Algebraic Combinatorics and interactions

The cellular Ansatz

Chapter 4

TASEP and the Catalan garden

(Totally ASymmetric Exclusion Process)

Ch4b Catalan alternative tableaux

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Febrary 1, 2013

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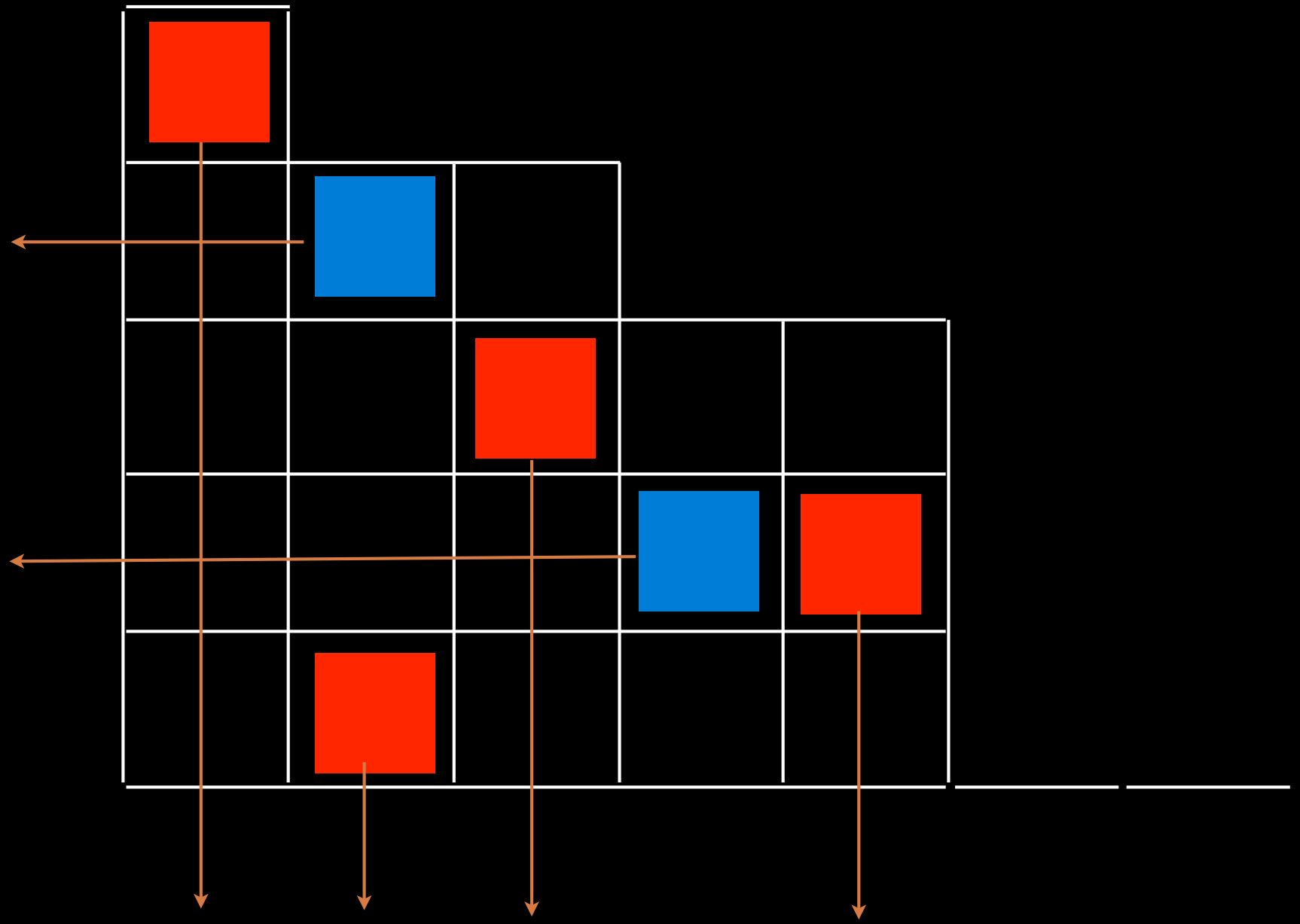
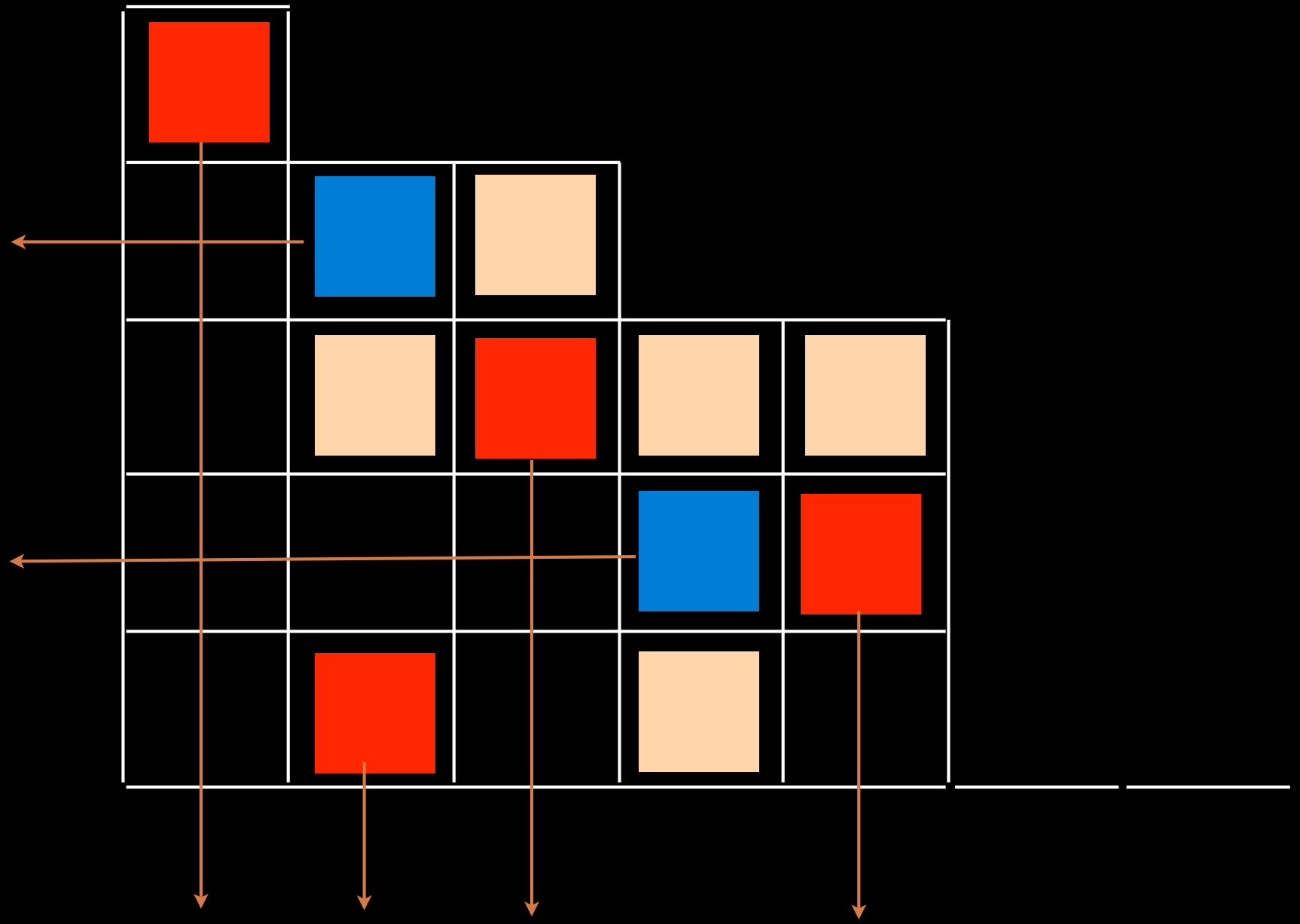
Catalan alternative tableaux

