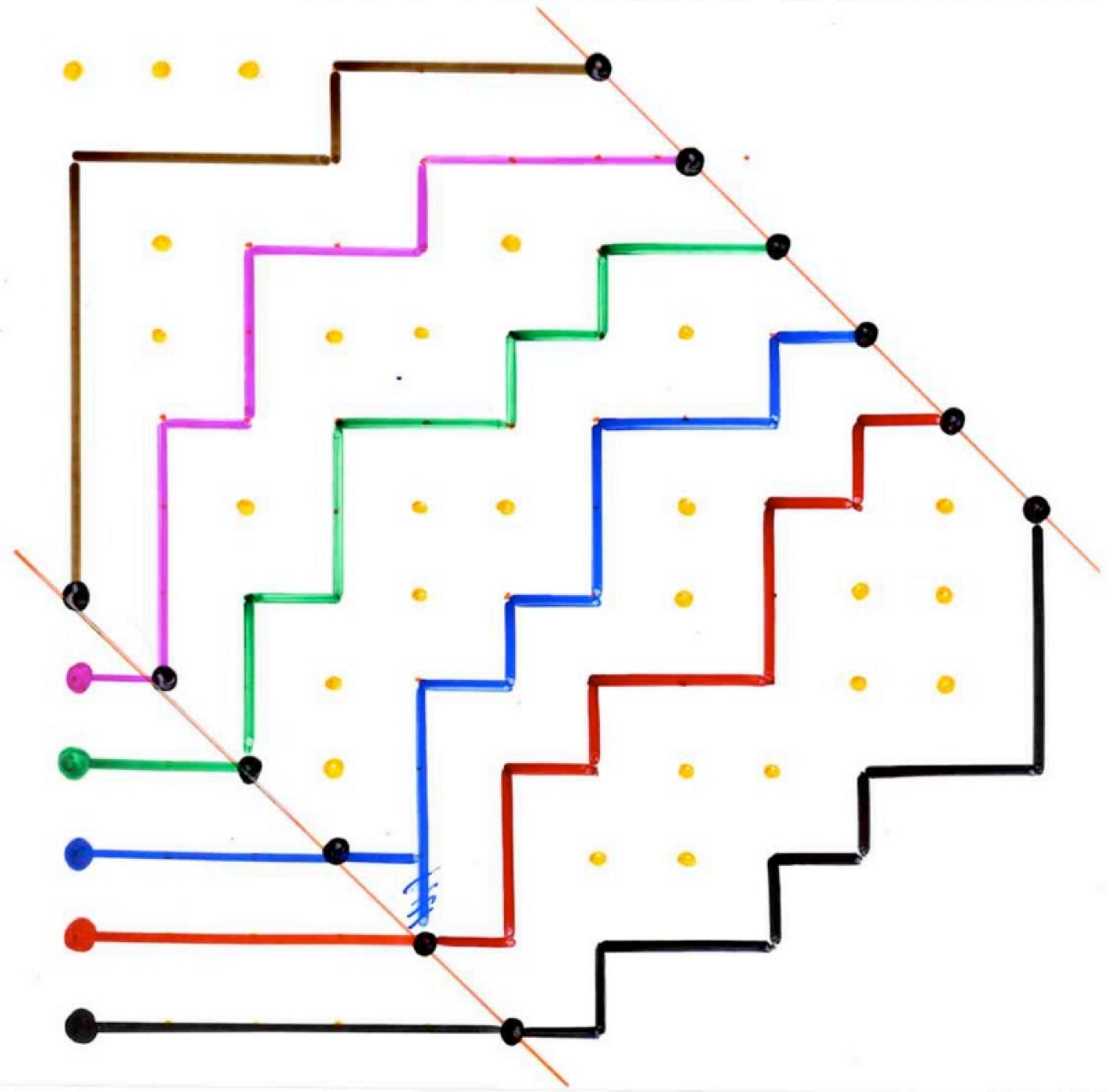
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			