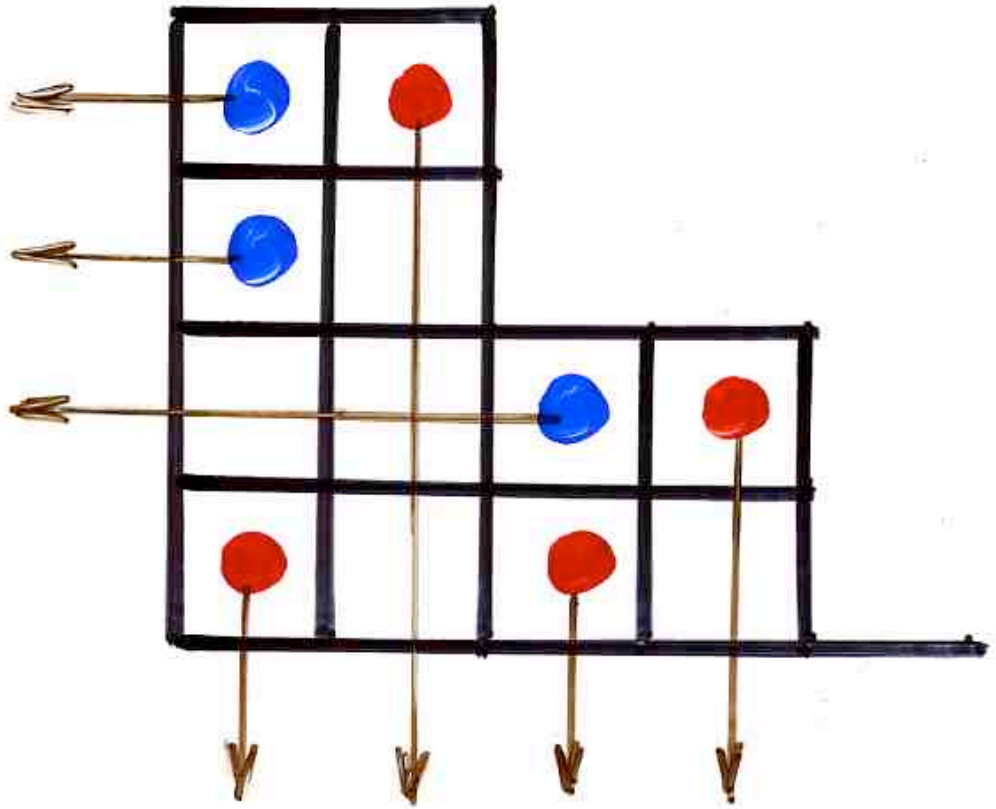


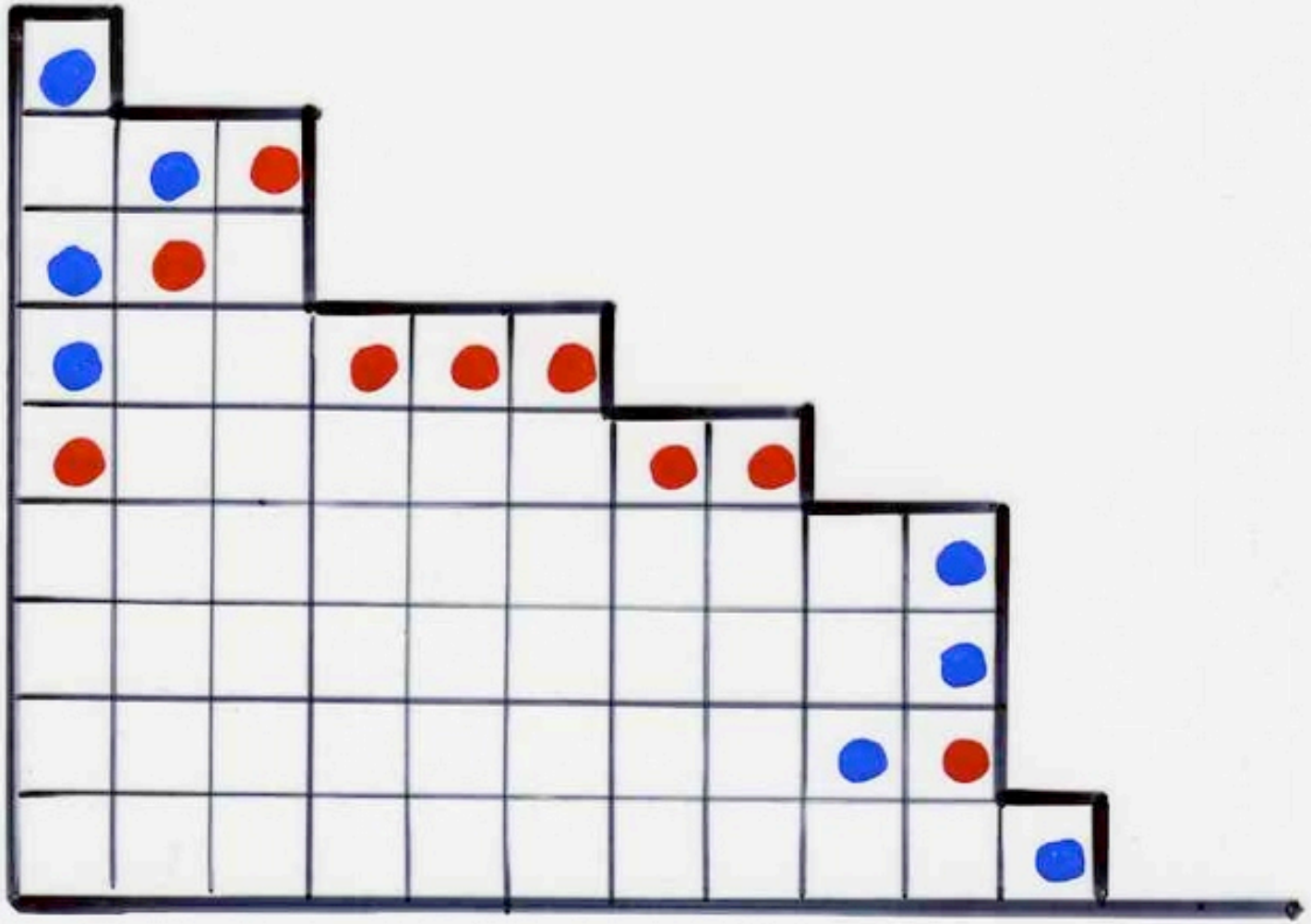


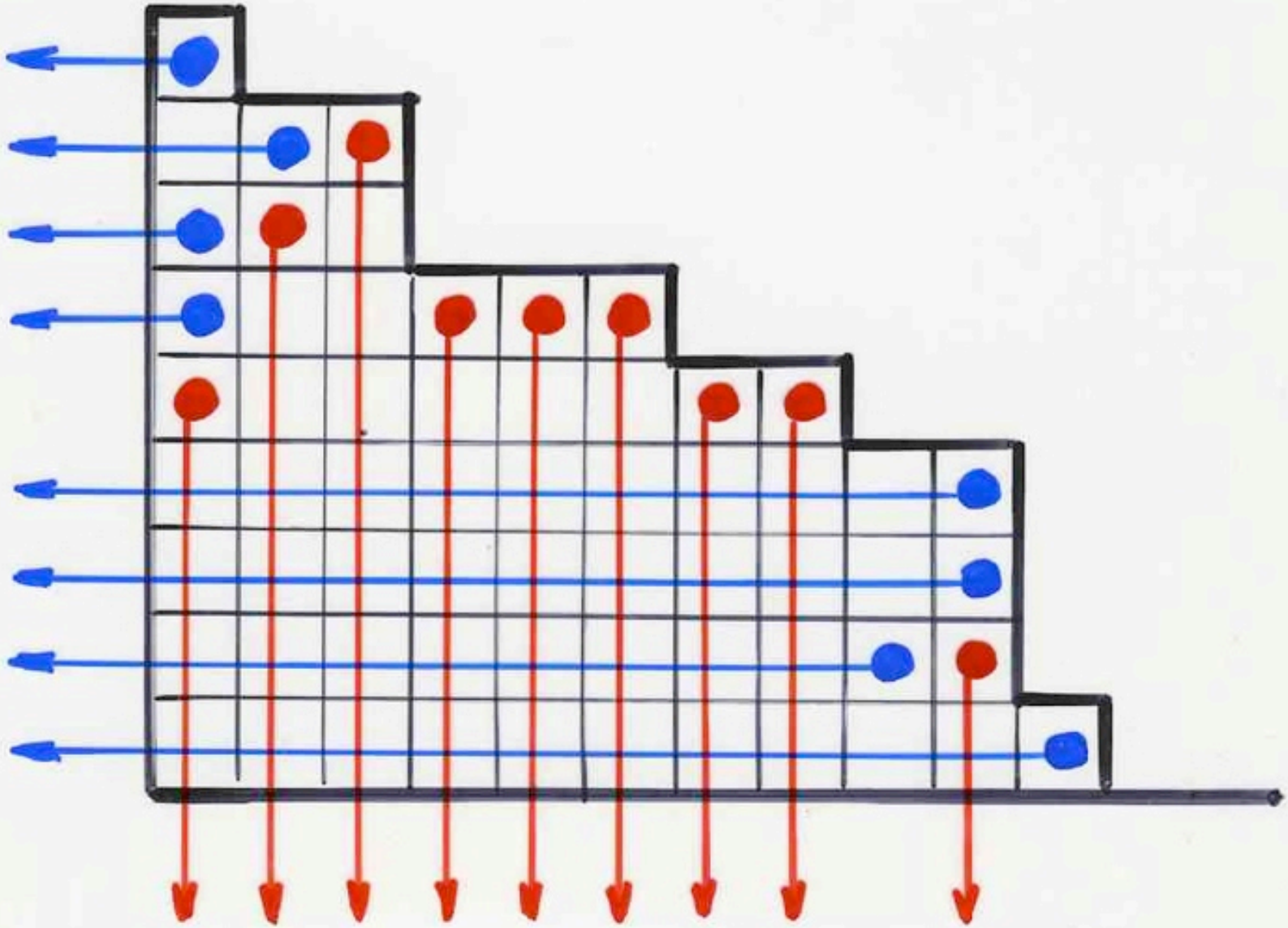
Def Catalan alternative tableau T

alt. tab. without cells ⊠

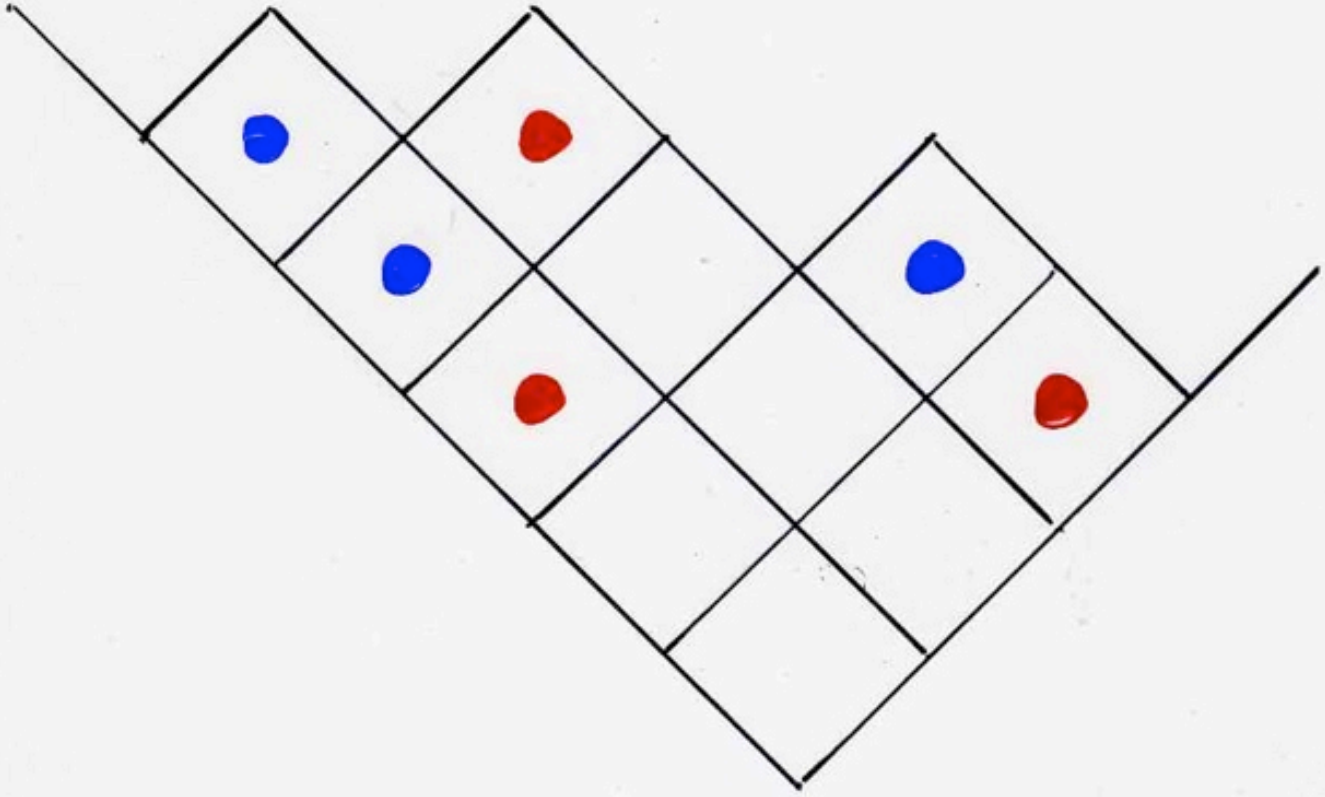
i.e: every empty cell is below a red cell or on the left of a blue cell

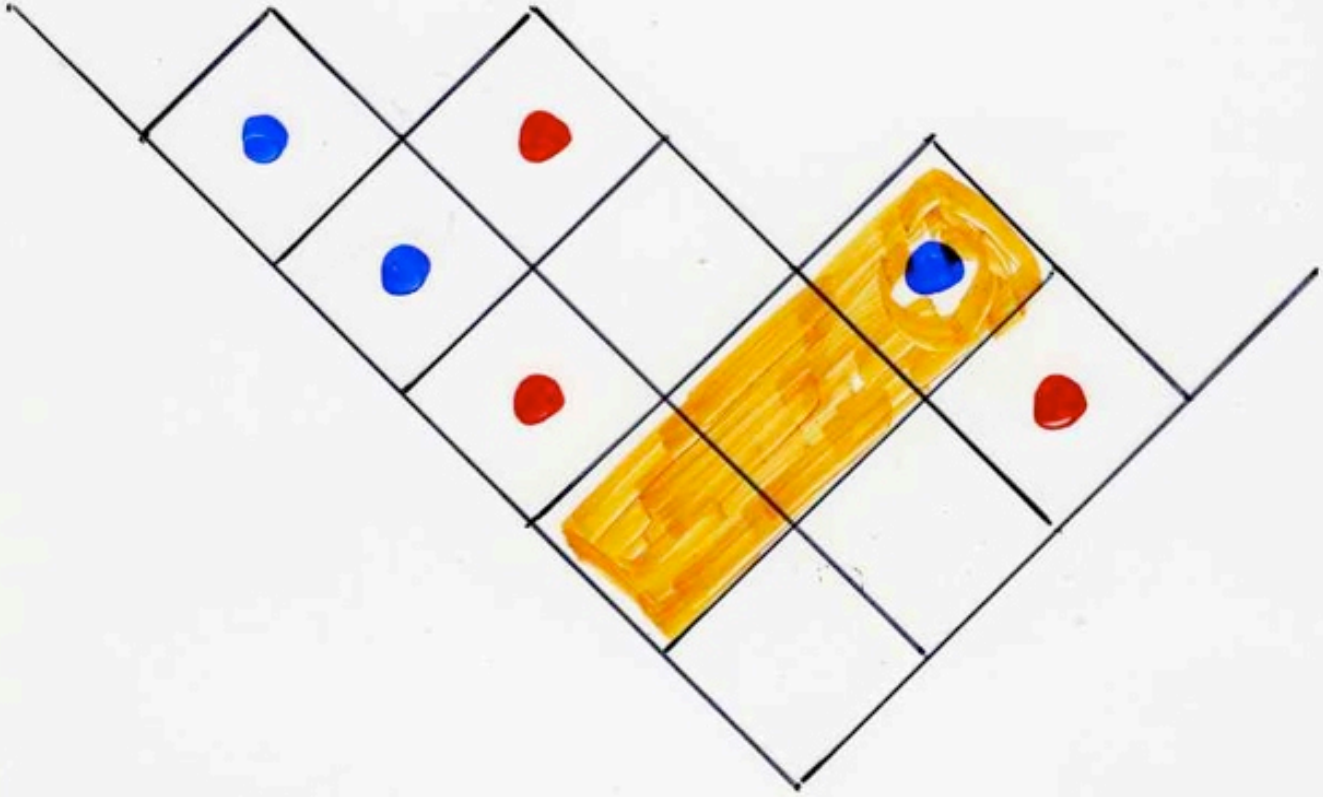


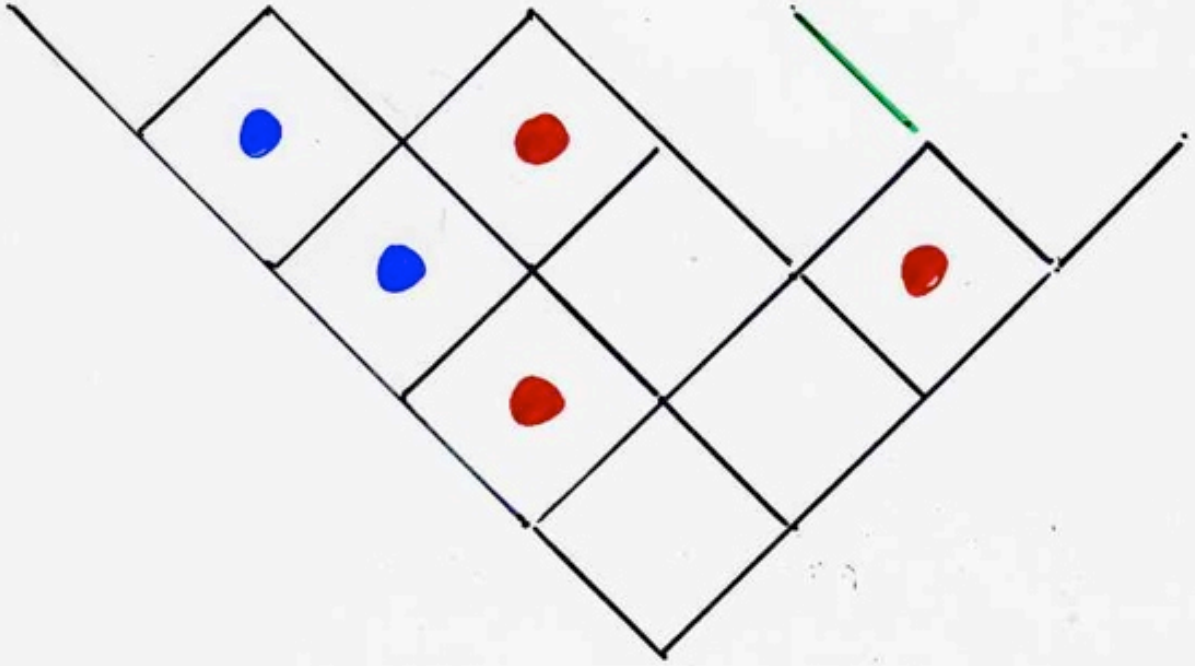


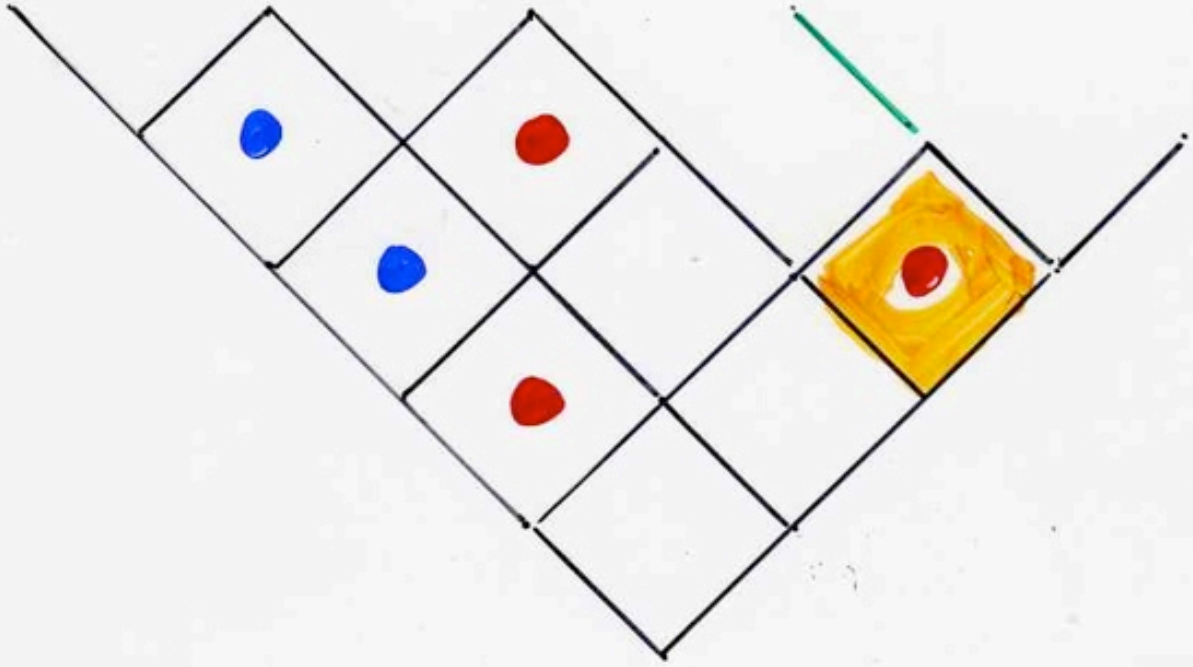


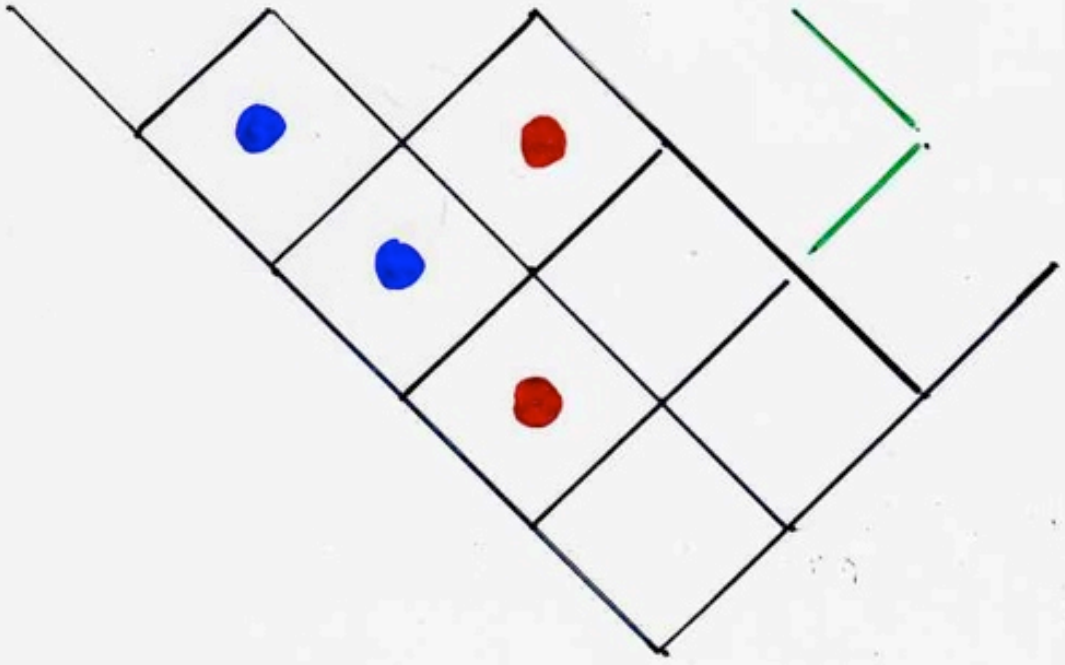
Bijection
alternative Catalan tableaux
binary trees

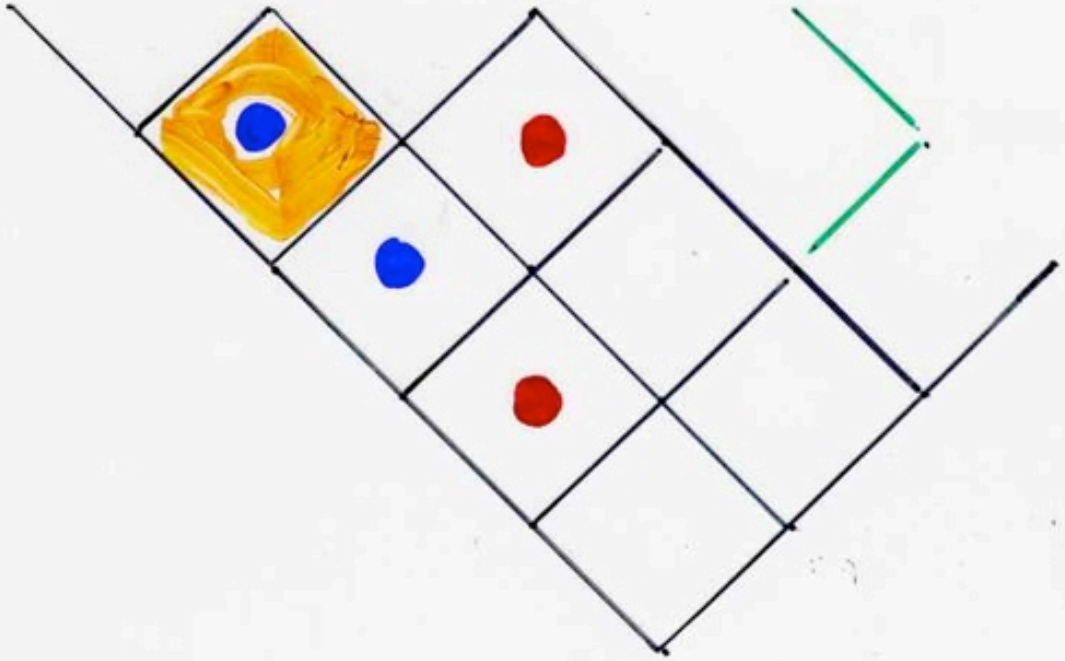


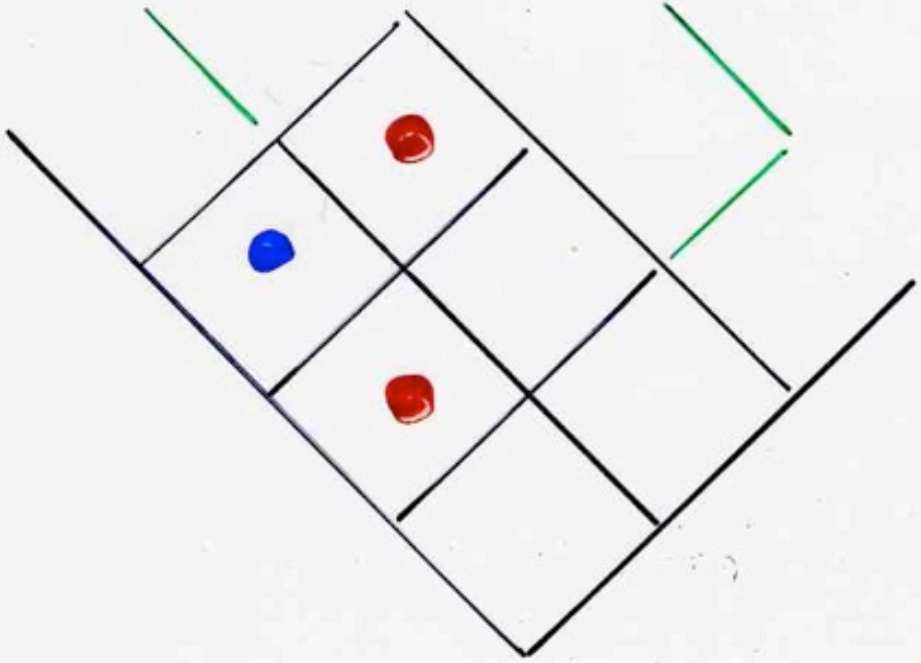


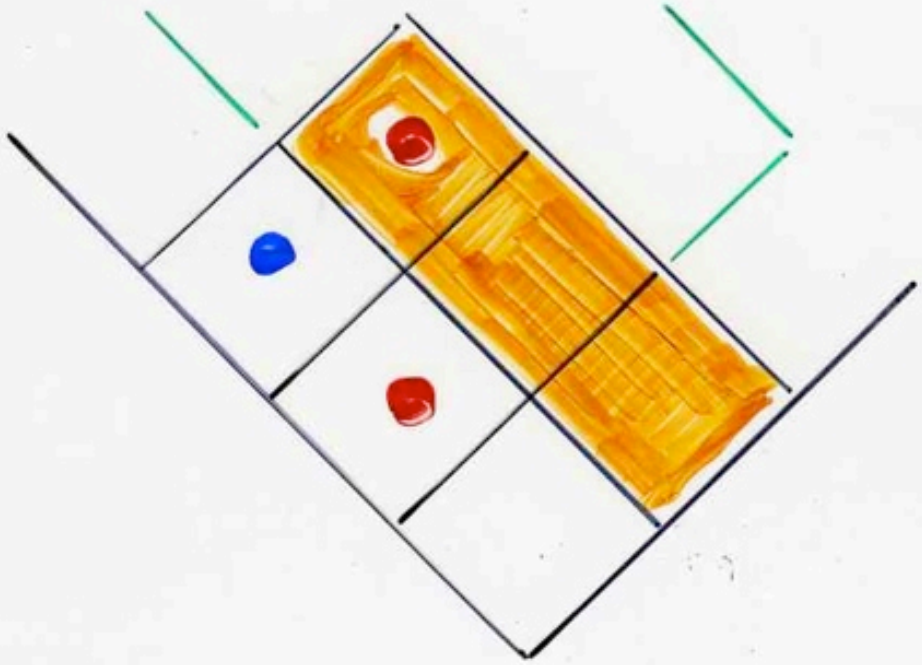


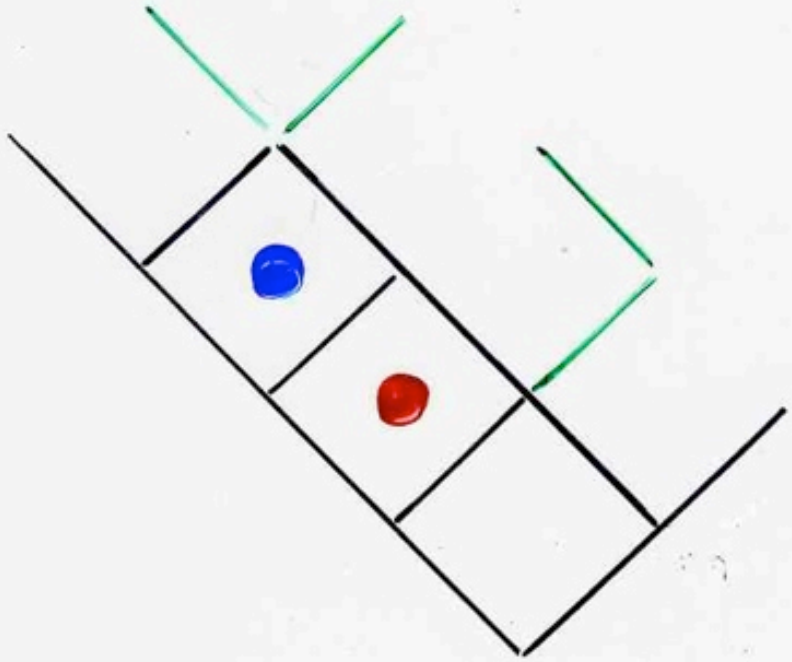


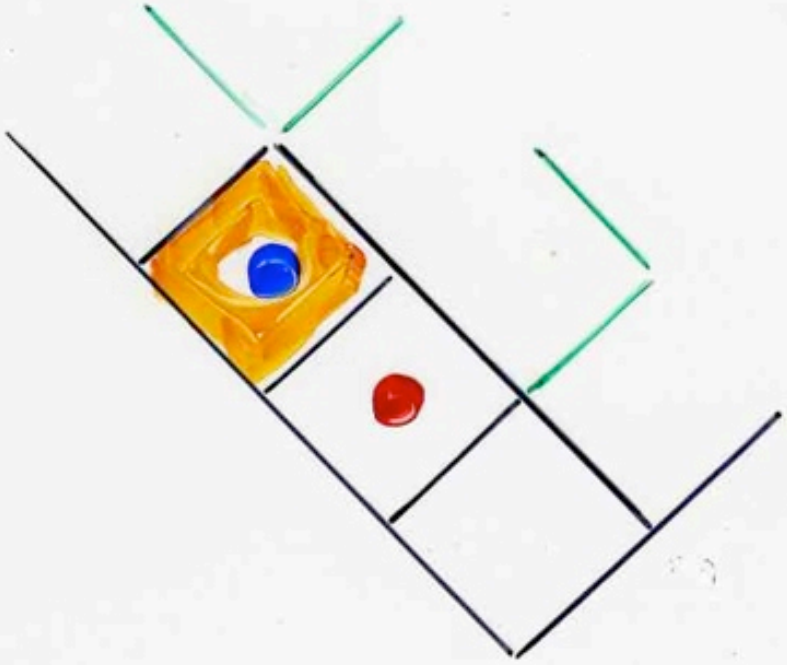


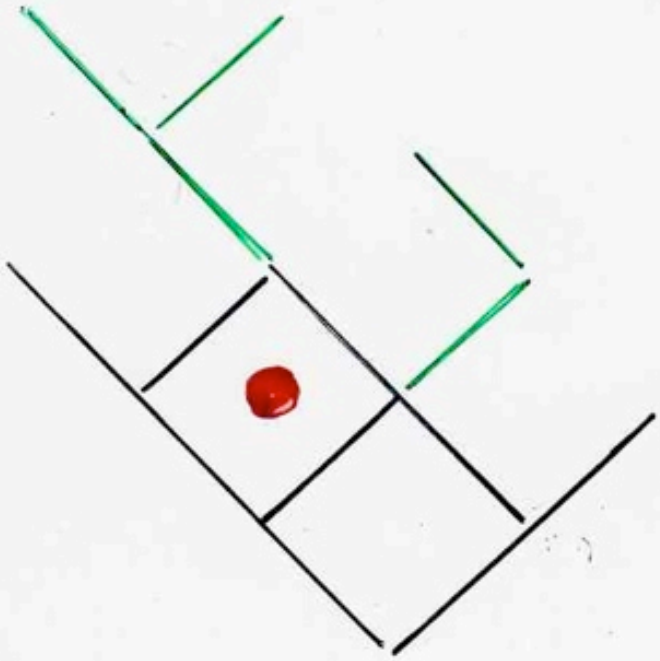


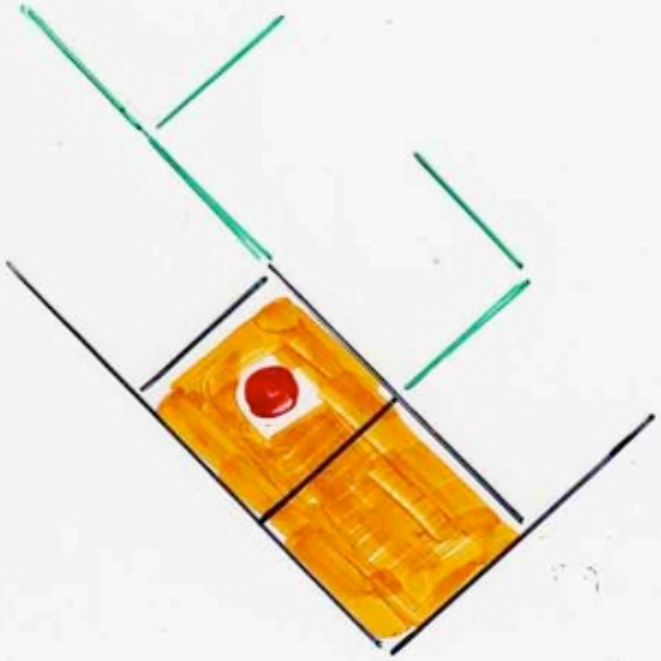


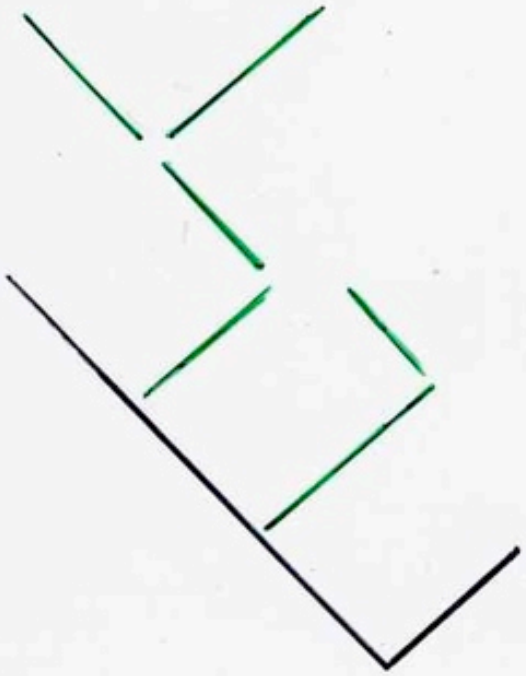






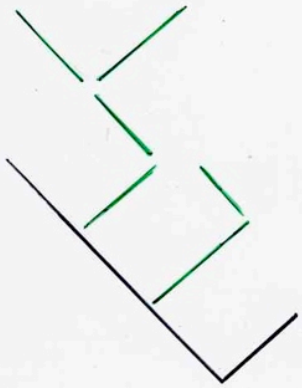


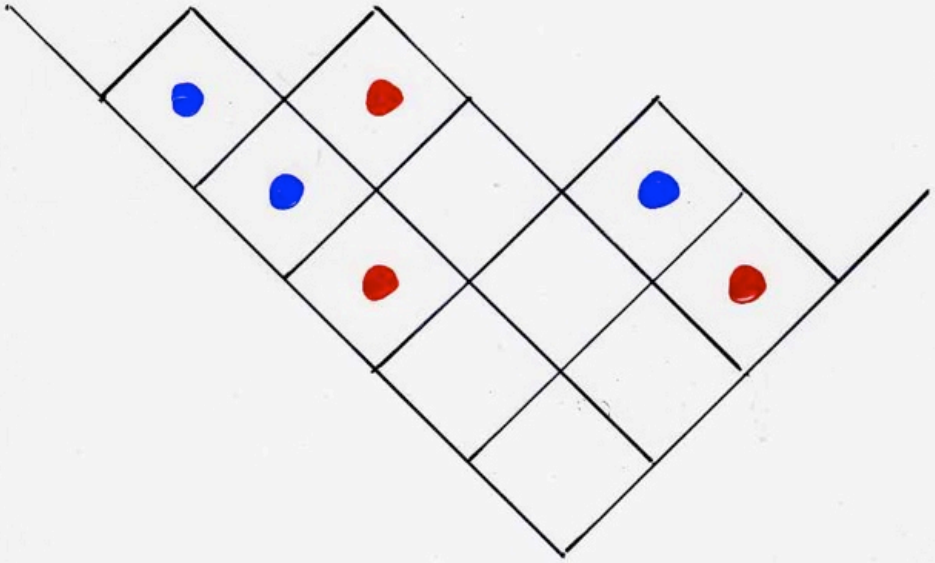




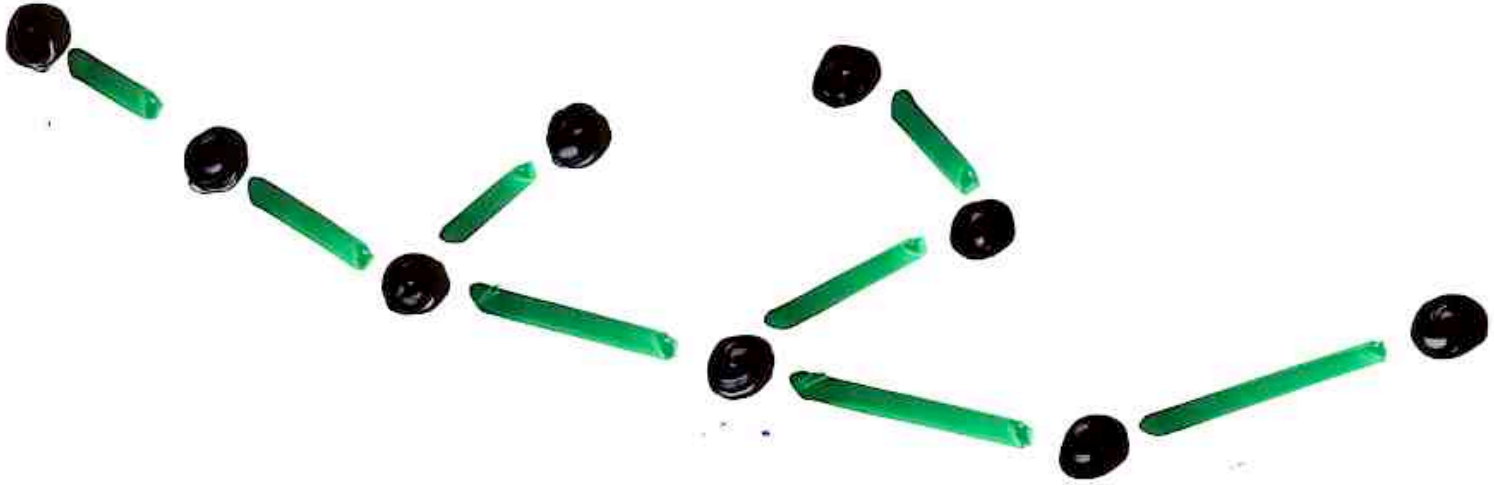
Bijection

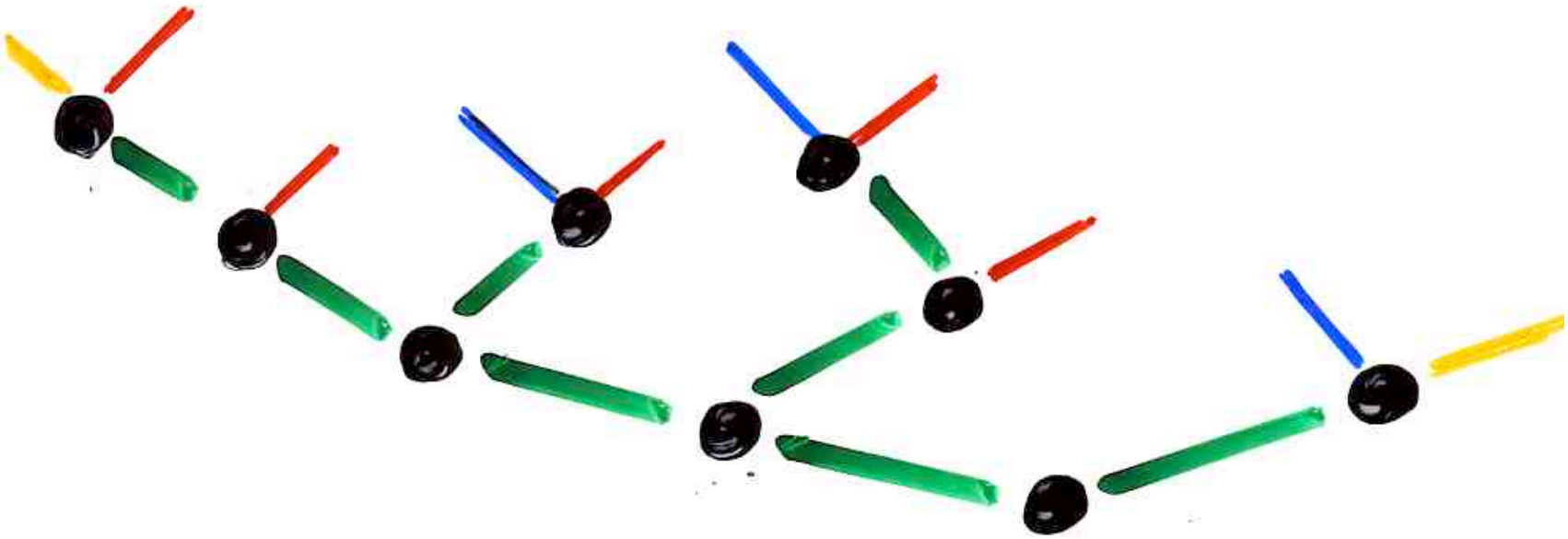
tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes