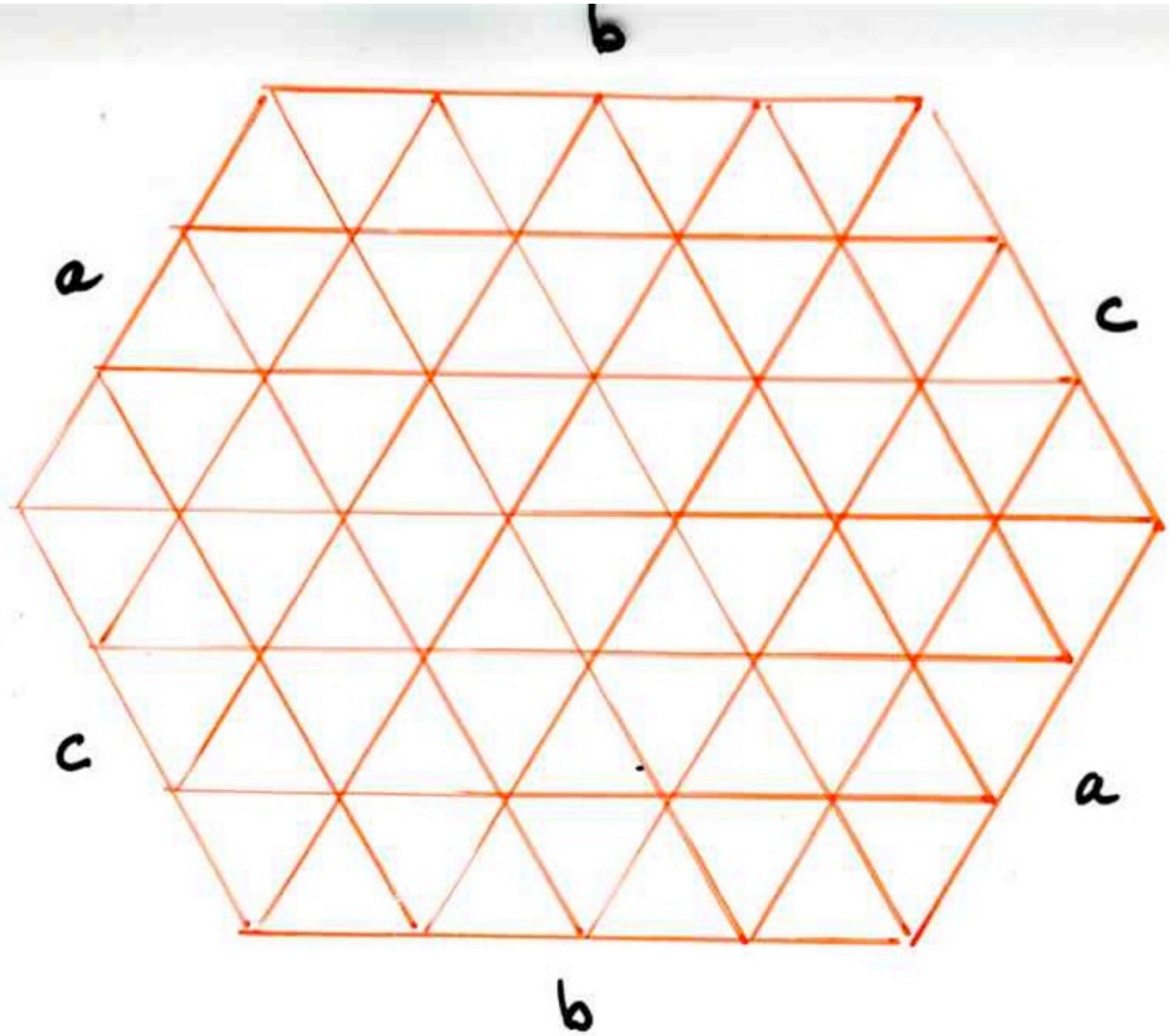


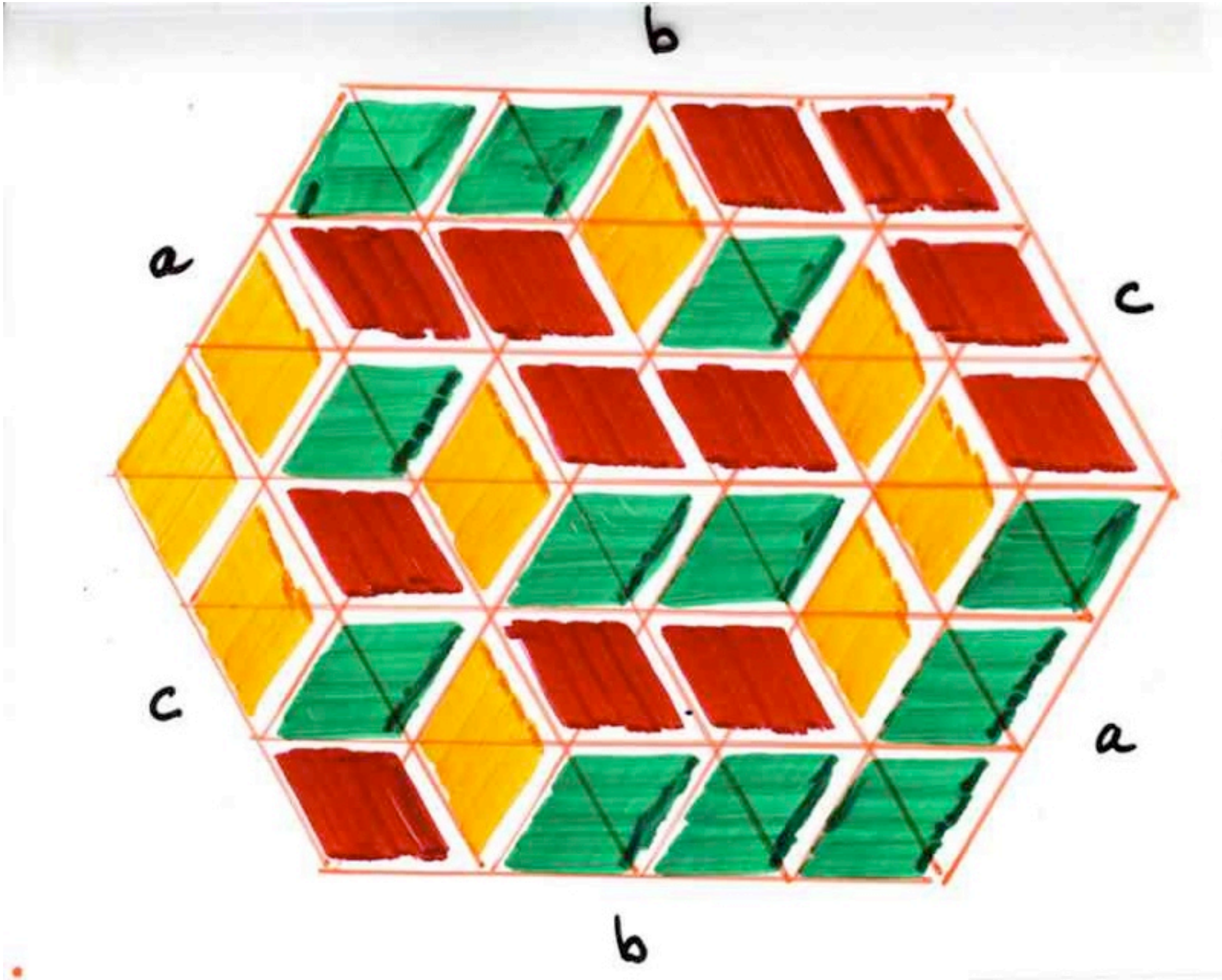


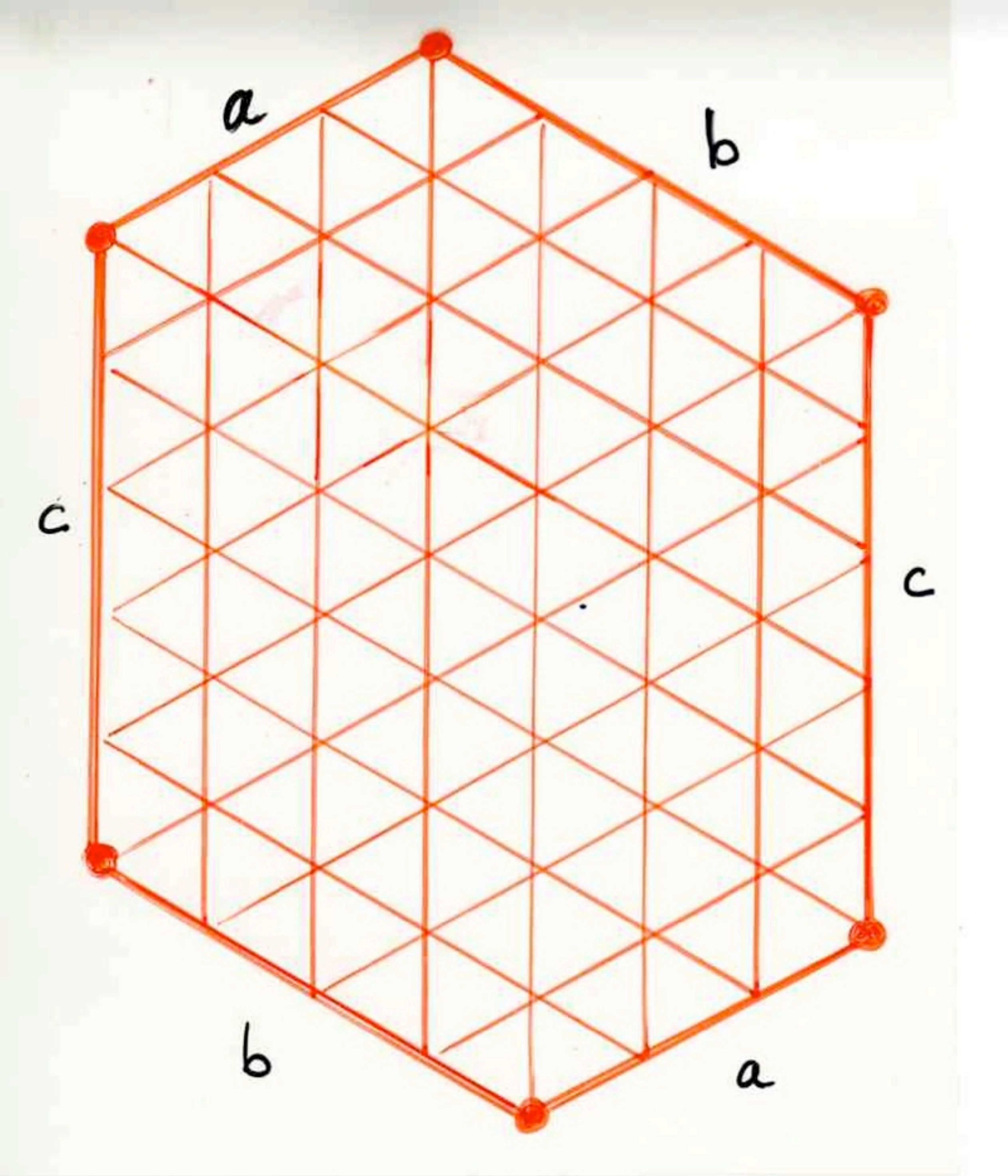


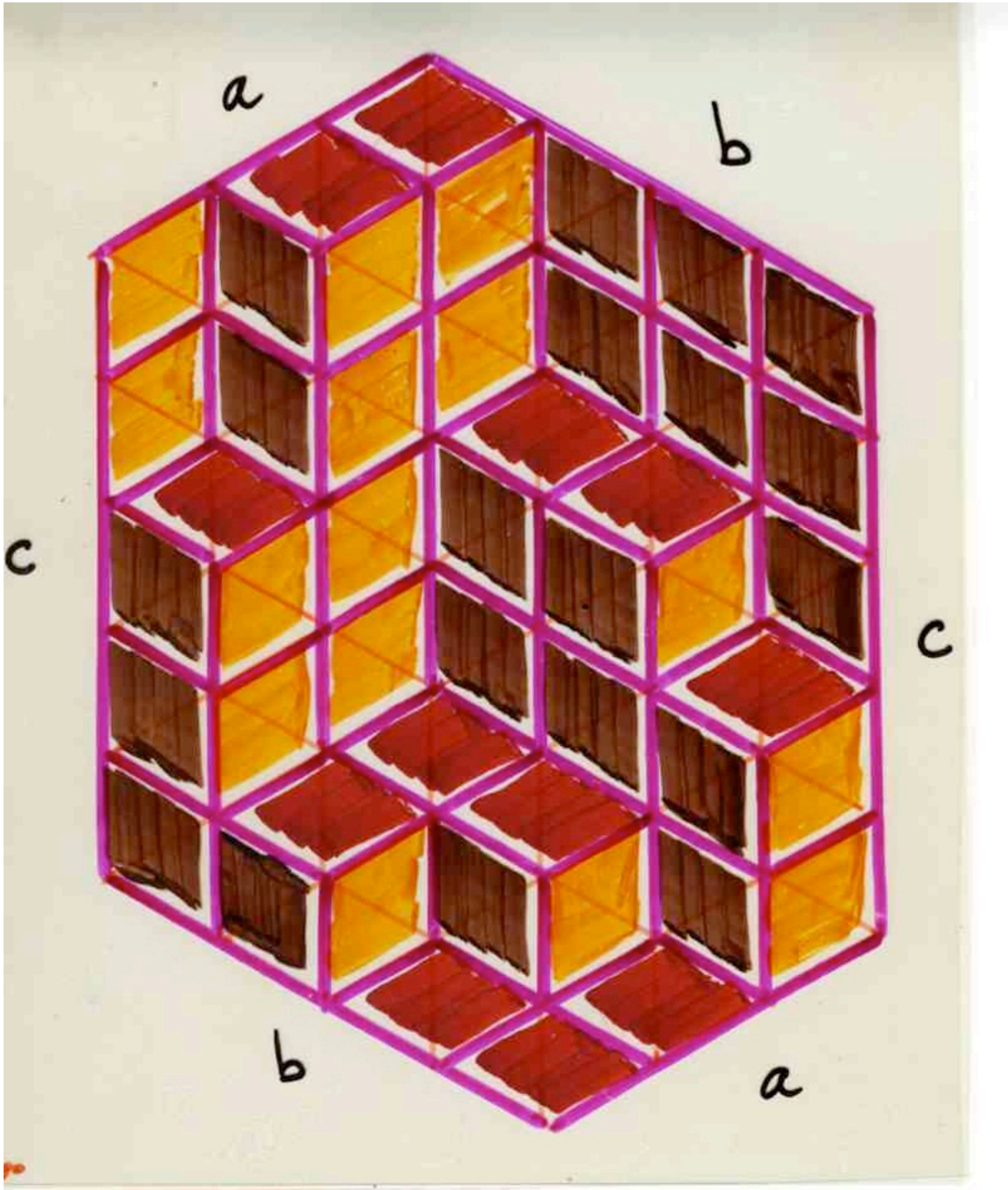


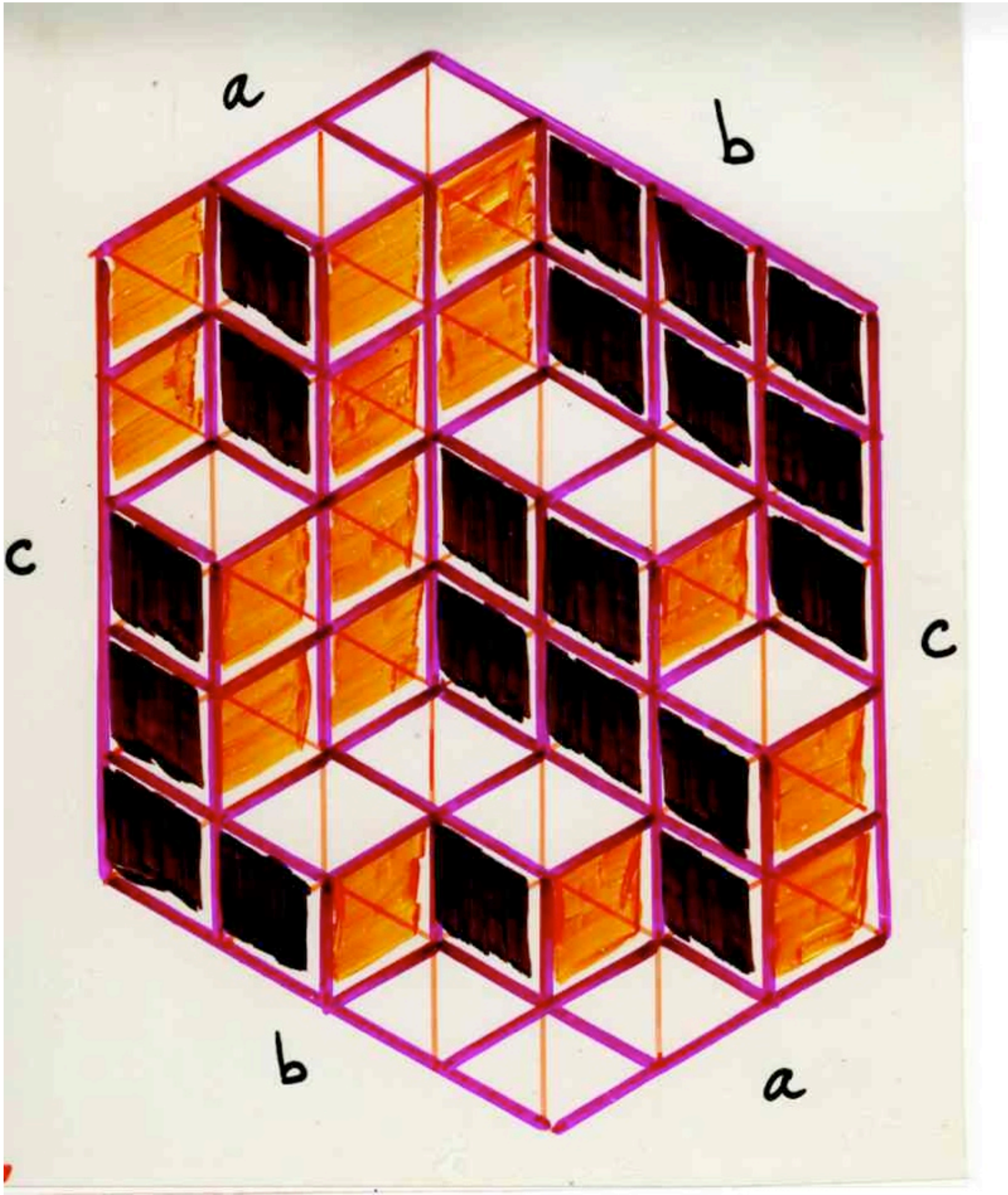
§5 Tilings