profil (bord) du diagramme de Ferrers ⟷ canopée





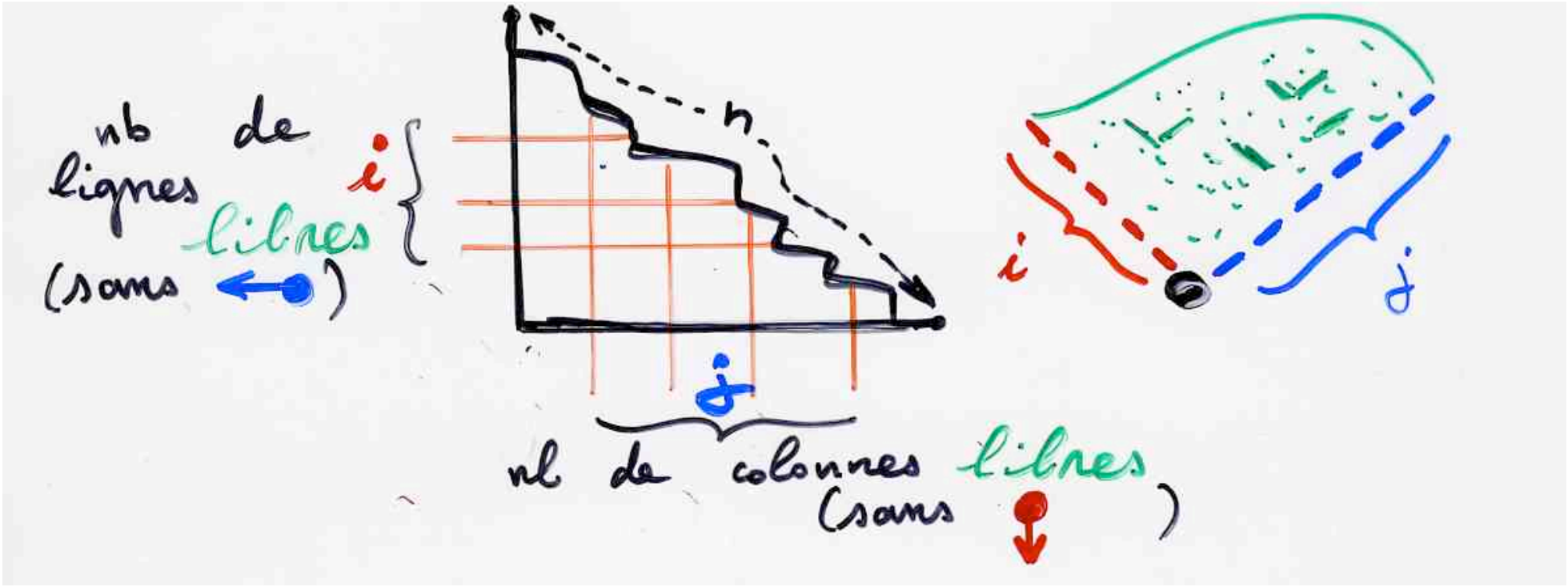
canopy of a binary tree

$c(B) = - - + - + - - +$

Bijection

tableaux alternatifs de Catalan taille n ⟷ arbres binaires n arêtes

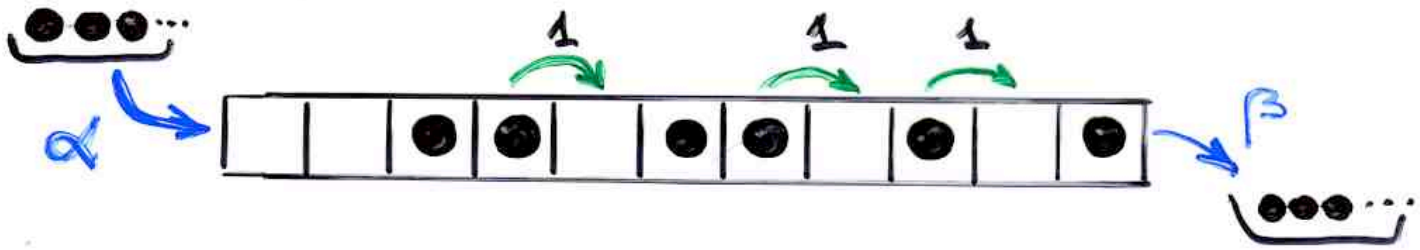
profil (bord) du diagramme de Ferrers ⟷ corrompée



stationary probabilities for TASEP

TASEP

"totally asymmetric exclusion process"



Combinatorics of the TASEP

Shapiro, Zeilberger, 1982

TASEP

Brak, Essam (2003), Duchi, Schaeffer, (2004),
Angel (2005), X.G.V., (2007)

$$P_n(\tau_1, \cdots, \tau_n) = f_n(\tau_1, \cdots, \tau_n) / Z_n$$

$$Z_n = \sum_{\tau} f_n(\tau_1, \cdots, \tau_n)$$

partition function

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

V column vector, W row vector

$$\begin{cases} DE = qED + D + E \\ (\beta D - \delta E)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \gamma D) = \langle W| \end{cases}$$

Then $f_n(\tau_1, \dots, \tau_n)$

Derrida, Evans, Hakim, Pasquier (1993)

"matrix ansatz"

D E matrices,

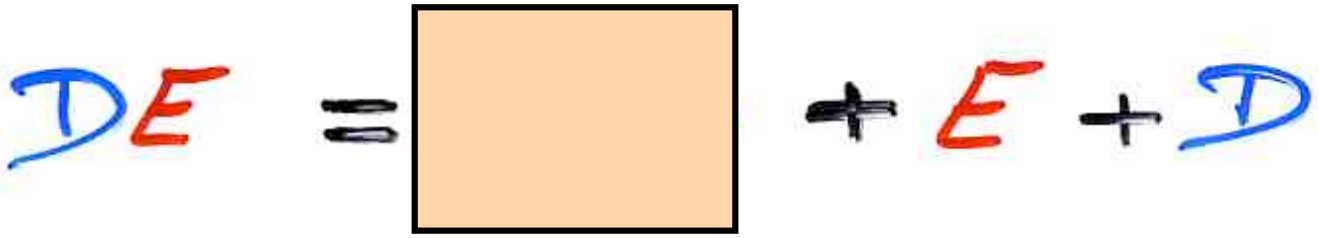
V column vector, W row vector

TASEP

$q=0$

$$\begin{cases} DE = \square + D + E \\ (\beta D - \square)|V\rangle = |V\rangle \\ \langle W|(\alpha E - \square) = \langle W| \end{cases}$$

Then $$f_n(\tau_1, \dots, \tau_n) = \langle W | \prod_{i=1}^{n} (\tau_i D + (1-\tau_i) E) | V \rangle$$



$$DE = \boxed{} + E + D$$

$$w(E,D) = \sum_{T} \boxed{}\, E^{i(T)} D^{j(T)}$$

Catalan T alternative tableau with profile w

~~k(T) = nb of~~

i(T) = nb of rows without blue cell

j(T) = nb of columns without red cell

$$q = 0$$

$$\begin{cases} DE = \square + D + E \\ DV = \bar{\beta} V & \bar{\beta} = 1/\beta \\ WE = \bar{\alpha} W & \bar{\alpha} = 1/\alpha \end{cases}$$

$q = 0$

$$WE^i D^j V = \bar{\alpha}^i \bar{\beta}^j \underbrace{WV}_{1}$$

Cor. The stationary probability associated to the state $\tau = (\tau_1, \dots, \tau_n)$ (PASEP) is

$$\text{proba}_\tau(q; \alpha, \beta) = \frac{1}{Z_n} \sum_{T} \square \, \alpha^{-f(T)} \beta^{-u(T)}$$

Catalan alternative tableaux profile τ

$\begin{cases} f(T) \\ u(T) \\ L(T) \text{ nb of cells} \end{cases}$ nb of $\begin{pmatrix} \text{rows} \\ \text{columns} \end{pmatrix}$ without $\begin{pmatrix} \bullet \\ \bullet \end{pmatrix}$ cell $\square$

$$s = (\tau_1, \ldots, \tau_n)$$

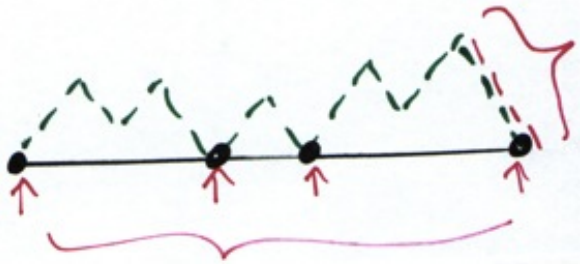
$$P_n(s; \alpha, \beta) = \frac{1}{Z_n} \sum_{B} \bar{\alpha}^{lb(B)} \bar{\beta}^{rb(B)}$$

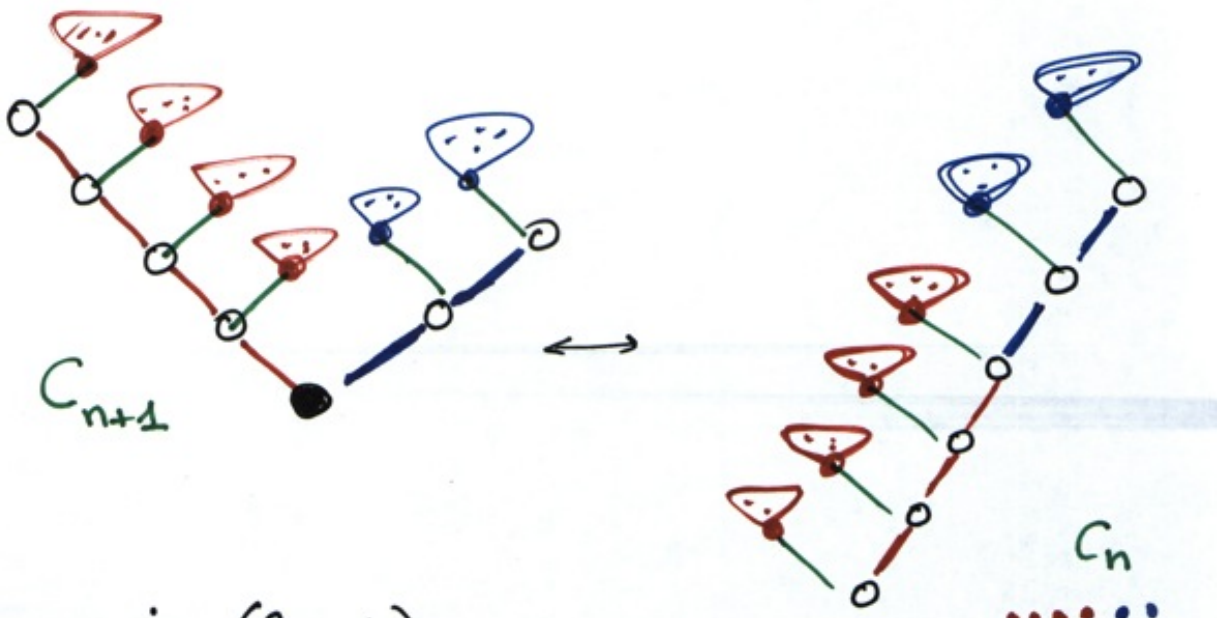
binary trees
canopy s

$$Z_n = \sum_{i=1}^{n} \frac{i}{2n-i} \binom{2n-i}{n} \frac{\bar{\alpha}^{(i+1)} - \bar{\beta}^{(i+1)}}{\bar{\alpha} - \bar{\beta}}$$

partition function

"ballot" numbers





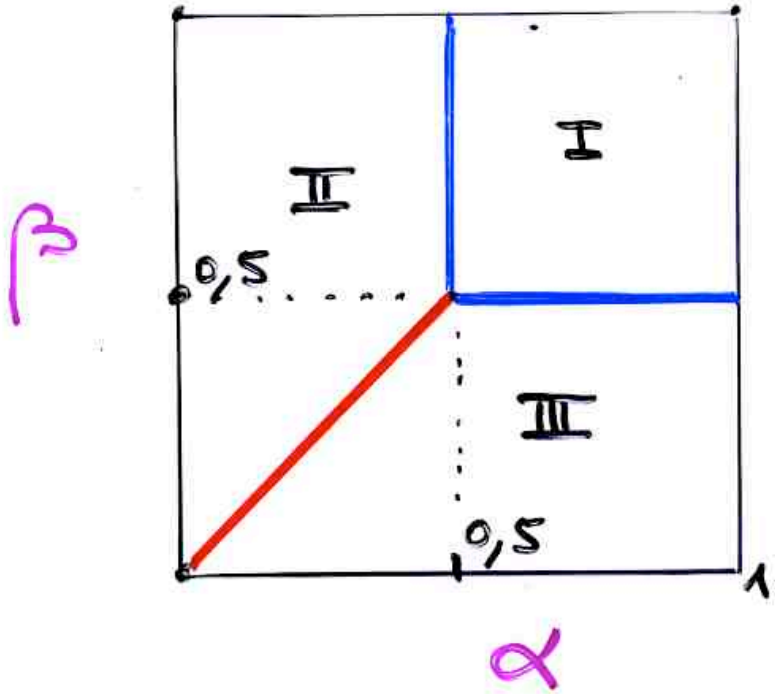
$$\frac{i}{2n-i}\binom{2n-i}{n}$$

$$\left[\bar{\alpha}^{(i)} + \bar{\alpha}^{(i-1)}\bar{\beta} + \ldots + \bar{\alpha}\,\bar{\beta}^{(i-1)}\bar{\beta}^{(i)}\right]$$

$$\sum_n Z_n t^n = \frac{1}{(1-\bar{\alpha} f(t))} \times \frac{1}{(1-\bar{\beta} f(t))}$$

$f(t) = t$ (generating function of Catalan numbers)

$$= \frac{1-\sqrt{1-4t}}{2}$$



$n \to \infty$

$\rho = \langle \tau_i \rangle$ = taux moyen d'occupation

i loin des bords

(I) $\rho = 1/2$

(II) $\rho = \alpha$

(III) $\rho = 1 - \beta$

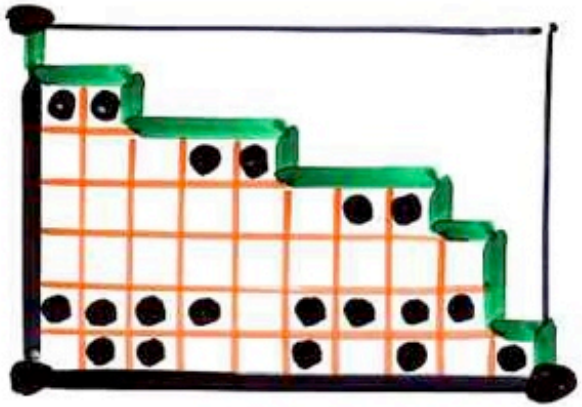
$$\langle \tau_i \rangle_N = \frac{\langle W | (D+E)^{i-1} D (D+E)^{N-i} | V \rangle}{\langle W | (D+E)^N | V \rangle}$$

$$\begin{cases} DE = D + E \\ D|V\rangle = \beta |V\rangle \\ \langle W|E = \alpha \langle W| \end{cases}$$

Catalan permutation tableaux

Permutation Tableau

Ferrers diagram $F \subseteq k \times (n-k)$ rectangle



filling of the cells with 0 and 1

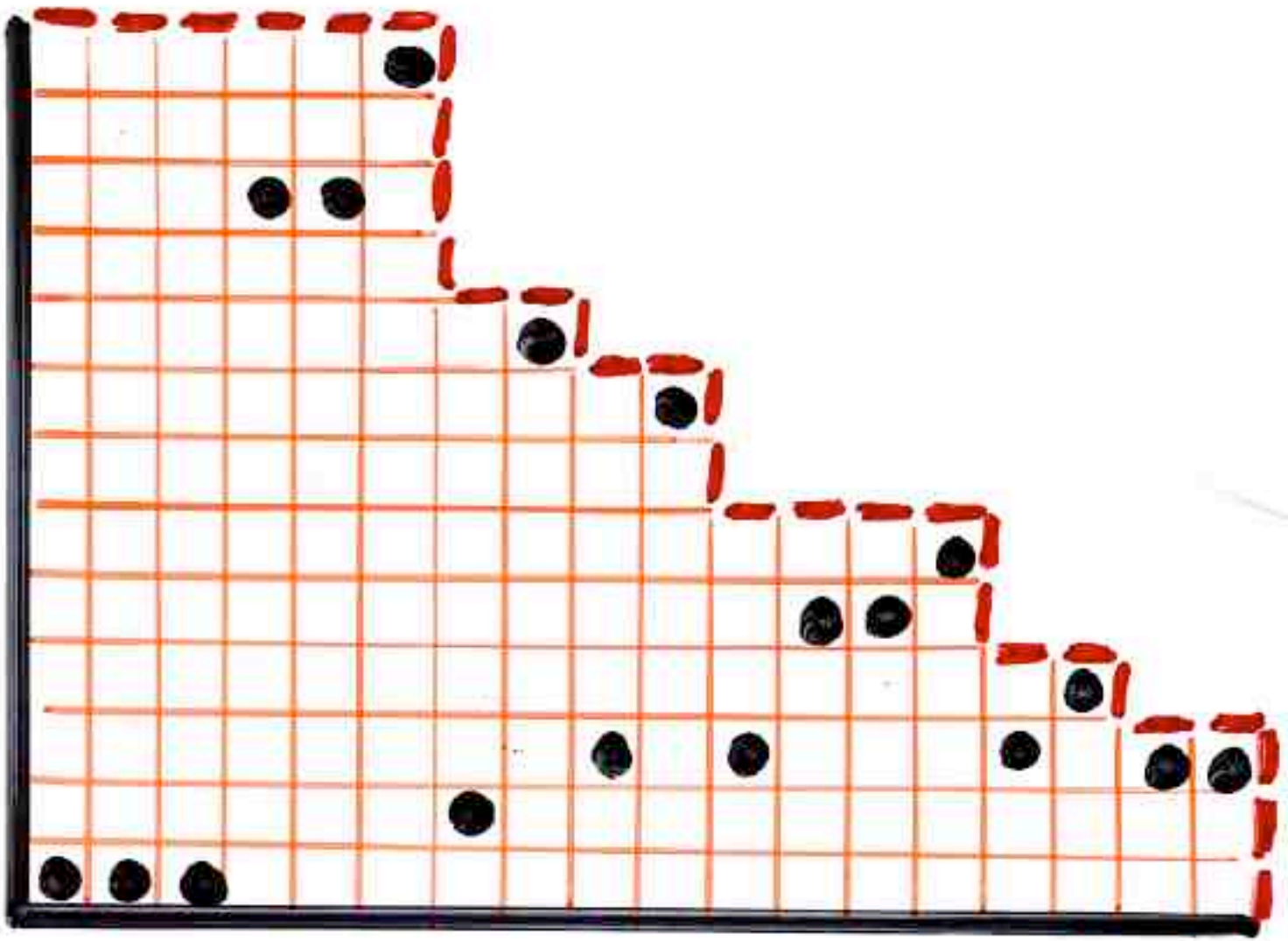
(i) in each column : at least one 1

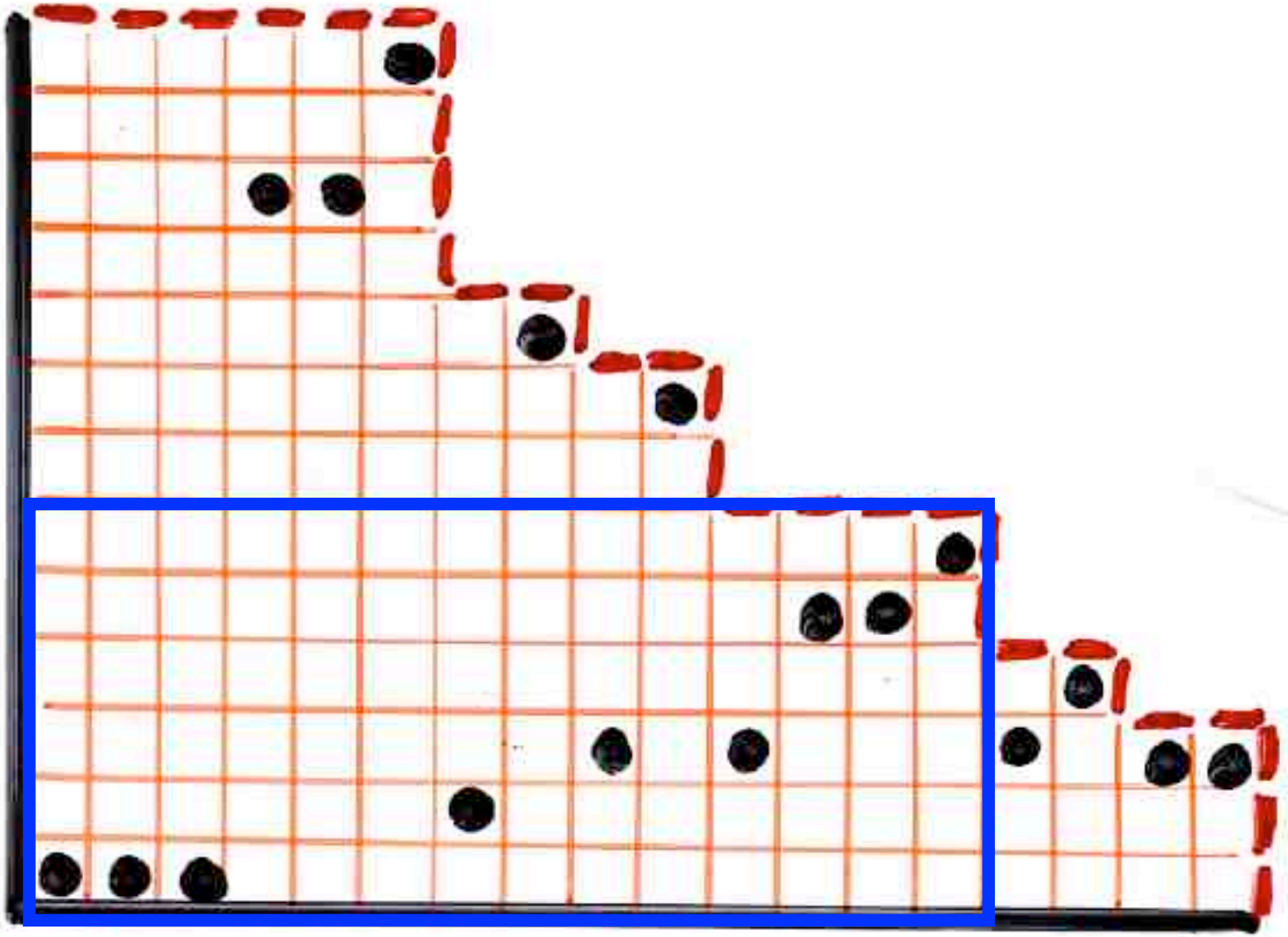
□ = 0 ■ = 1

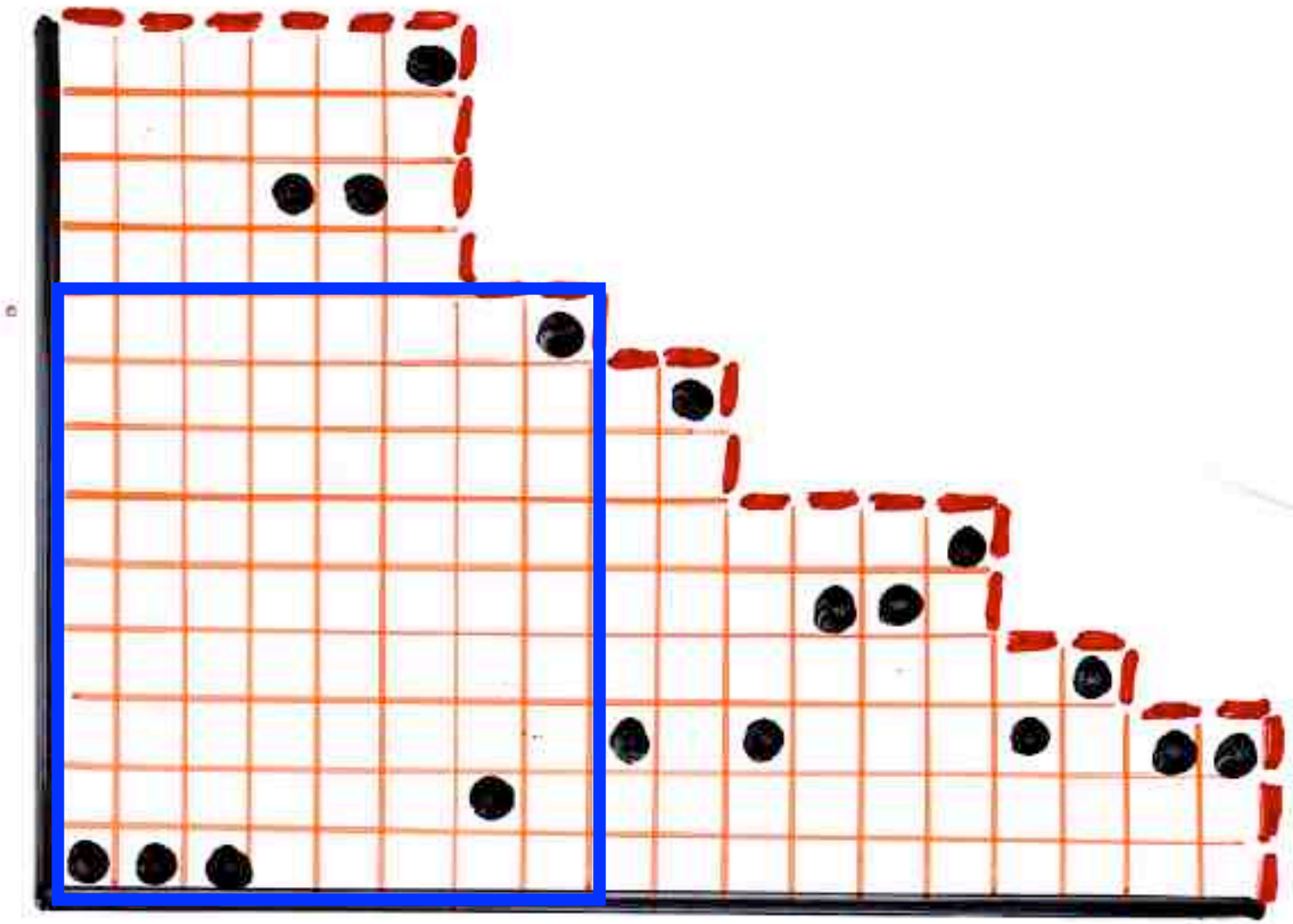
(ii) forbidden

Catalan permutation tableau

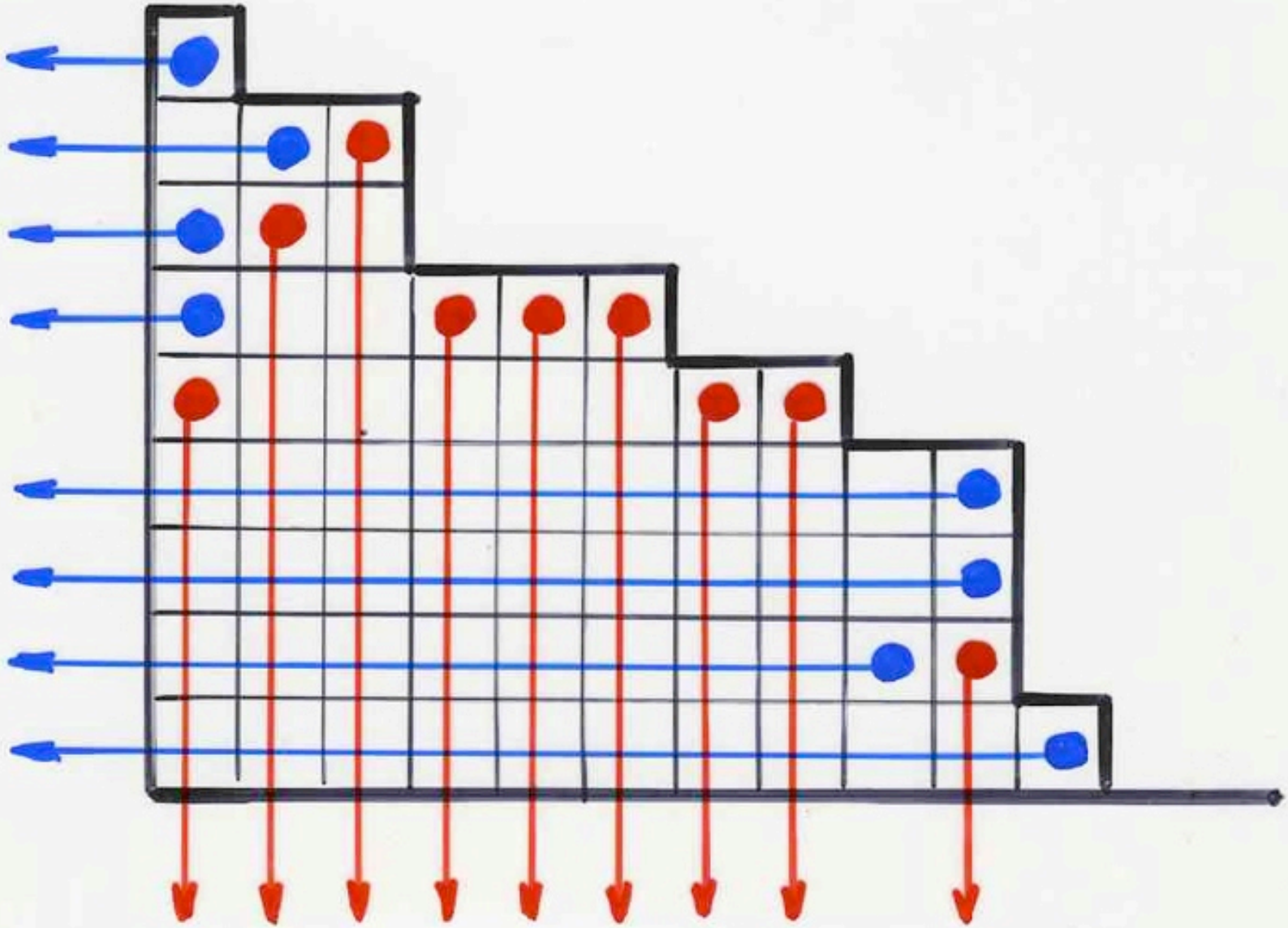
(iii) only one 1 in each column

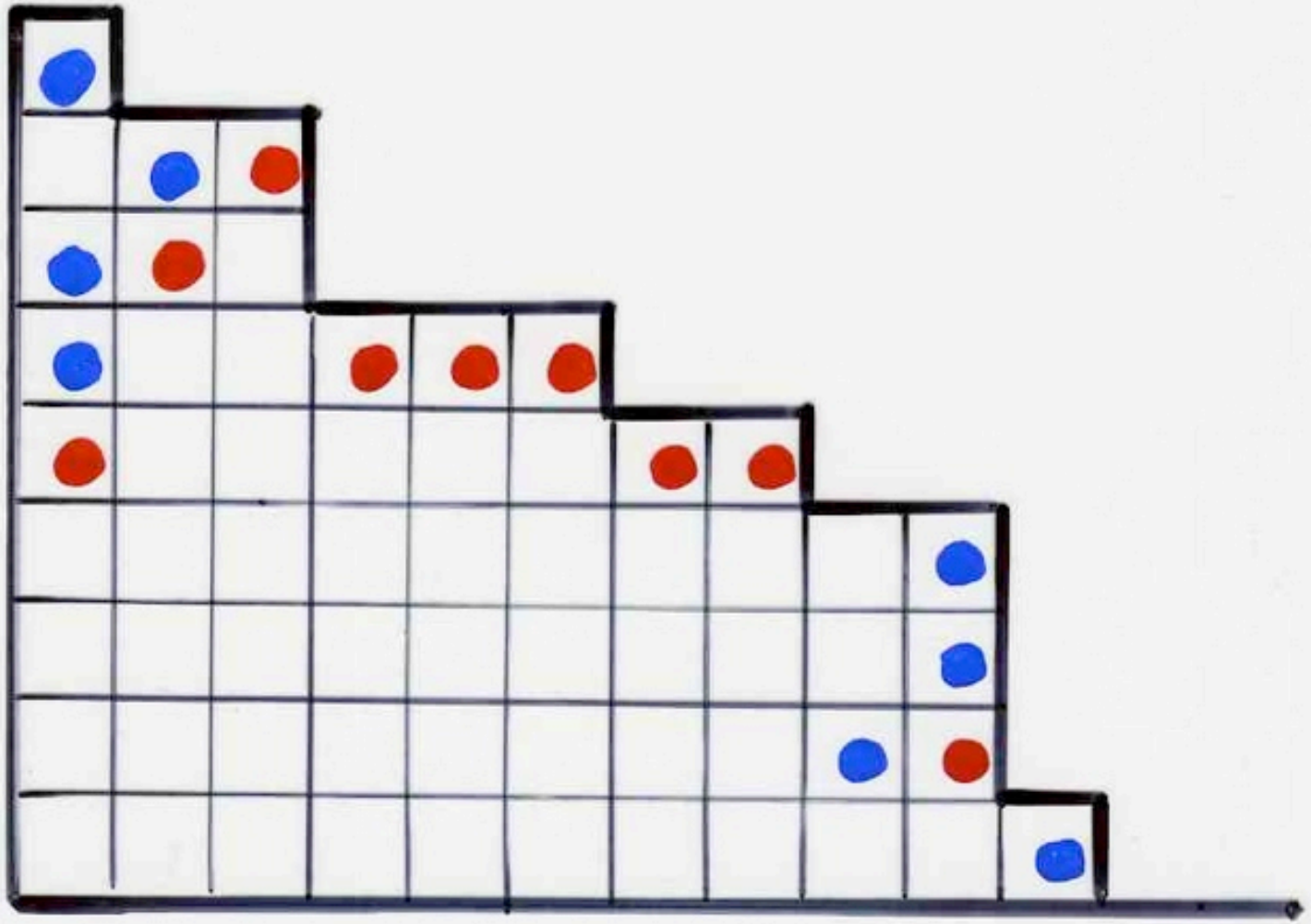


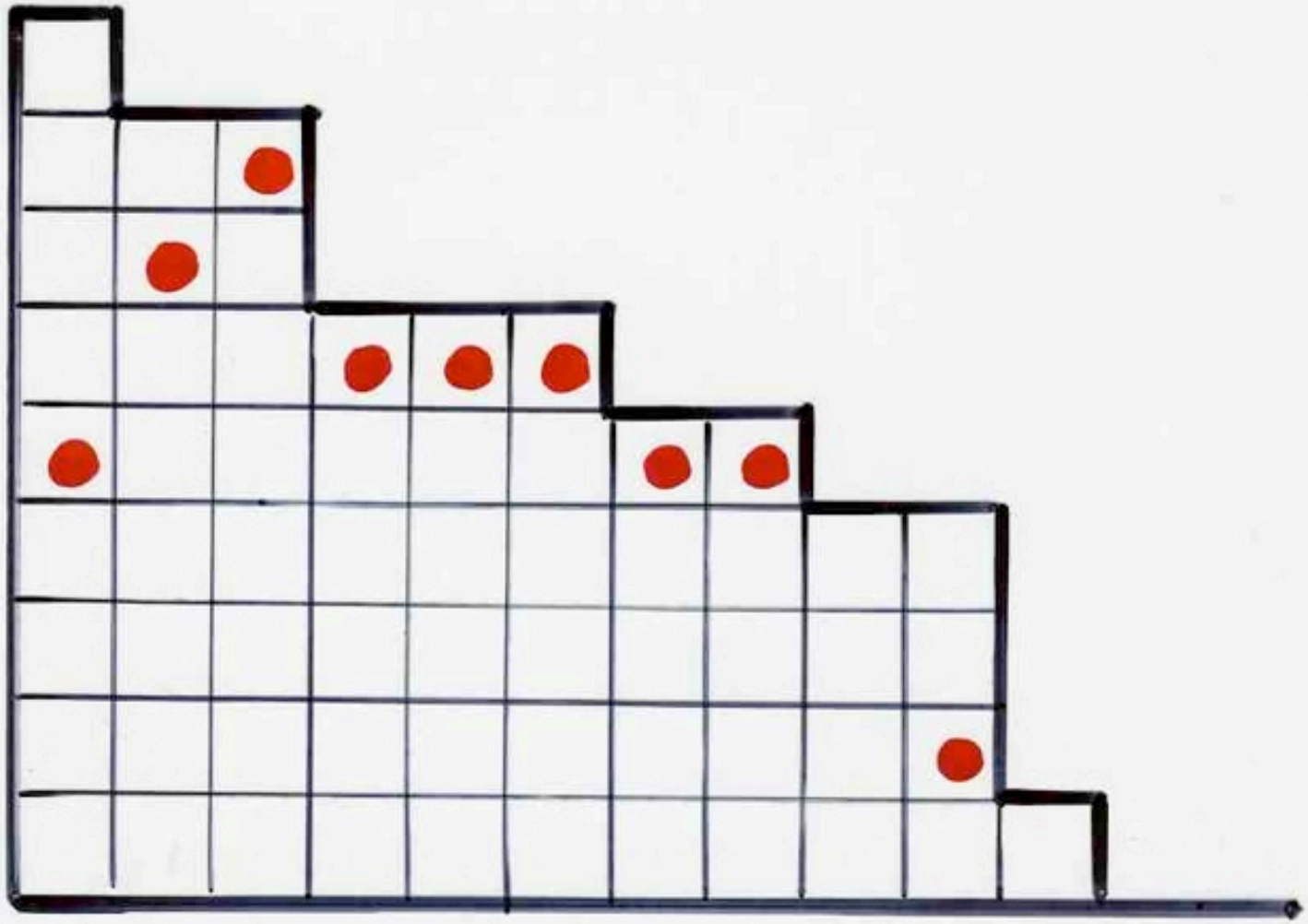




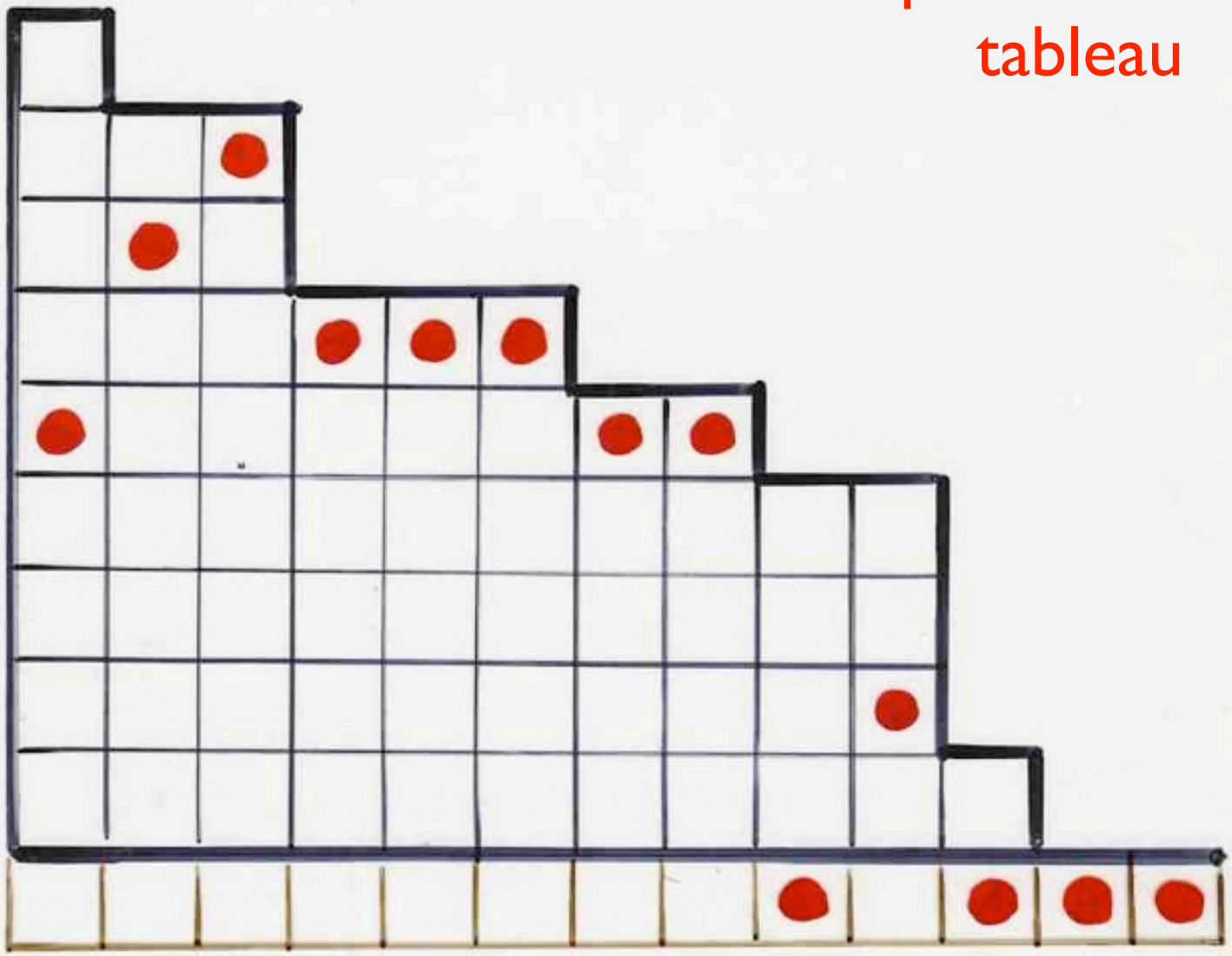
Characterization of alternative Catalan tableaux