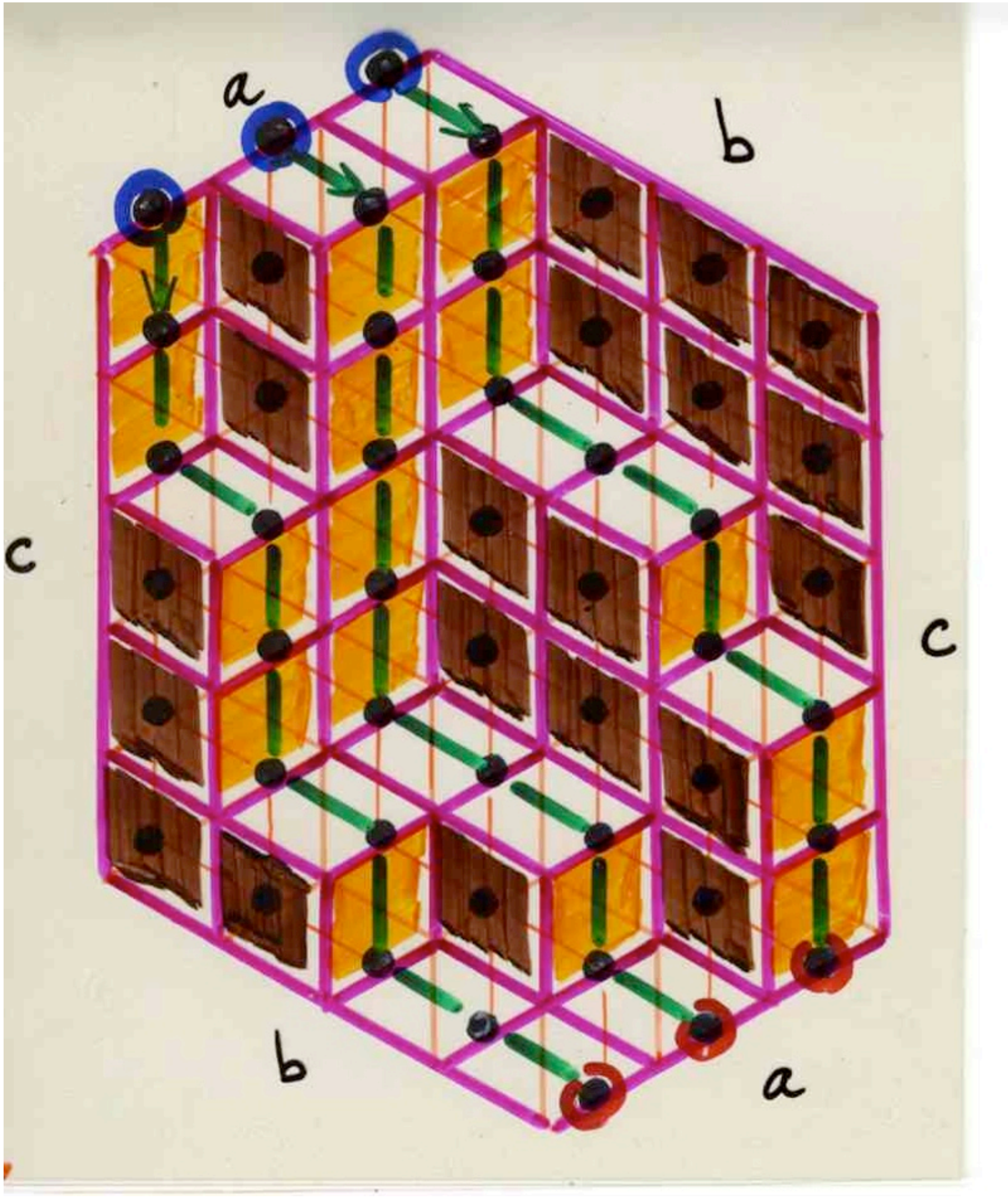

b
a
c
c
a
b


b
a
c
c
a
b

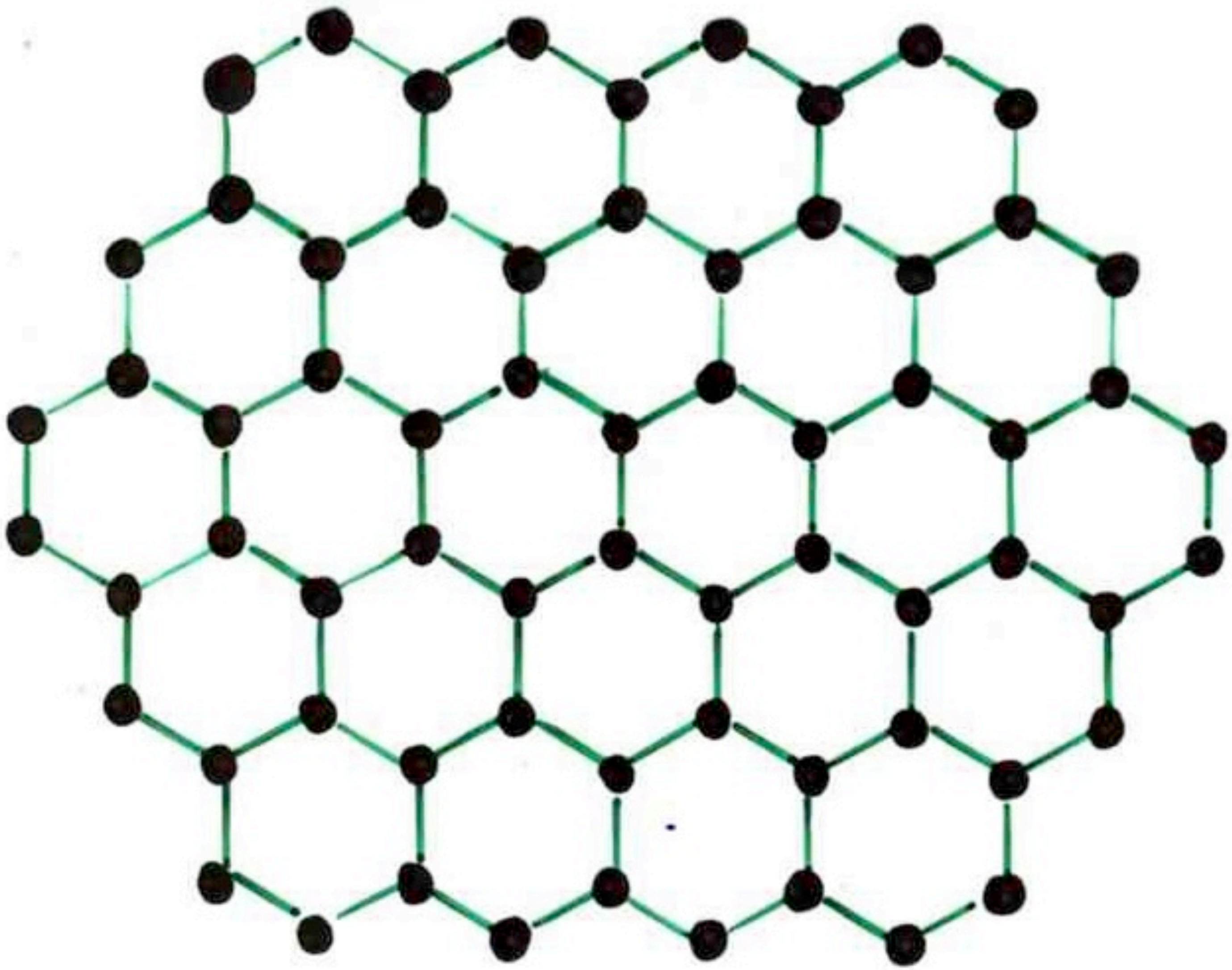

a
b
c
c
b
a

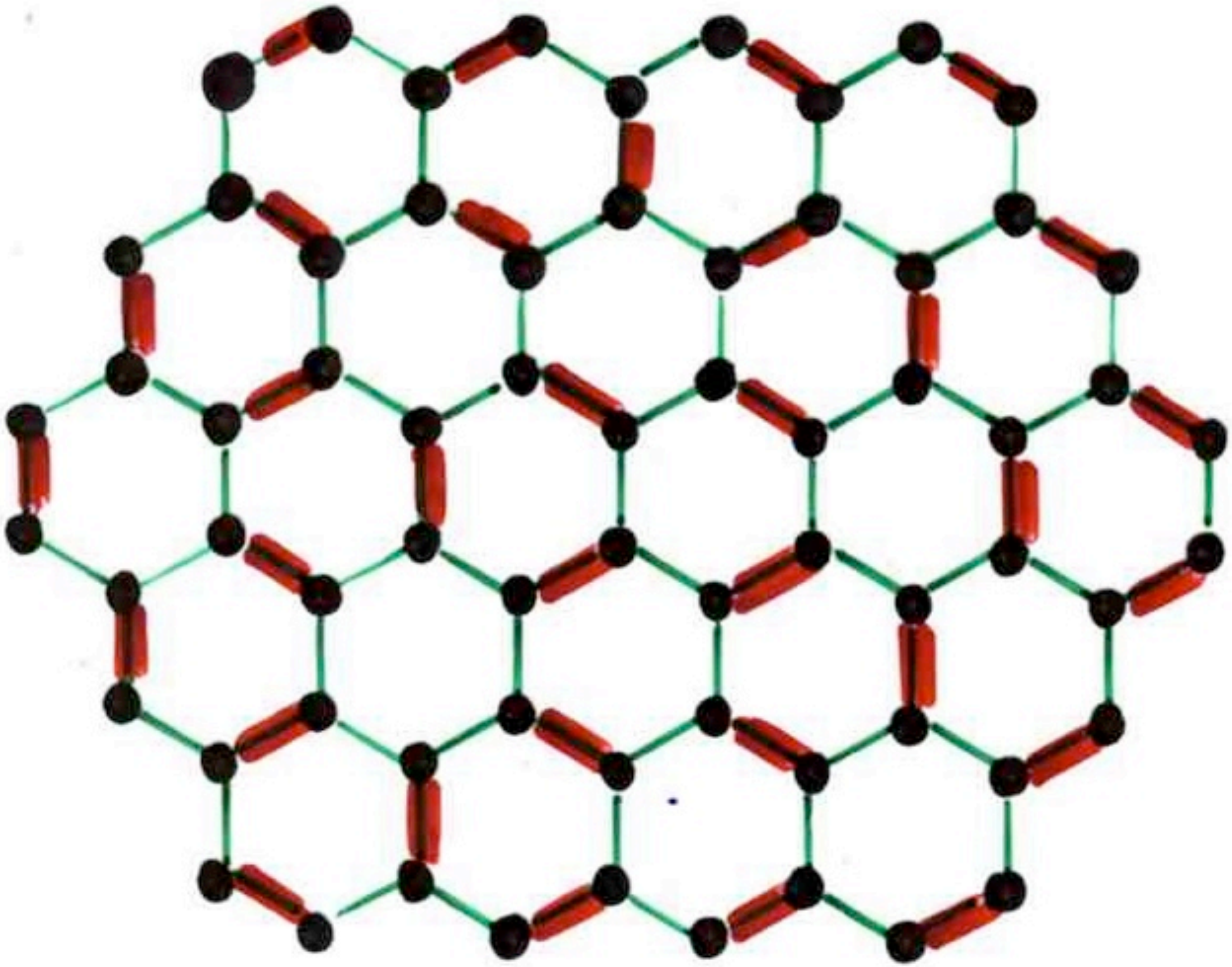

a
b
c
c
b
a

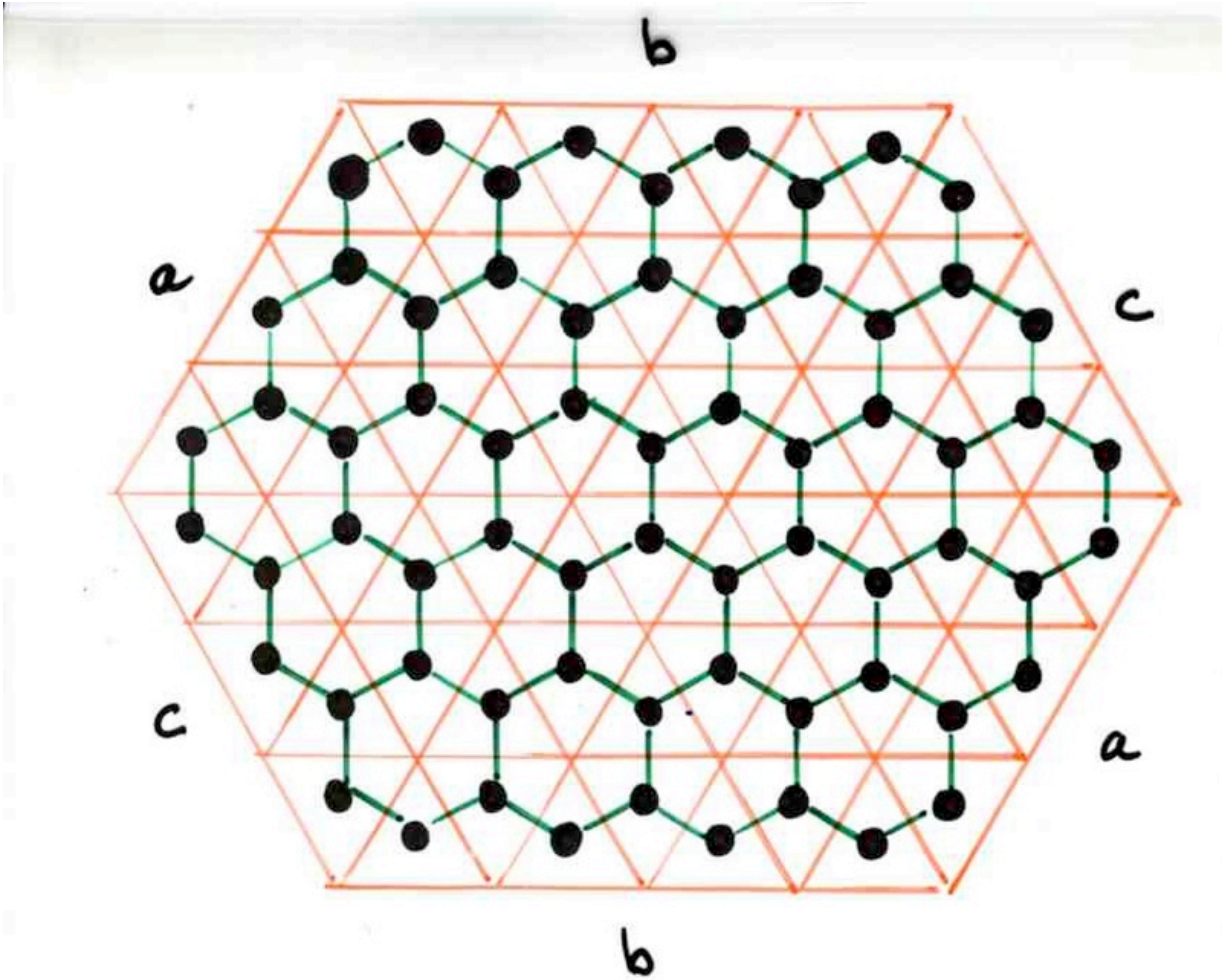

a
b
c
c
b
a


a
b
c
c
b
a

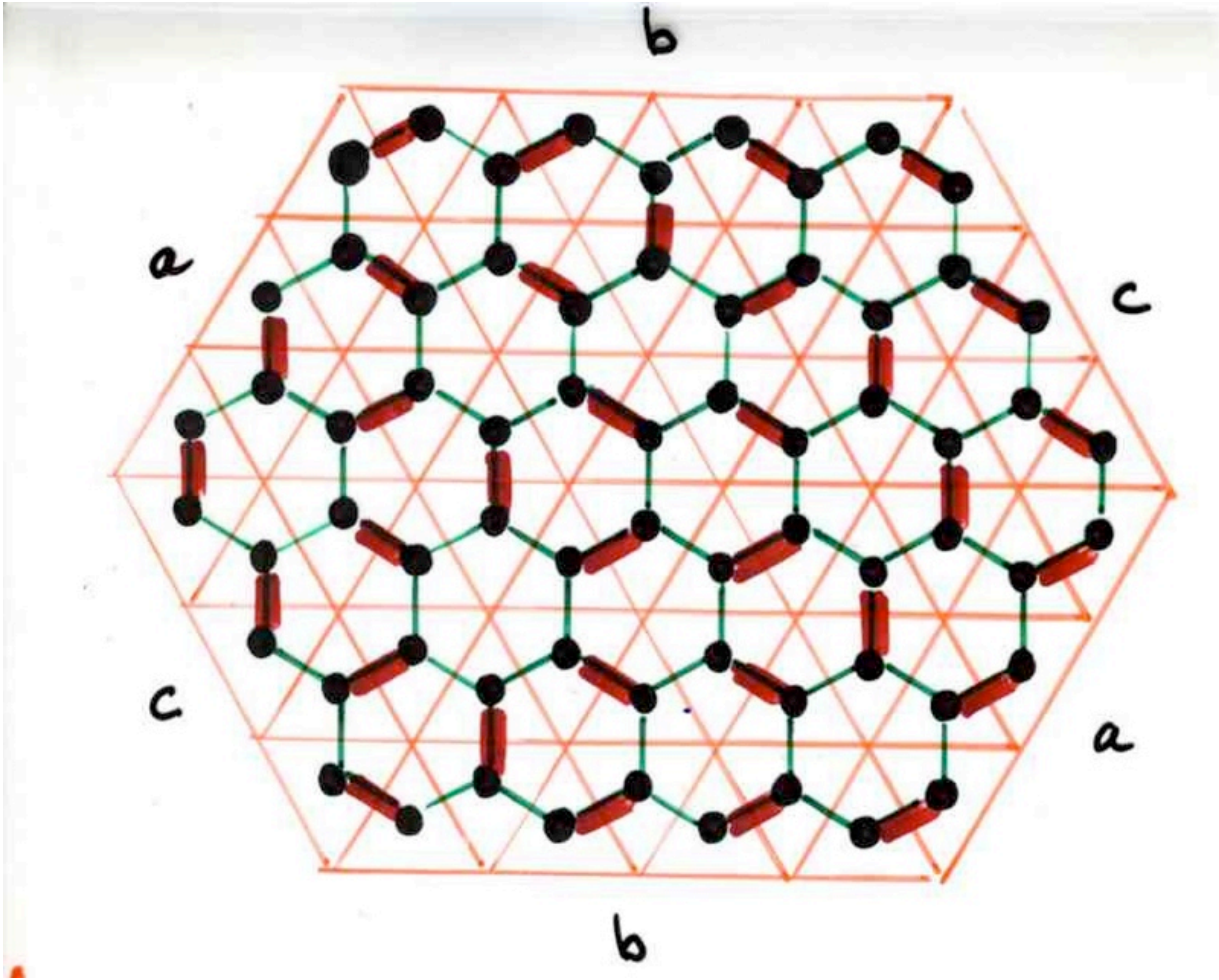
§6 Perfect matchings

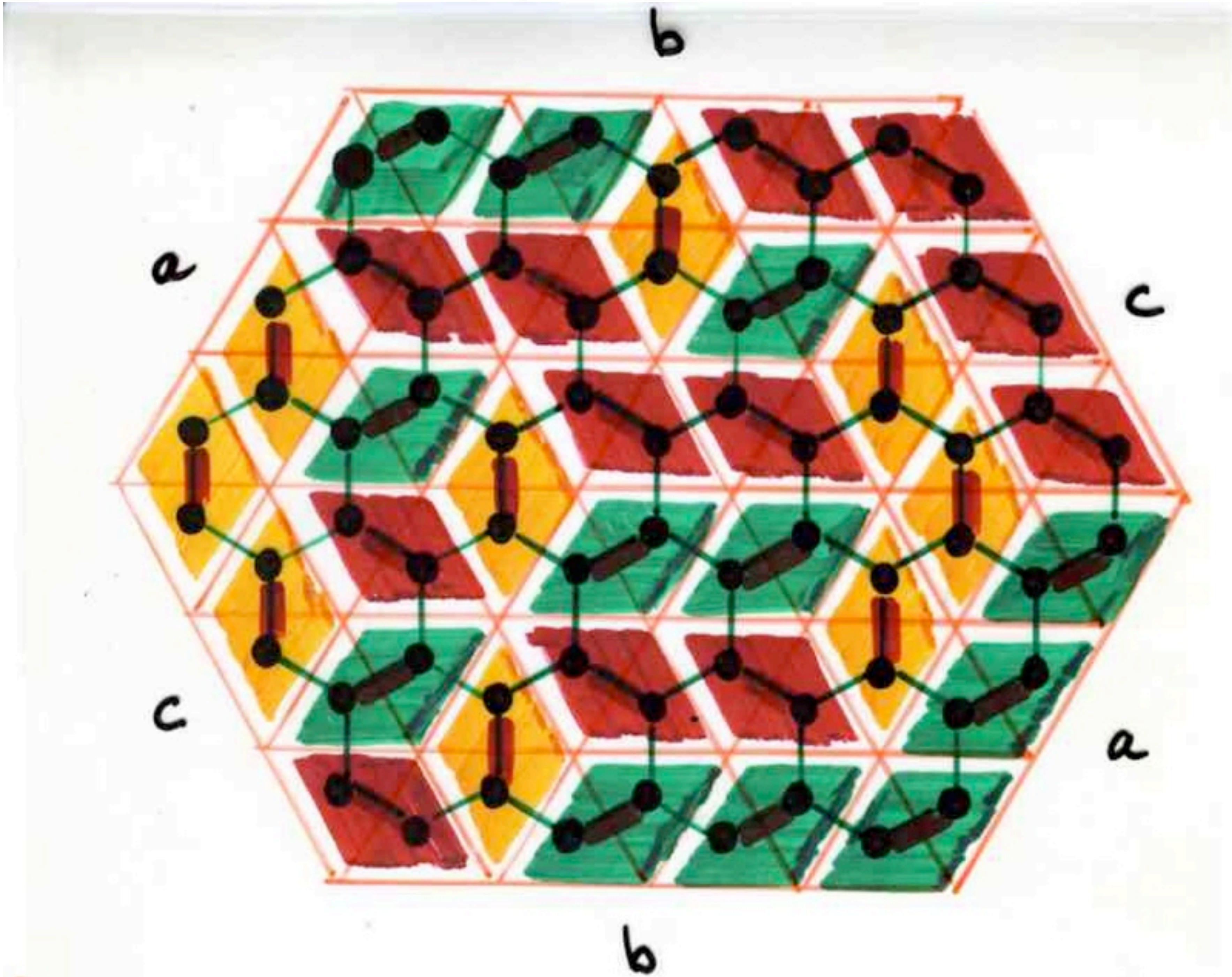






b
a
c
c
a
b


b
a
c
c
a
b


b
a
c
c
a
b

Quantum-chemical theory

resonance theoretic methods

Gutman (1980, --- 1990, ---)