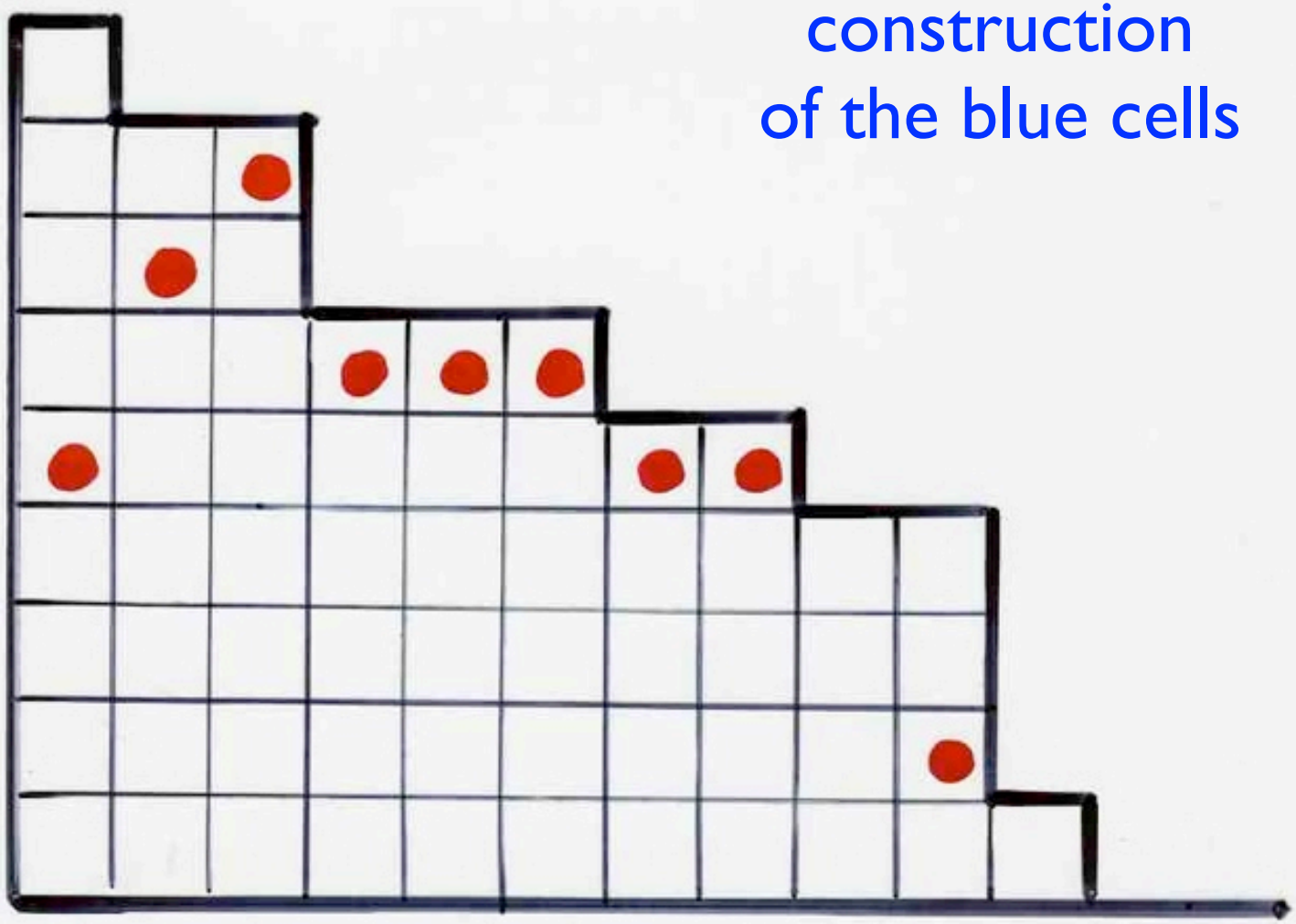


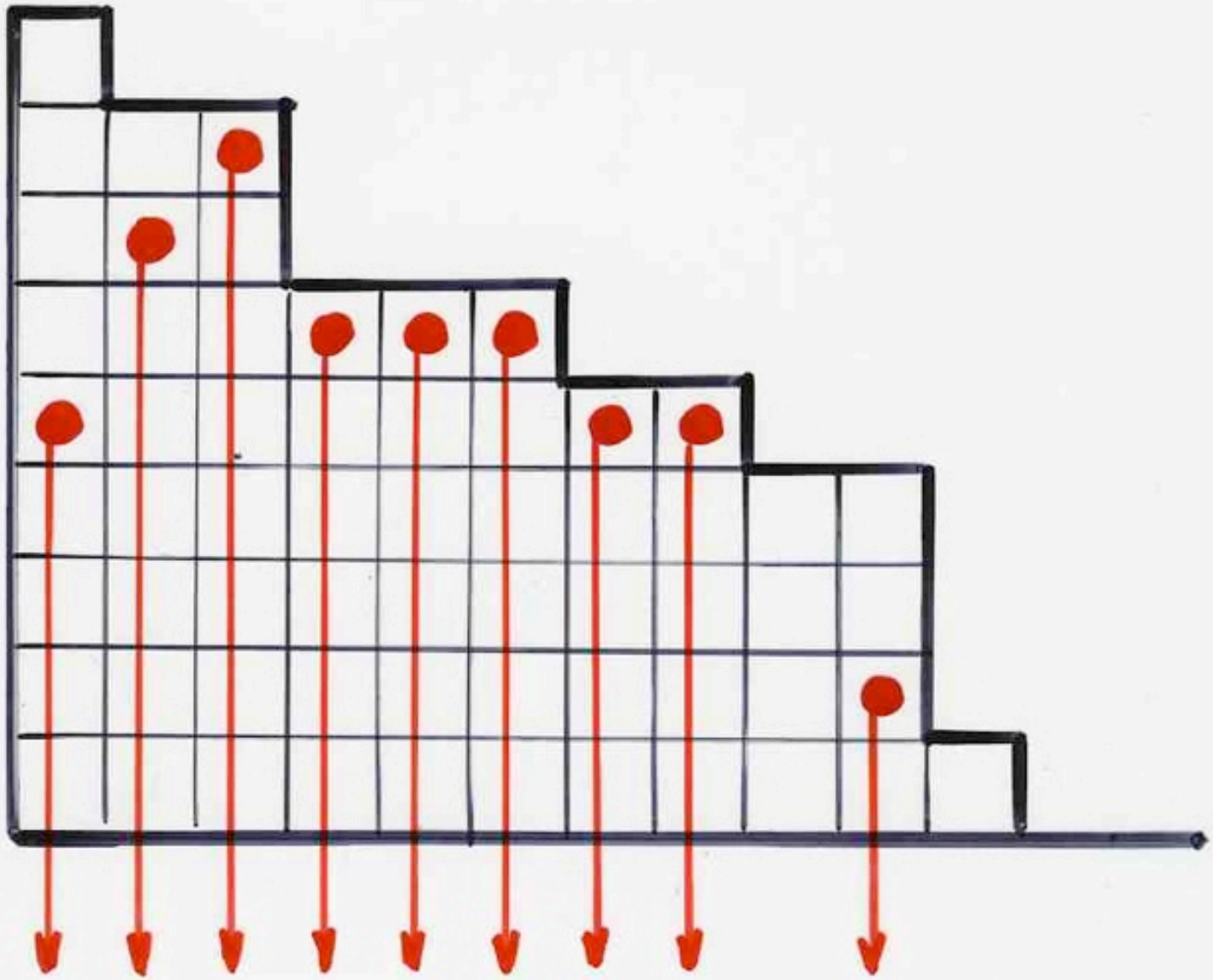


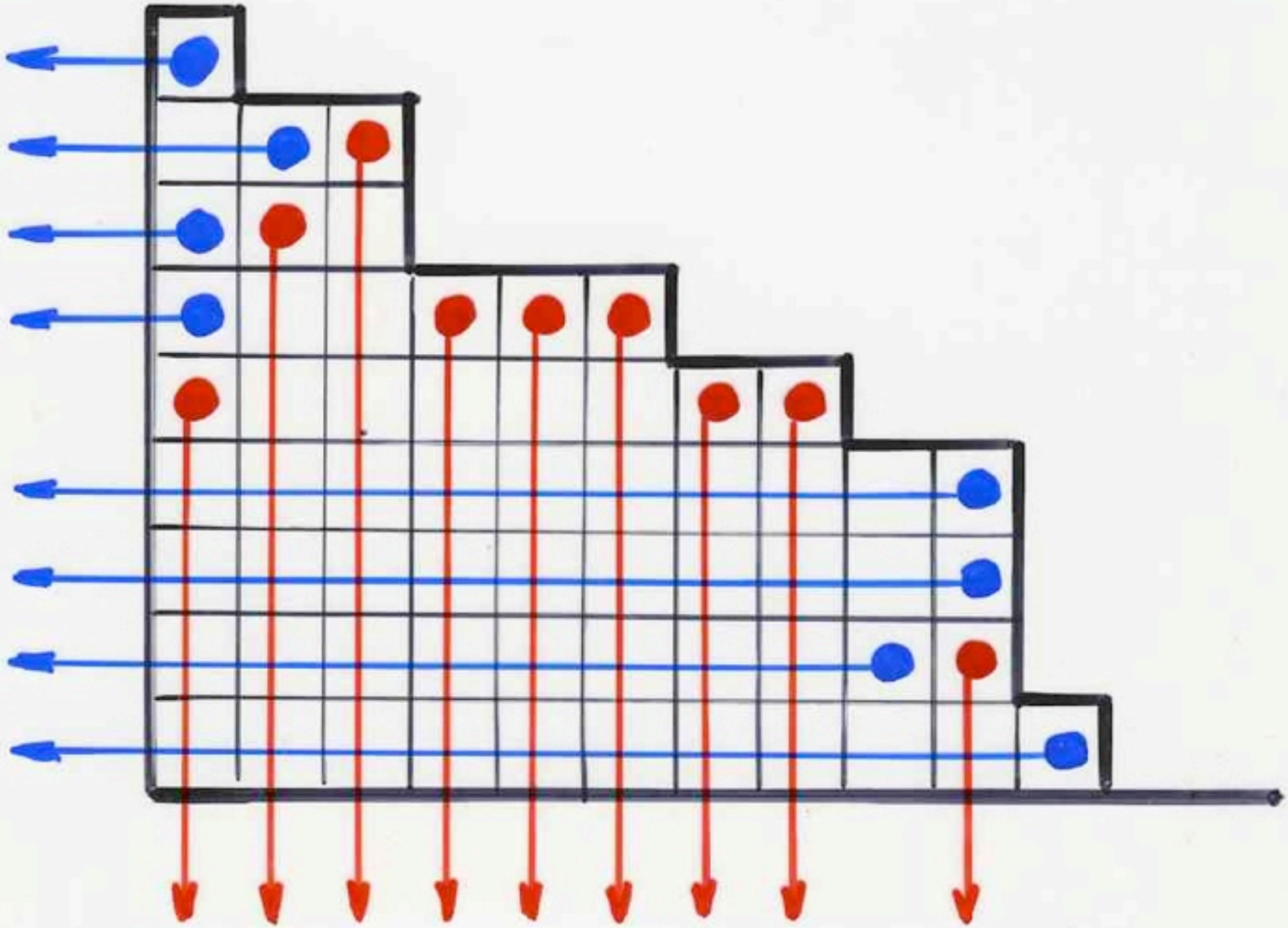
permutation tableau

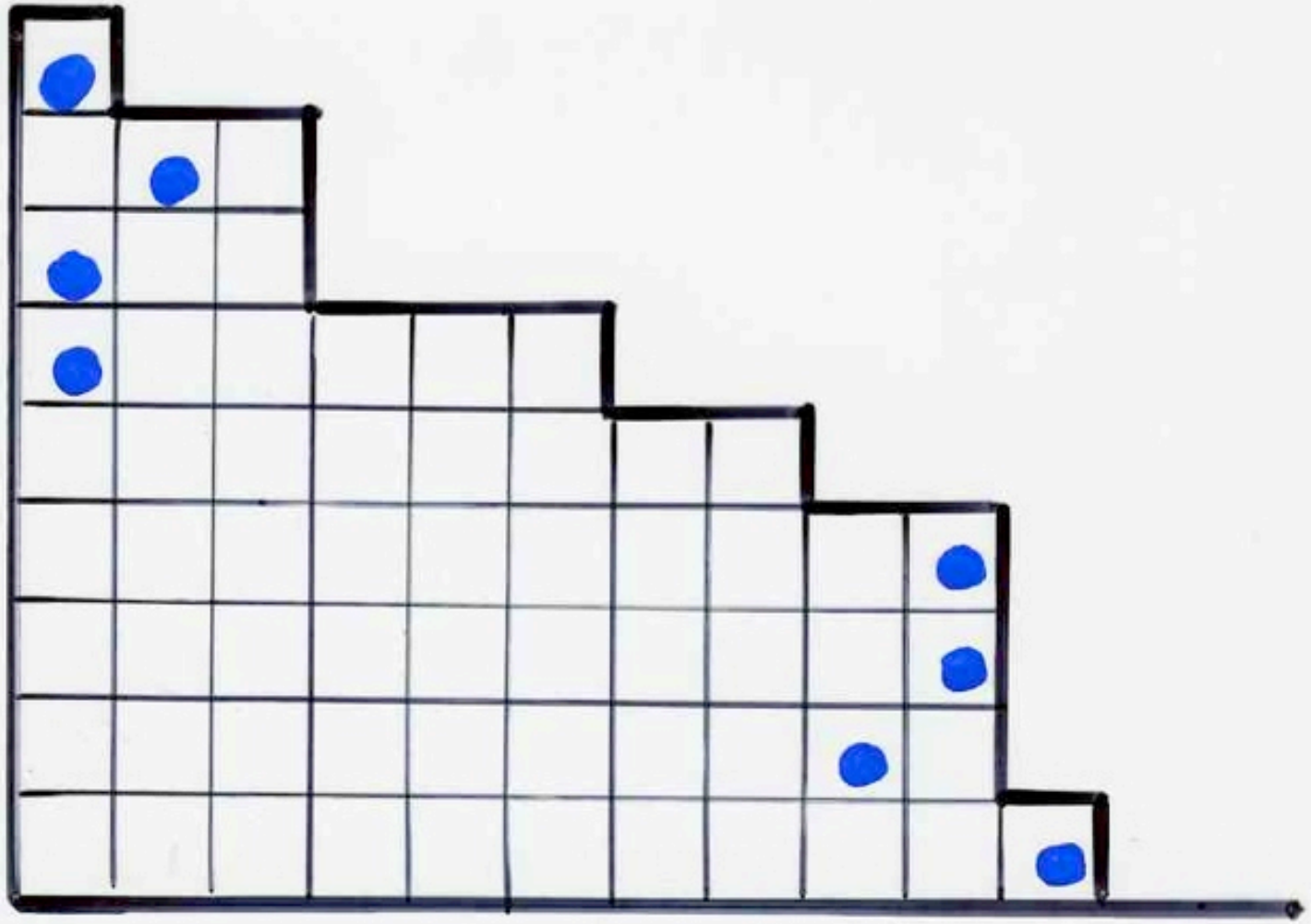


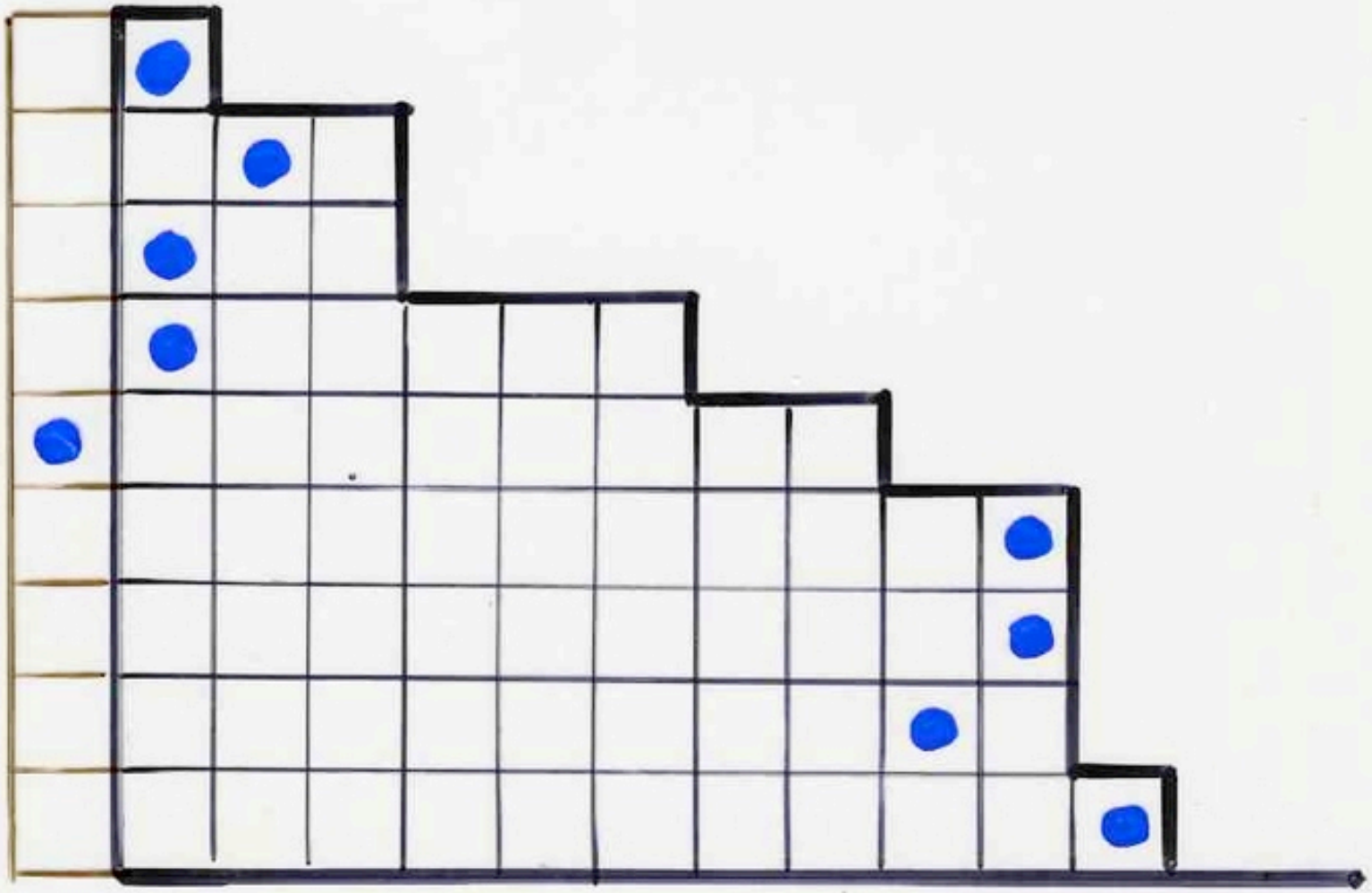
construction
of the blue cells

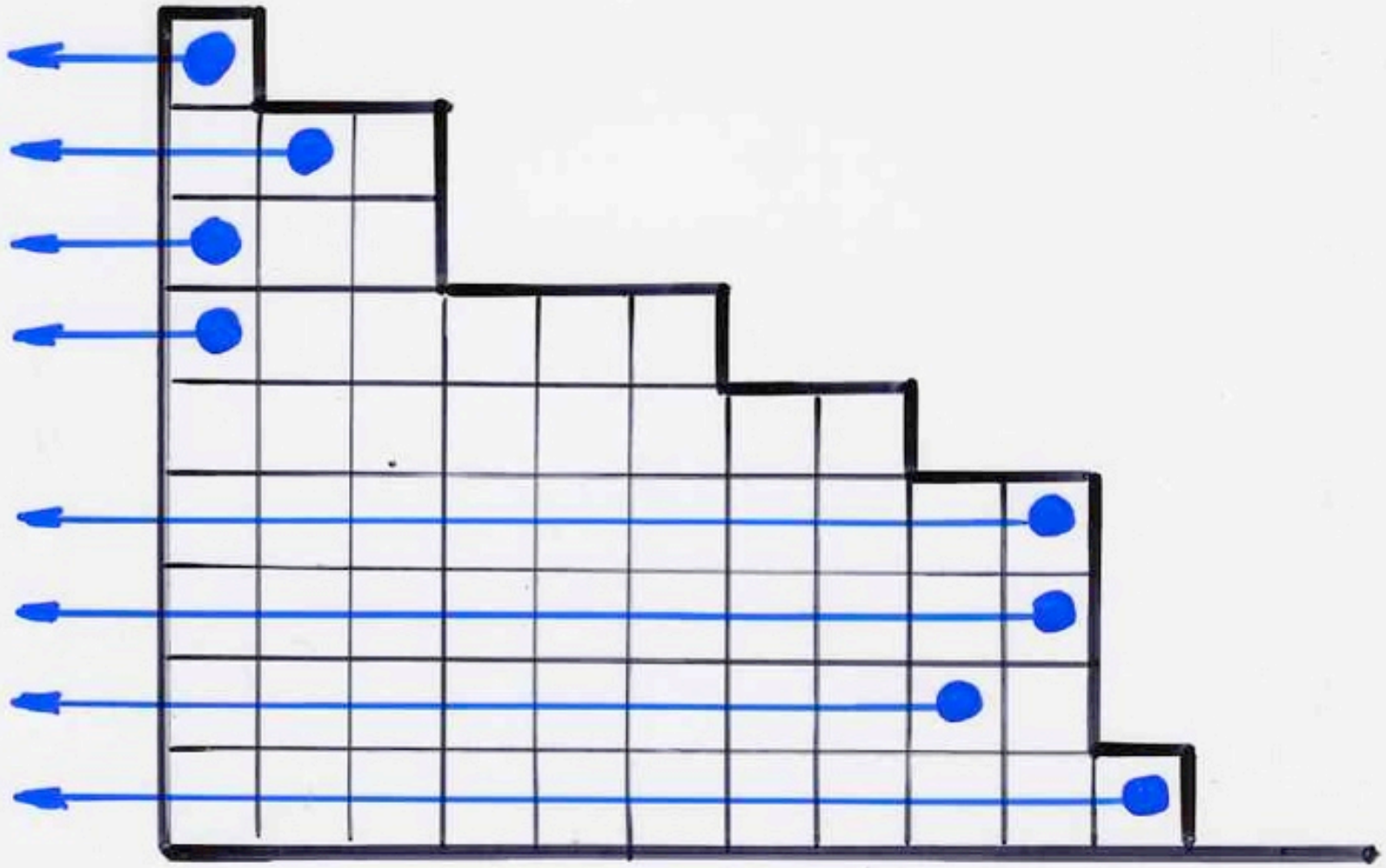


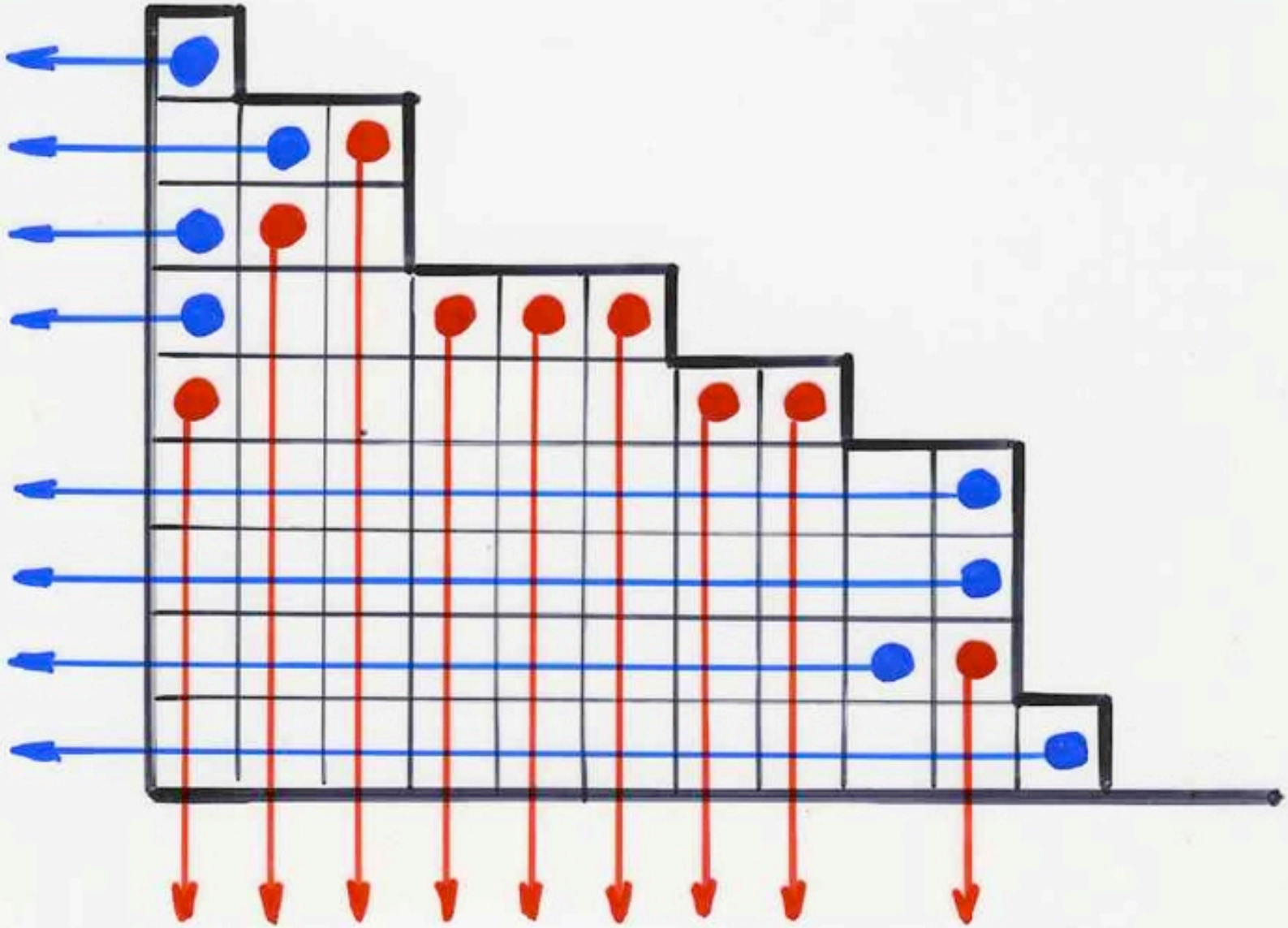


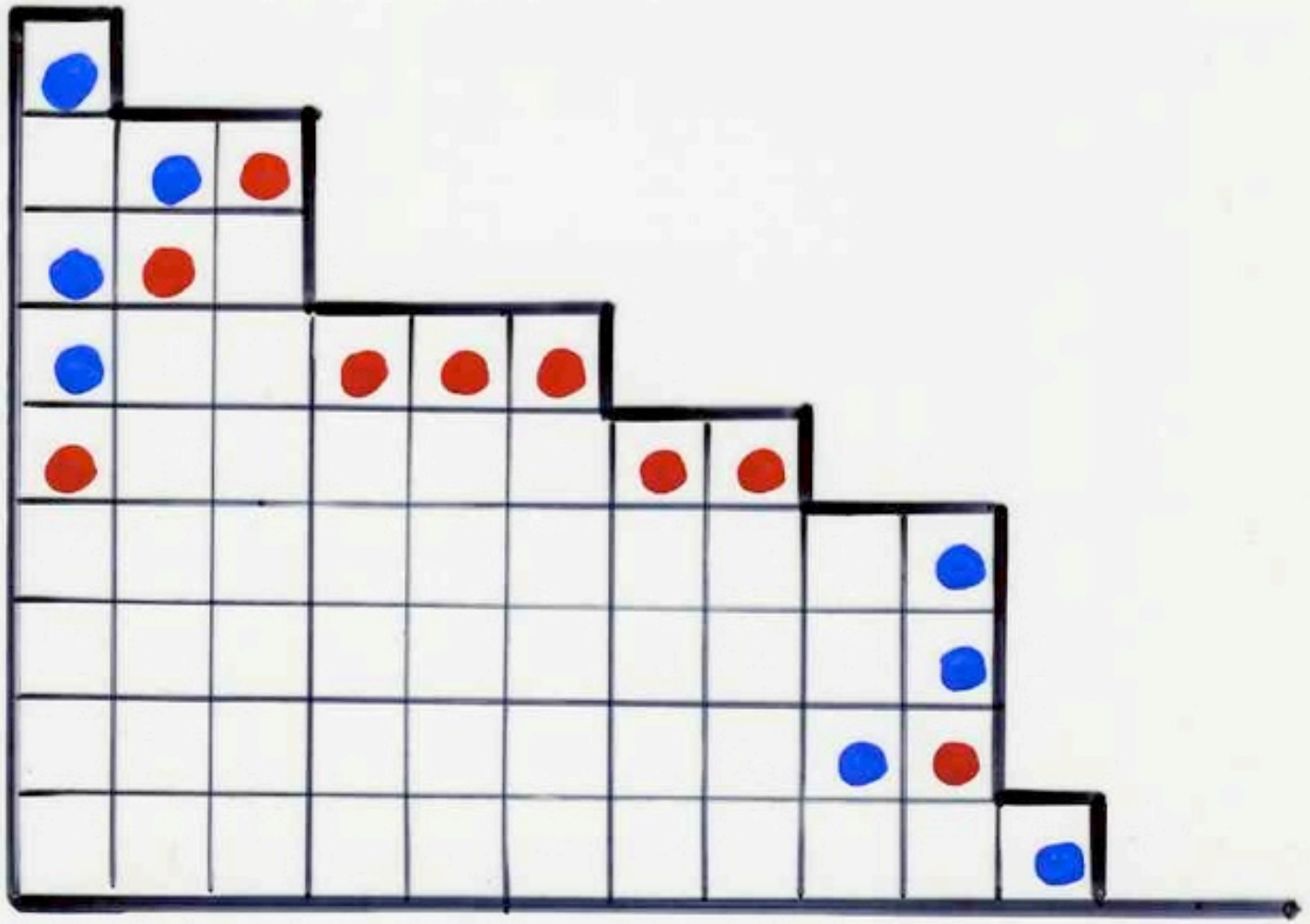








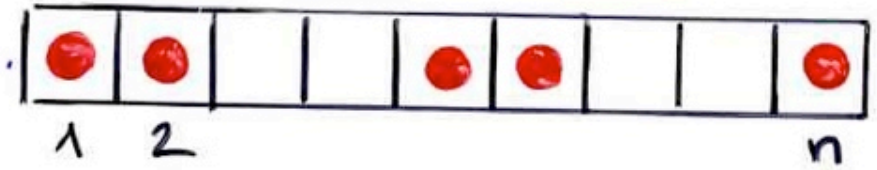


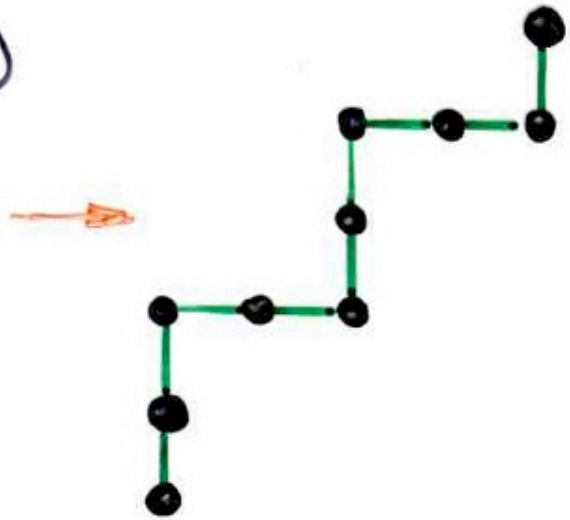


TASEP with Catalan tableaux

relation with Shapiro-Zeilberger interpretation

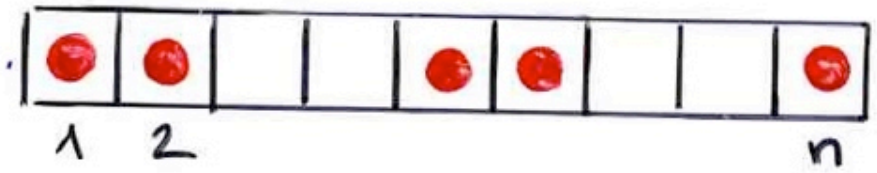
state $s = (\tau_1, \ldots, \tau_n)$

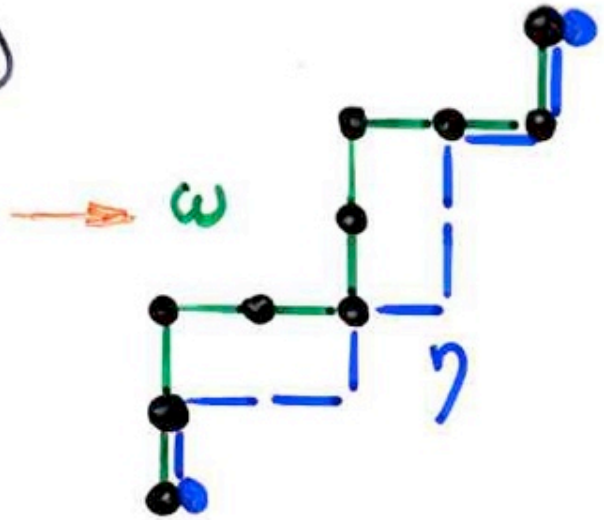




$P_n(s) =$

state $s = (\tau_1, \dots, \tau_n)$

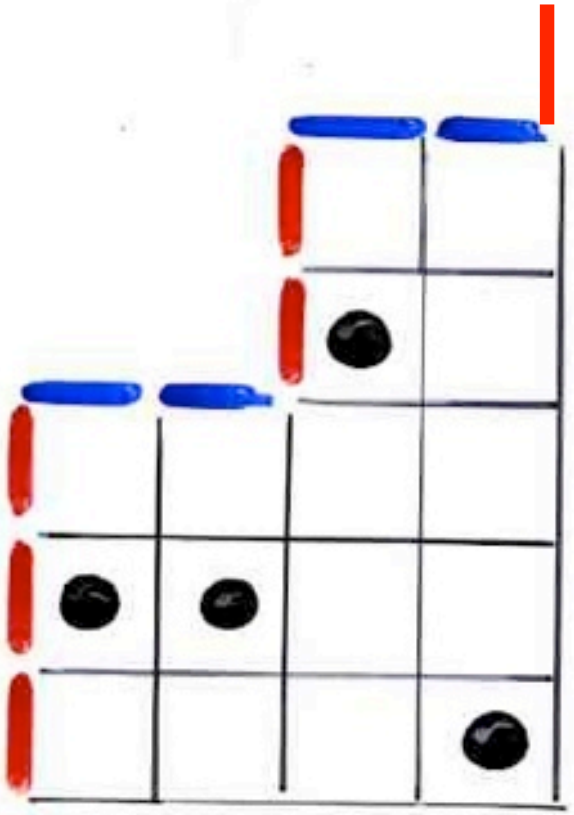




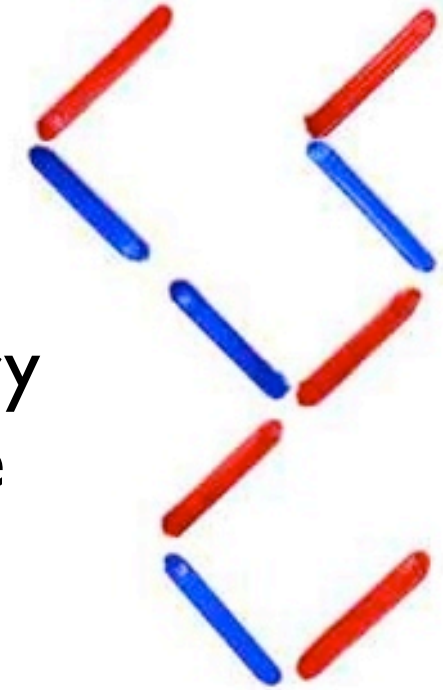
$$P_n(s) = \frac{1}{C_{n+1}} \left(\begin{array}{l} \text{number of paths } \eta \\ \text{below the path } \omega \\ \text{associated to } s \end{array} \right)$$

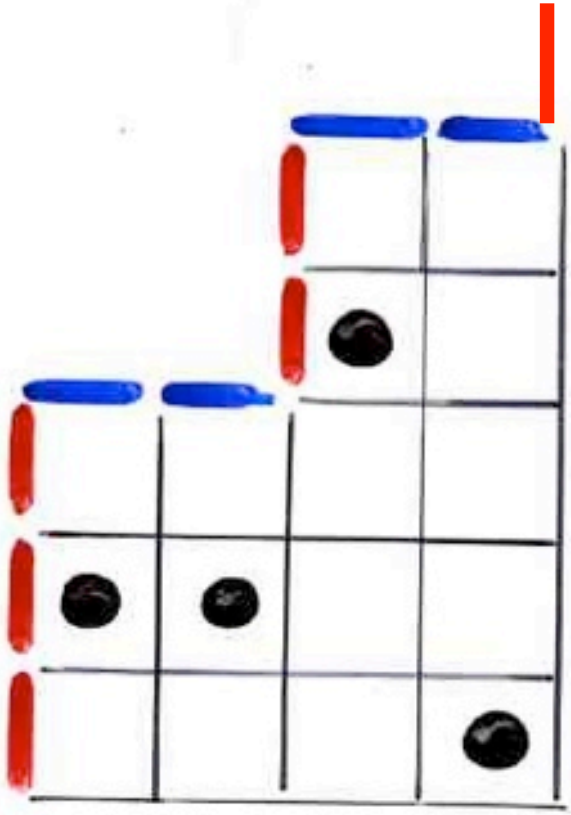