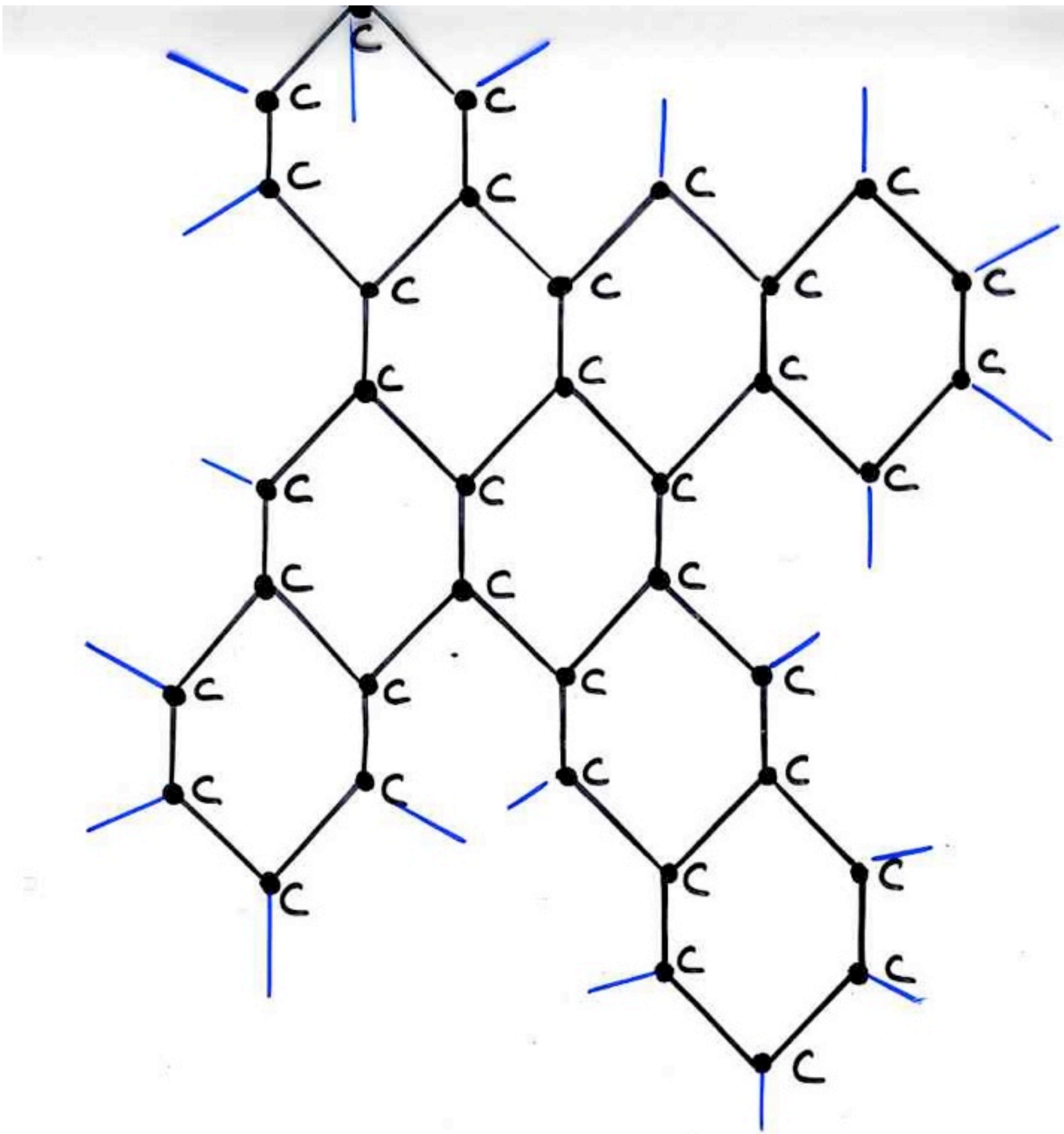
Klein, Hite, Seitz, Schmaltz (1986)

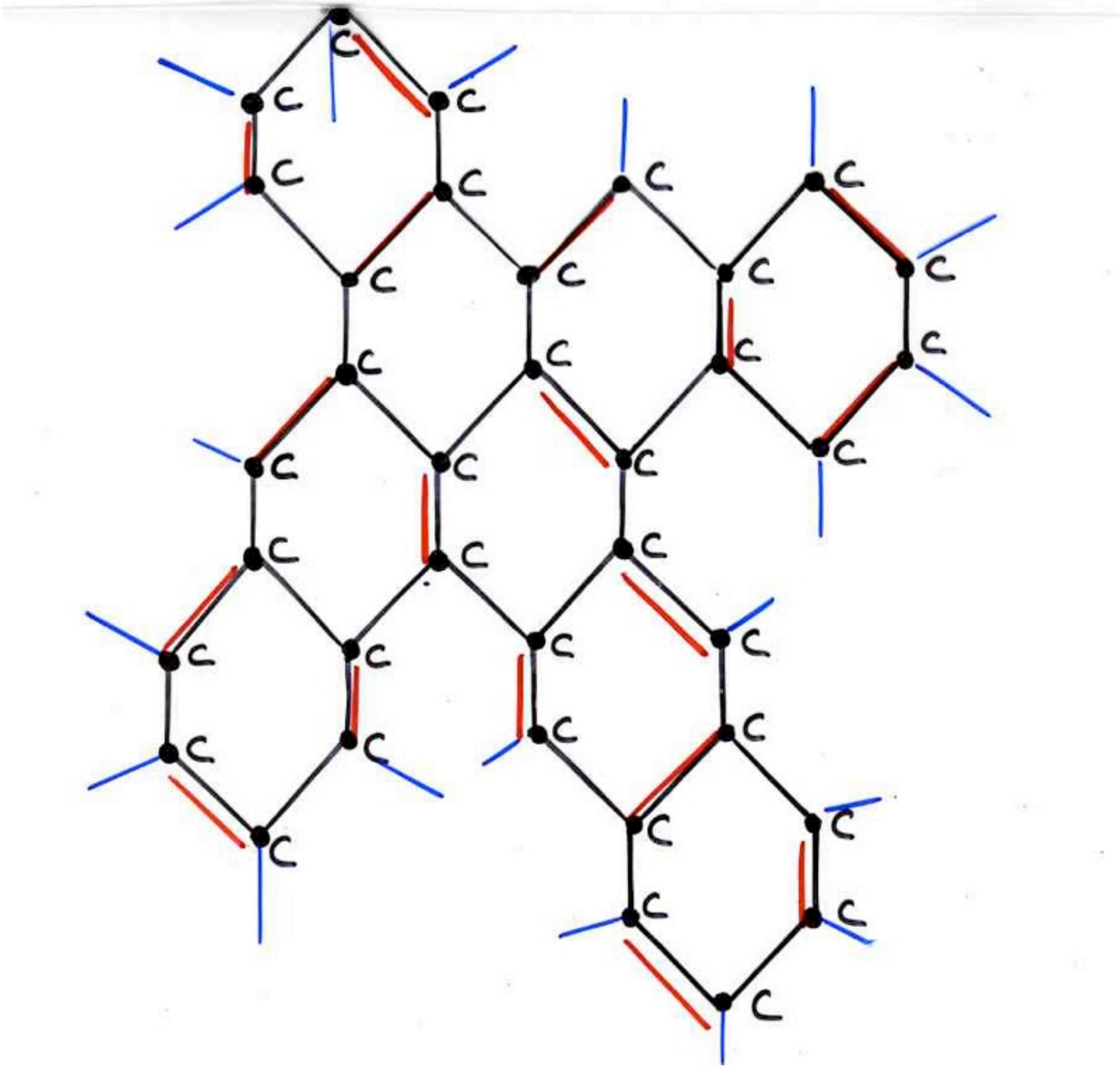
Randić, Nikolić, Trinajstić (1988)
--- (1990)

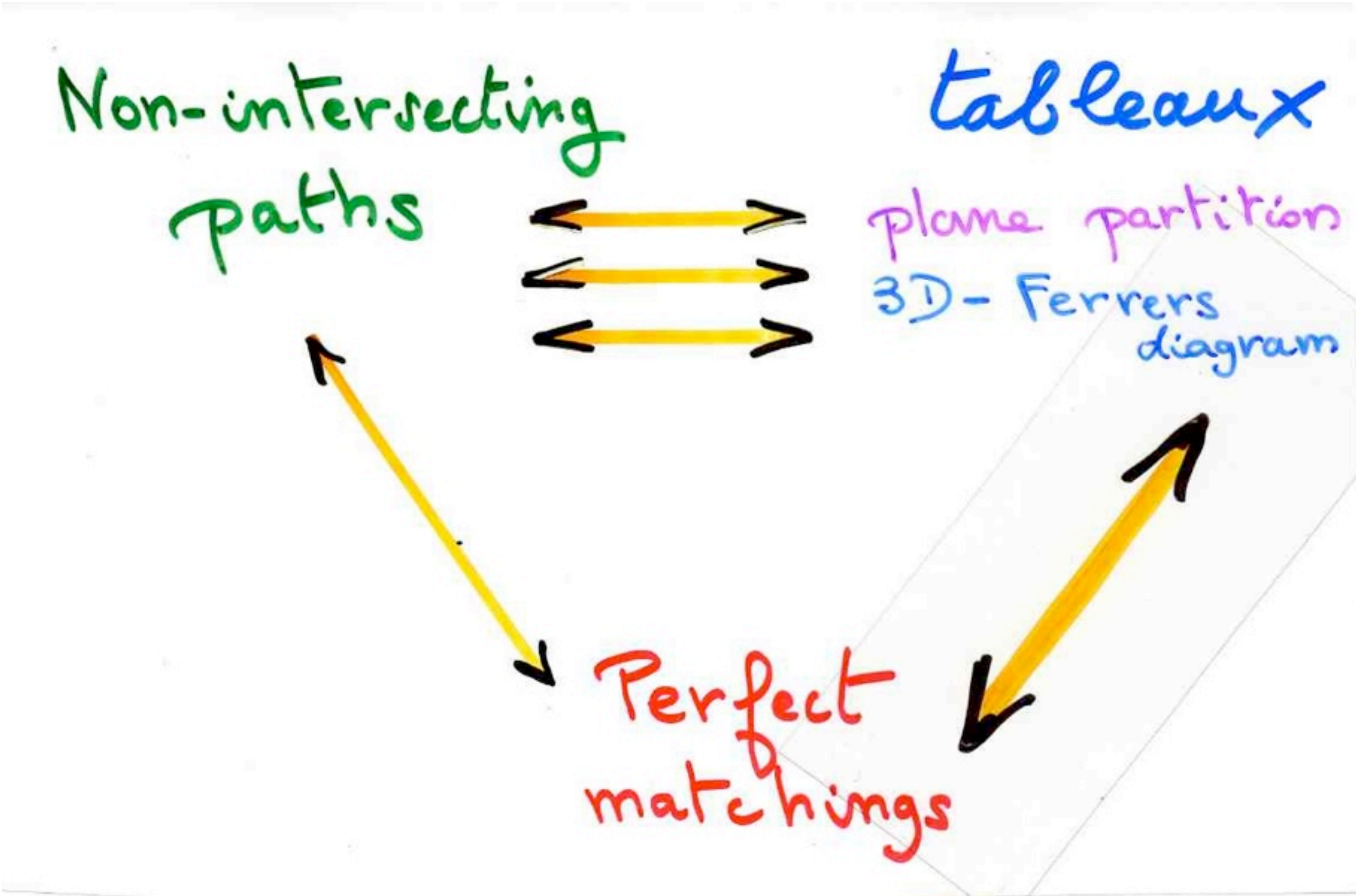
Jerman-Blažič, Živković (1991)

Zhang Fuji (1990, --) Hosoya (1986)

honeycomb graph


C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C
C


C


Non-intersecting paths
tableaux
plane partition
3D - Ferrers diagram
Perfect matchings

- dénombrement de
  couplages parfaits
- graphe planaire

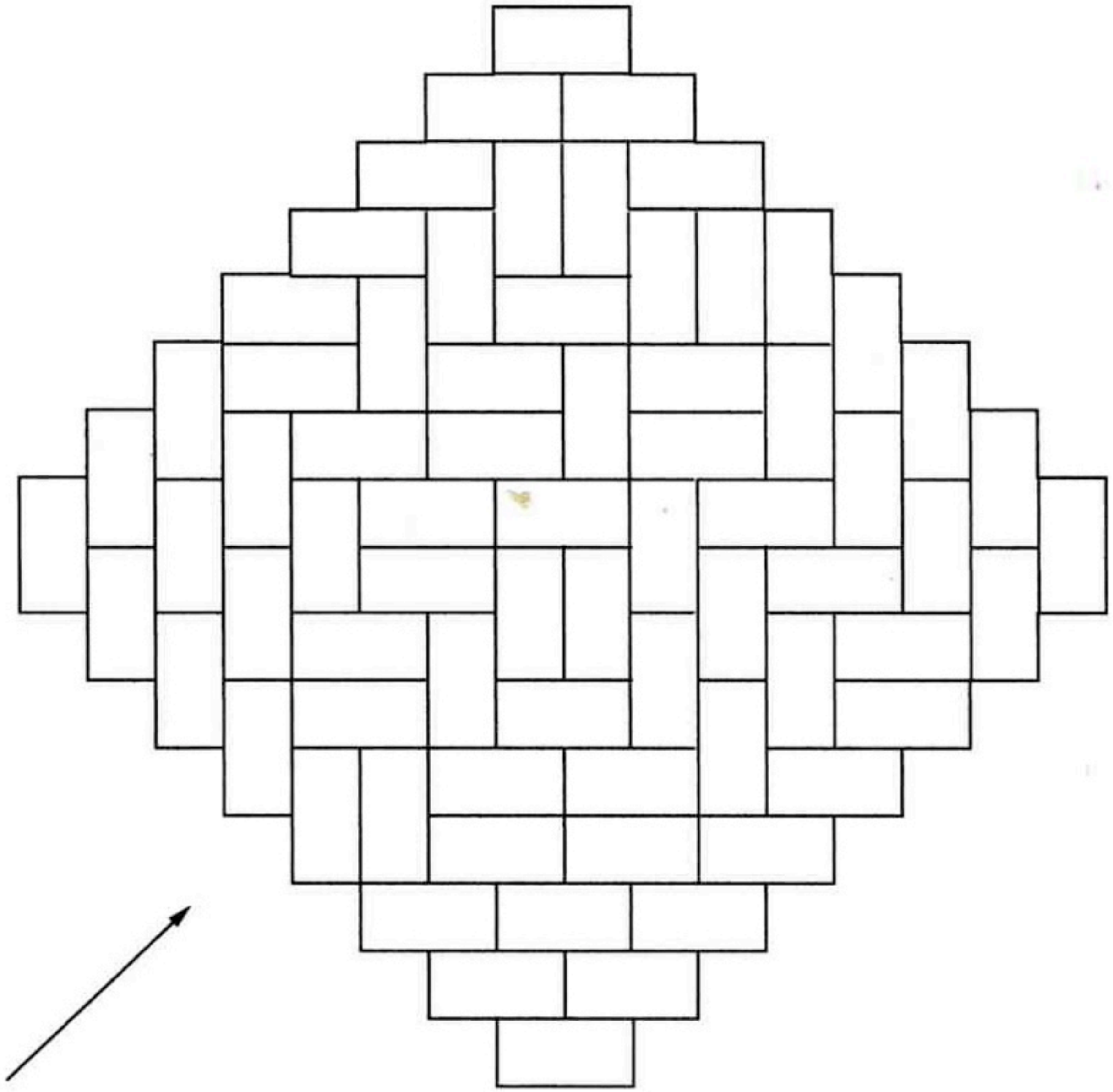
  méthode du Pfaffien

- modèle d'Ising (1925)

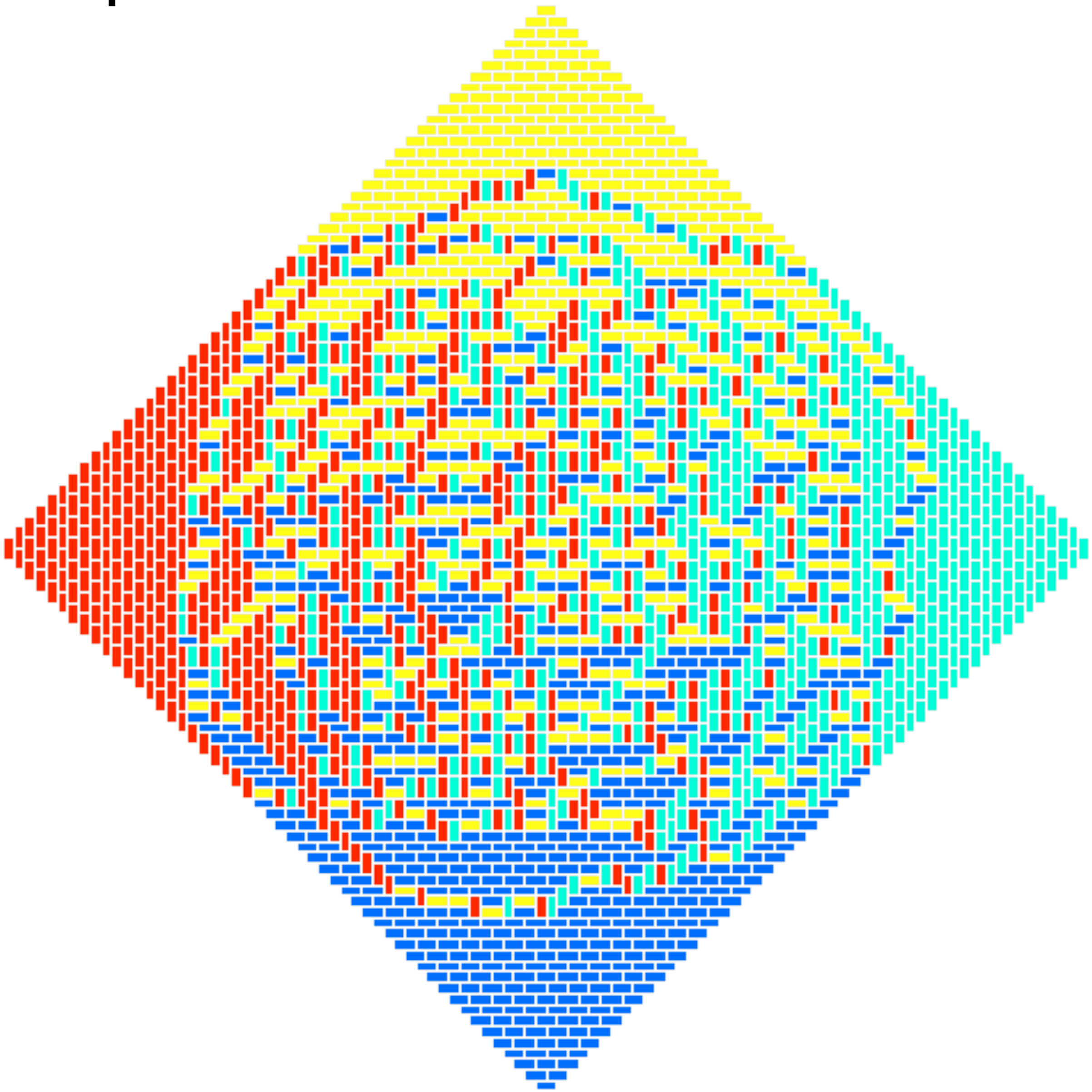
  Kasteleyn, Fisher, Temperley (1961, ...)

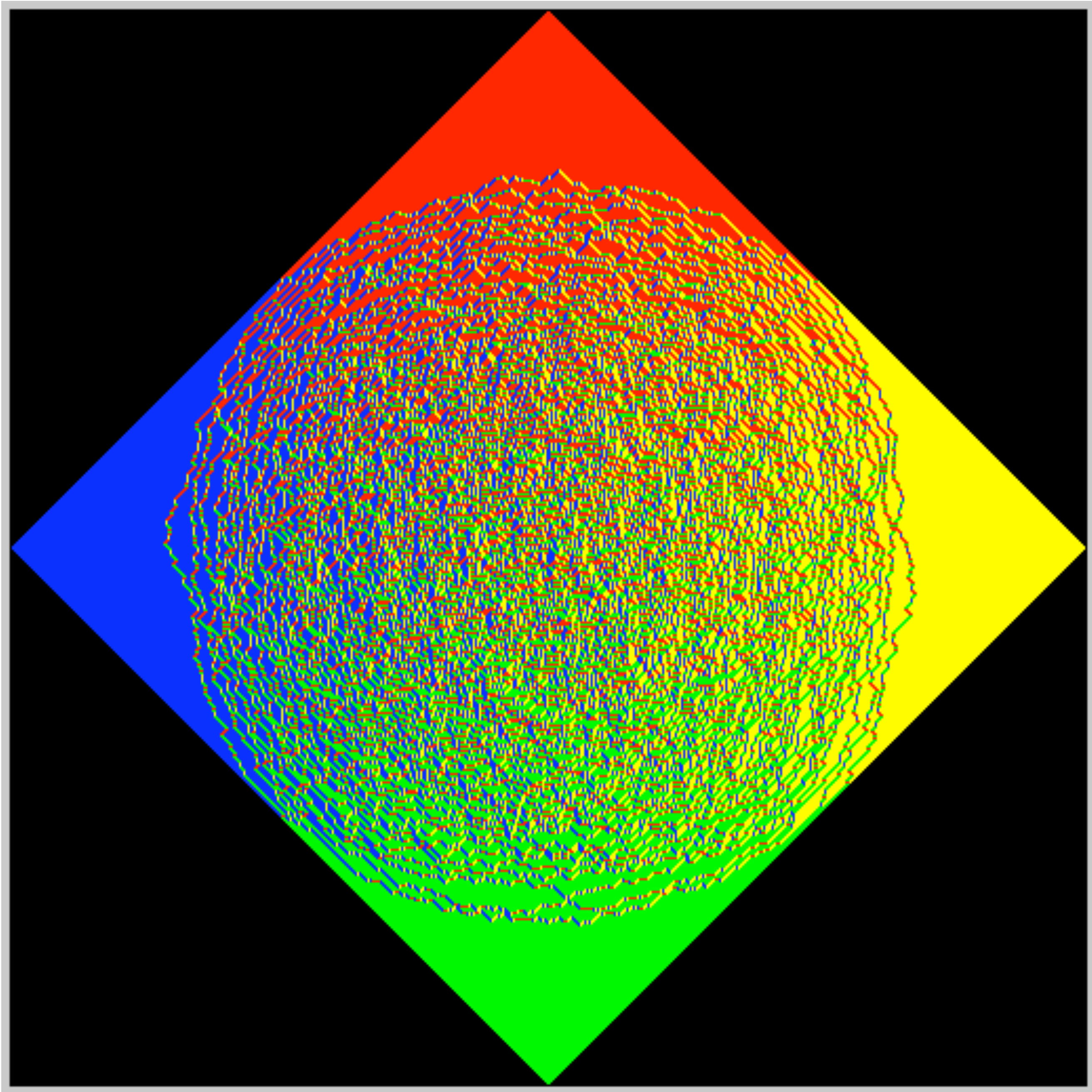
  Onsager (1944)

Aztec tilings



le cercle arctique





§1 LGV

§2 déterminants de Hankel de moments

§3 Déterminants binomiaux

§4 Tableaux de Young

Partitions planes

Partitions planes et chemins

§5 Pavages Aztec

§6 Couplages Pfaffien

§7 Fonctions de Schur

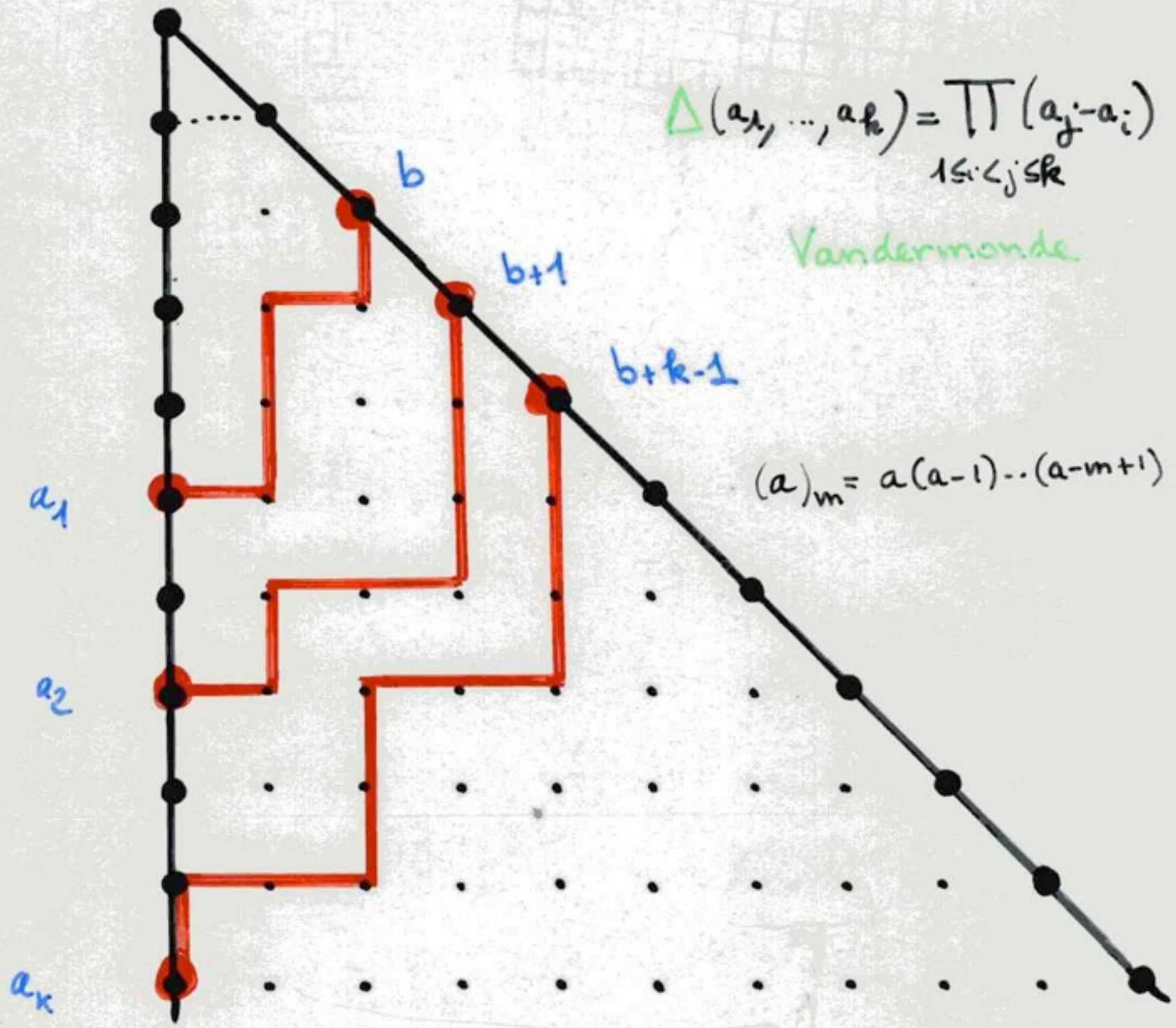
Compléments

Tableaux de Young

RSK Formule des équerres

compléments

$$\binom{a_1, \dots, a_k}{b, b+1, \dots, b+k-1} = \frac{(a_1)_b \cdots (a_k)_b}{b! \cdots (b+k-1)!} \Delta(a_1, \dots, a_k)$$

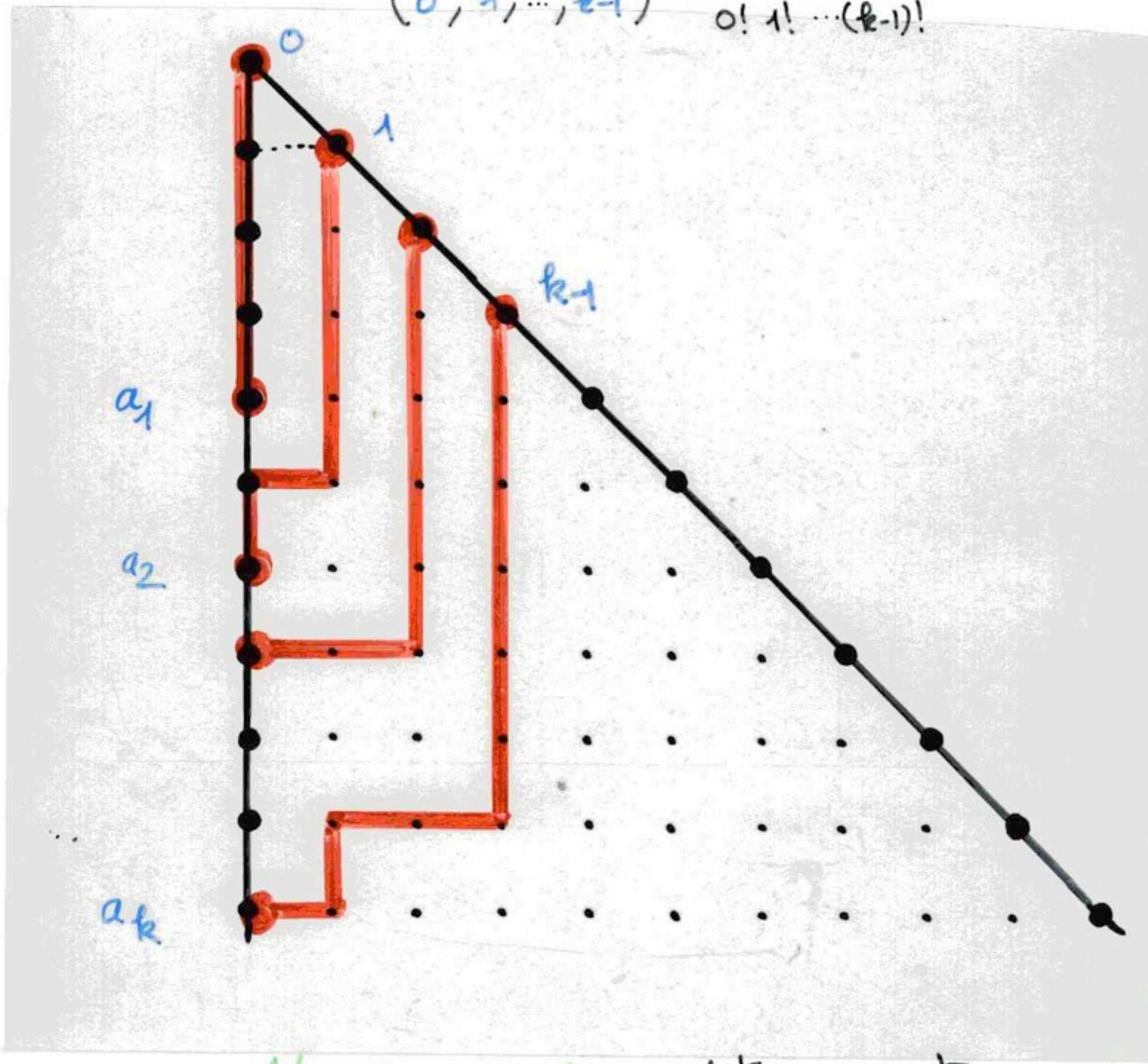


$$\Delta(a_1, \dots, a_k) = \prod_{1 \le i < j \le k} (a_j - a_i)$$

Vandermonde

$$(a)_m = a(a-1) \cdots (a-m+1)$$

$$\binom{a_1, a_2, \dots, a_k}{0, 1, \dots, k-1} = \frac{\Delta(a_1, \dots, a_k)}{0!\, 1! \cdots (k-1)!}$$



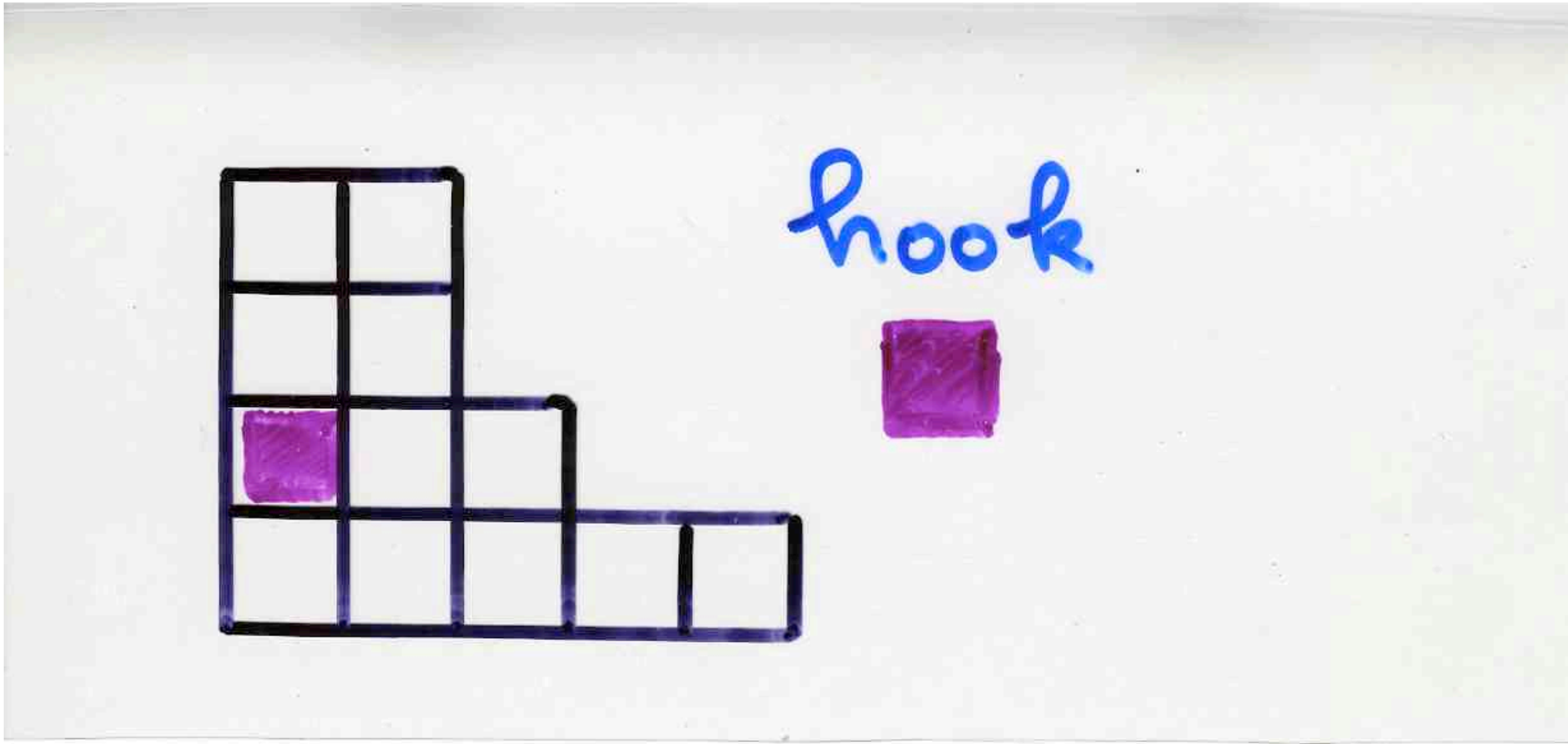
Vandermonde determinant

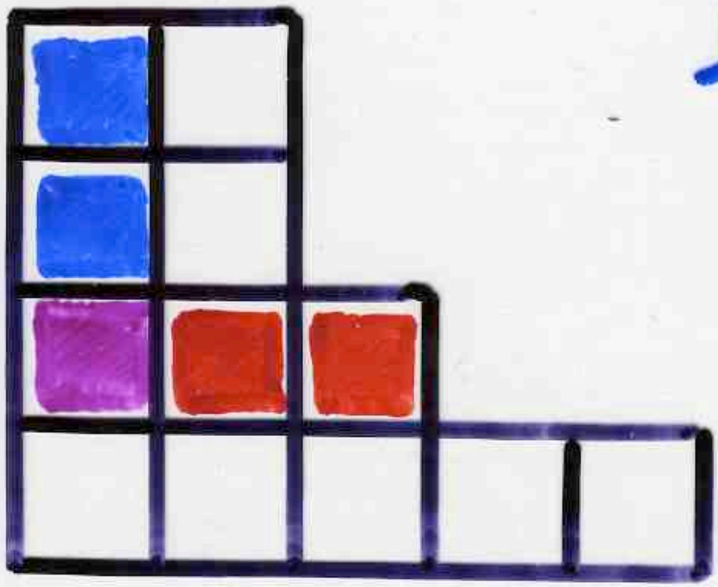
$$\Delta(a_1, \dots, a_k) = \prod_{1 \le i < j \le k} (a_j - a_i)$$

Formule des équerres

J.S. Frame, G. de B. Robinson et R.M. Thrall, 1954

..... Franzblau-Zeilberger, Remmel, Greene-Wilf, Krattenthaler, Novelli- Pak-Stoyanovski, ...


hook



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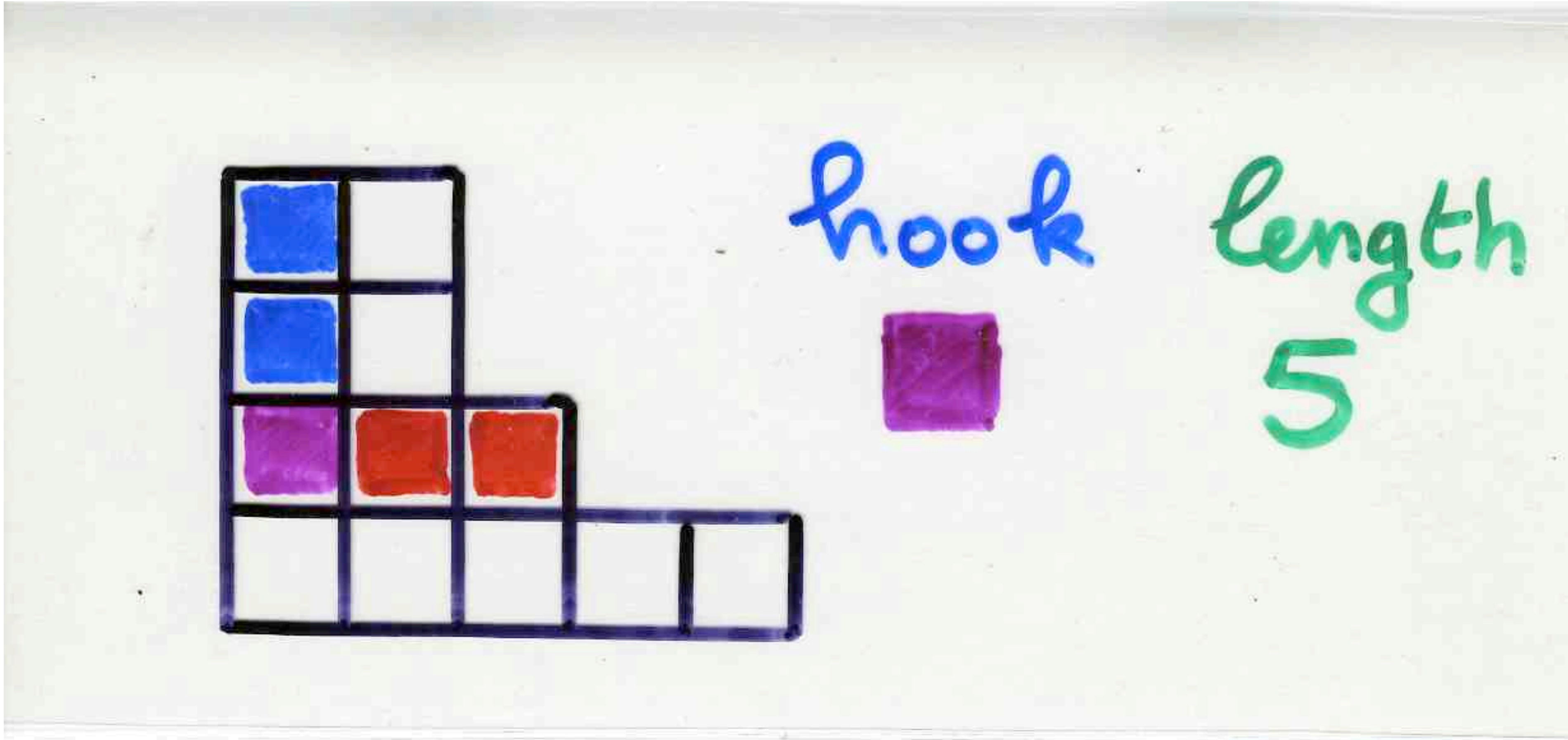
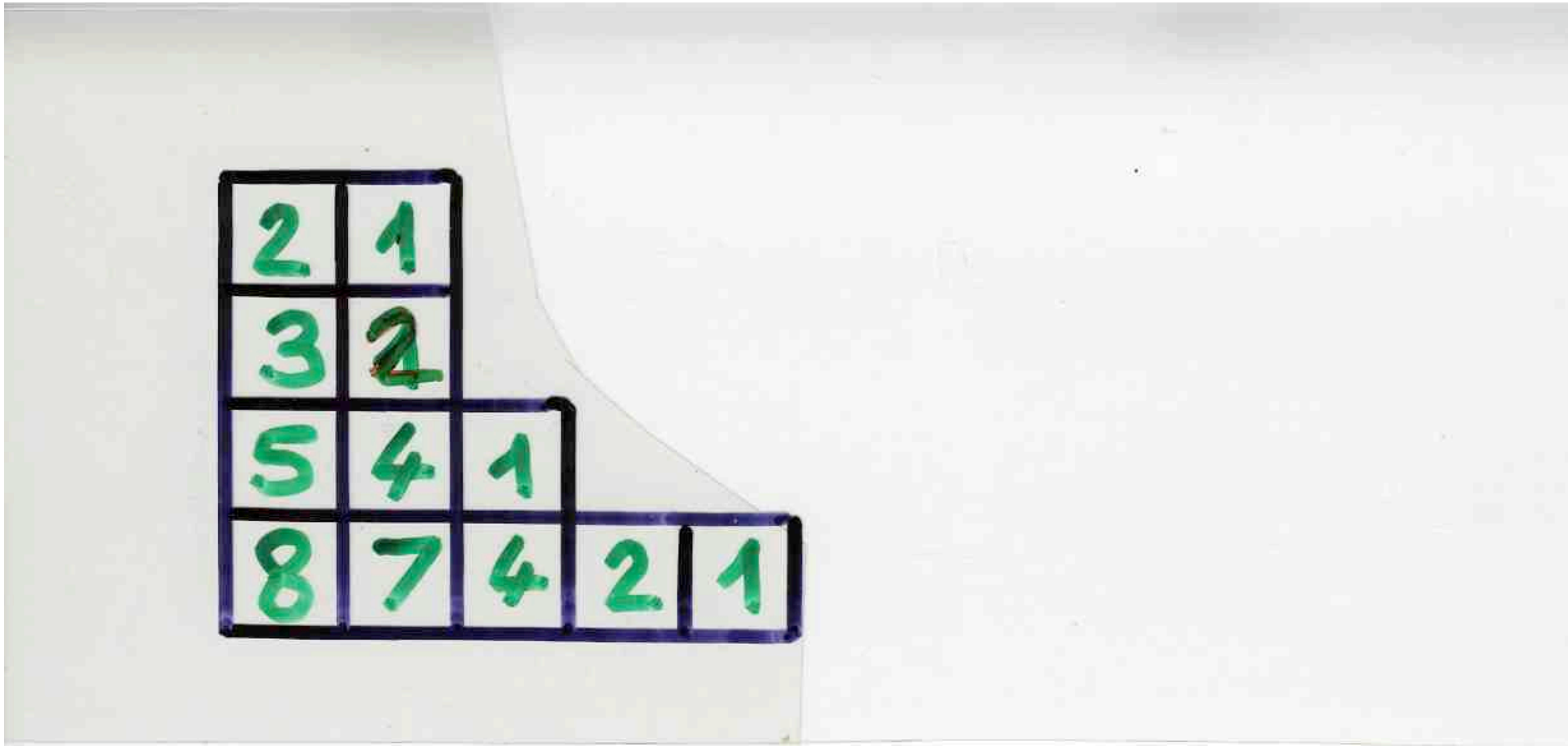
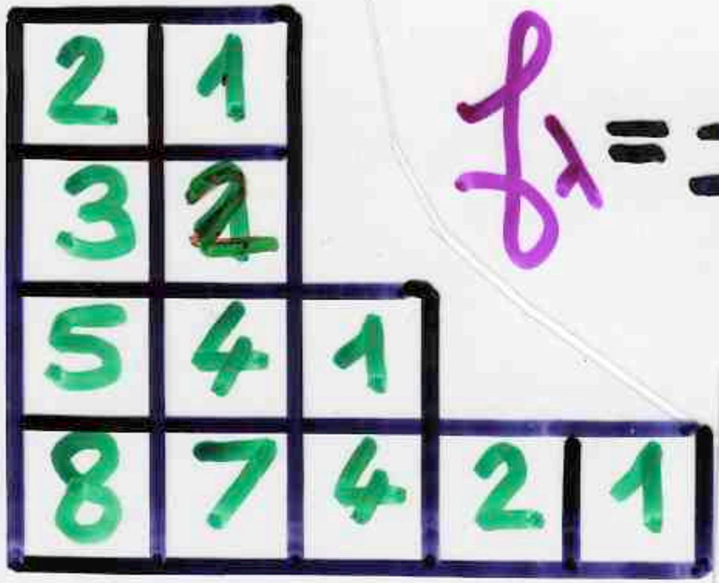
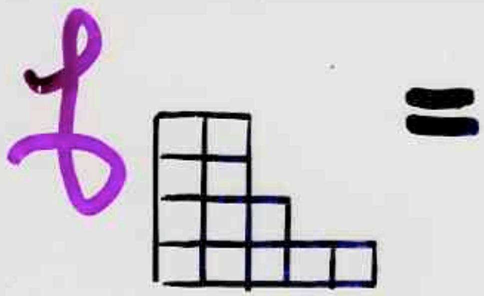

hook length
5

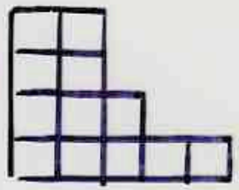

2 1
3 2
5 4 1
8 7 4 2 1

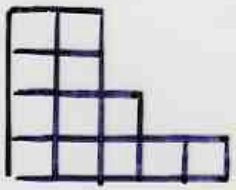


$$f_\lambda = \frac{n!}{\prod_x h_x}$$

hook length formula



f  $= \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{1^3\cdot 2^3\cdot 3\cdot 4^2\cdot 5\cdot 7\cdot 8}$

f  $= \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1^3 \times 2^3 \times 3 \times 4^2 \cdot 5 \cdot 7 \cdot 8}$

$= 3^4 \times 5 \times 11 = 4455$