Catalan alternating tableau

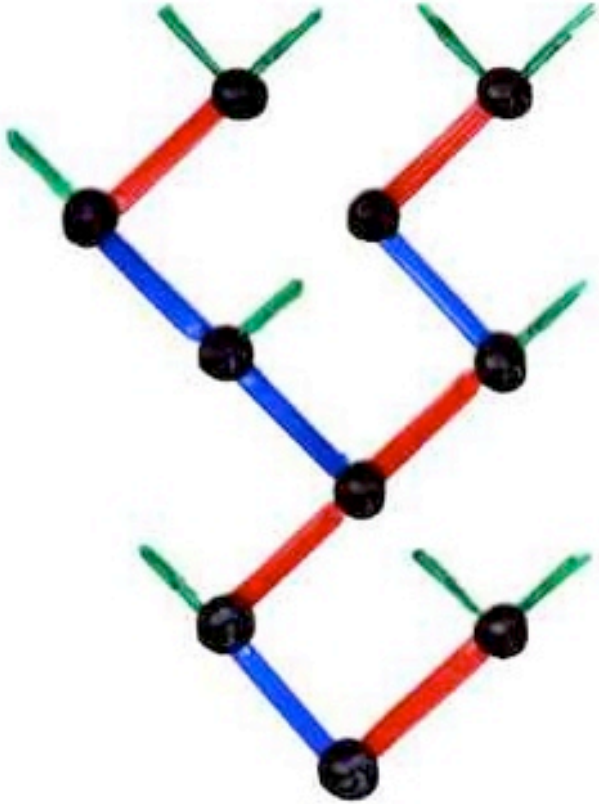
here Catalan permutation tableau

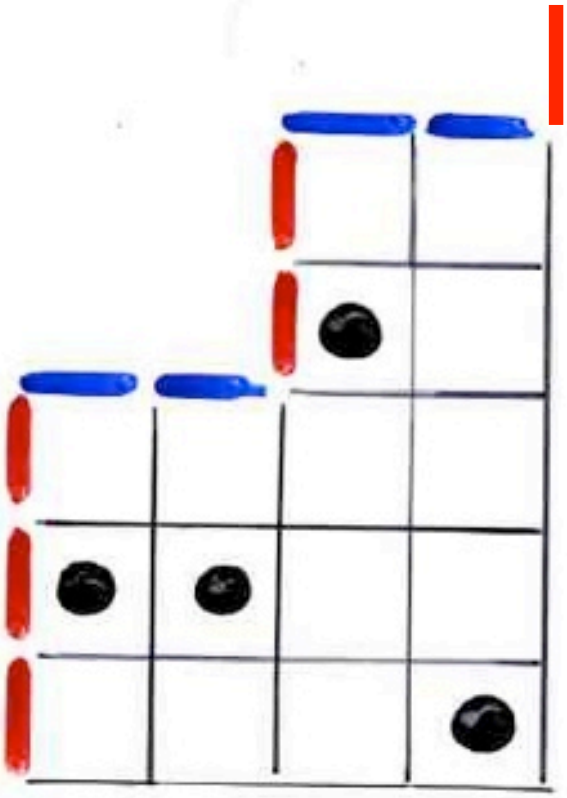


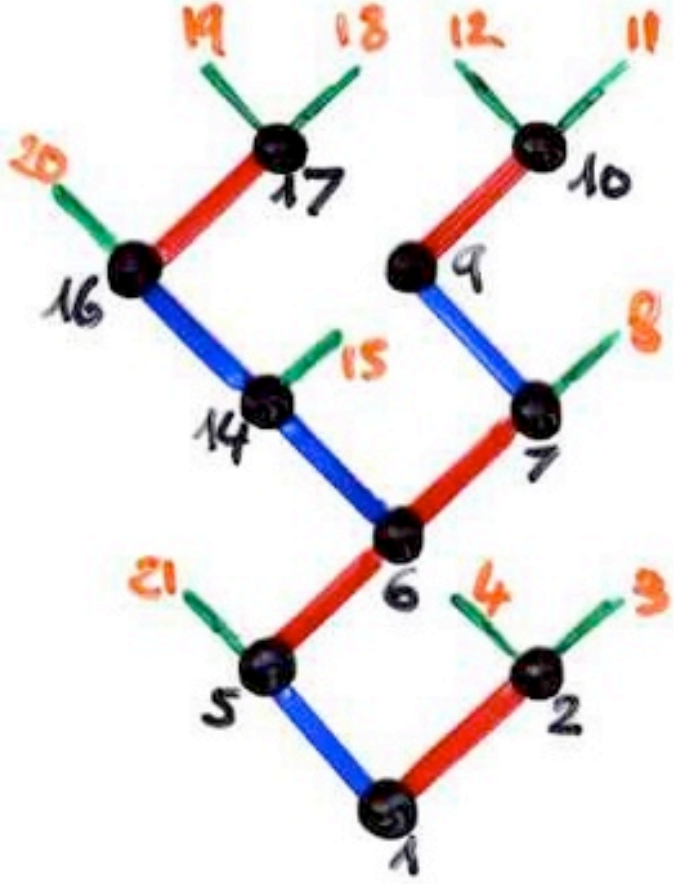
binary tree







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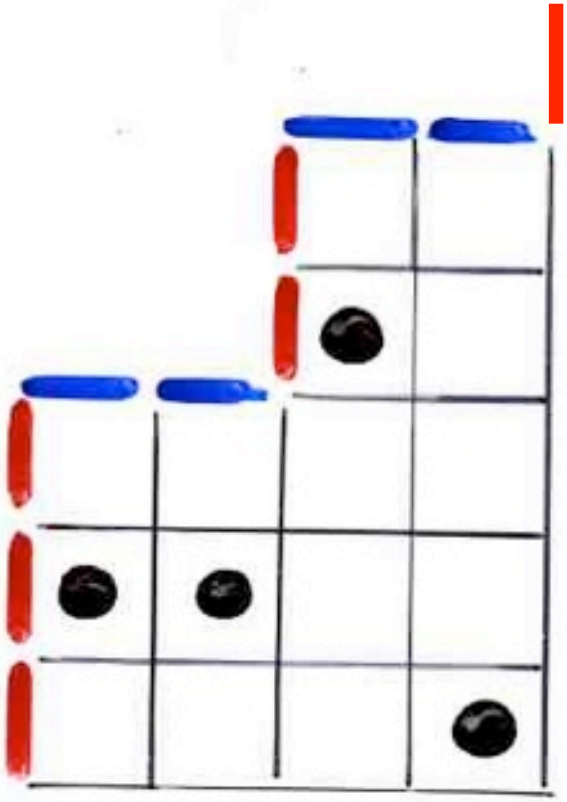

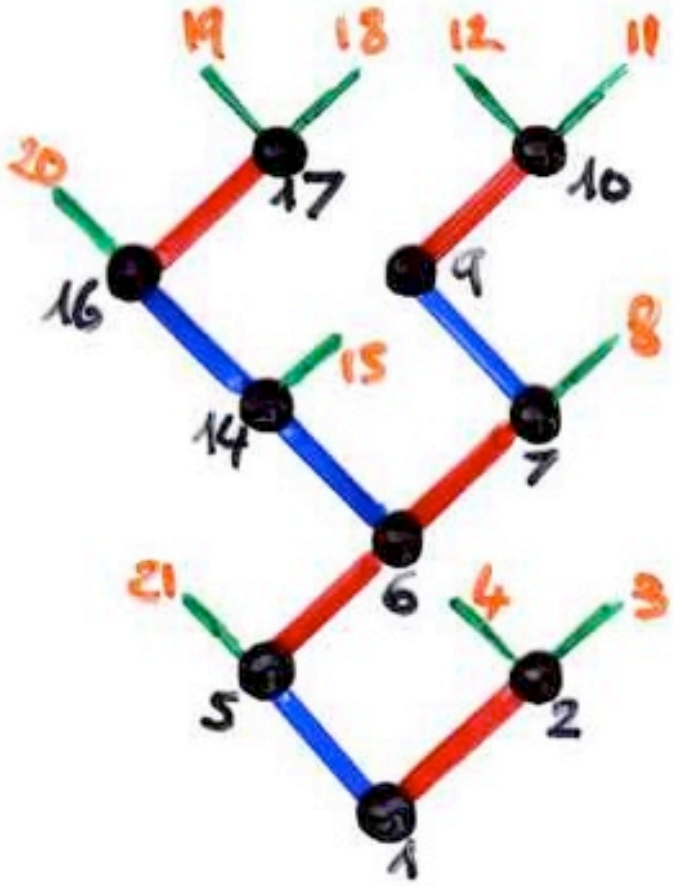
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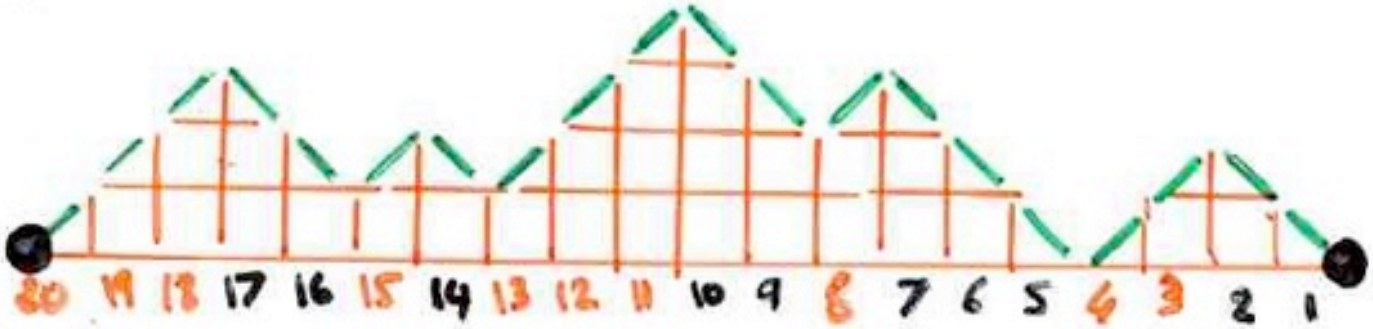
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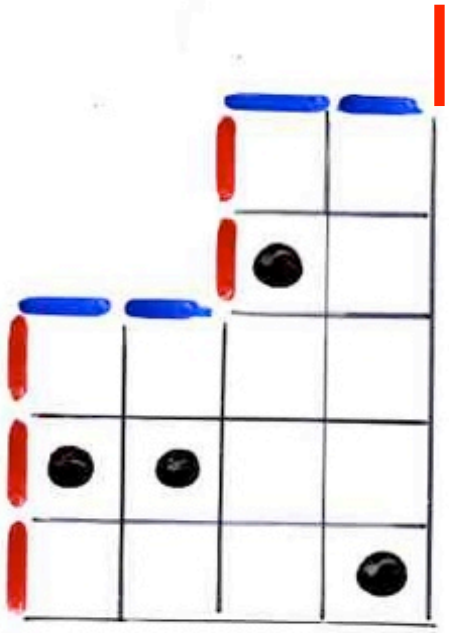

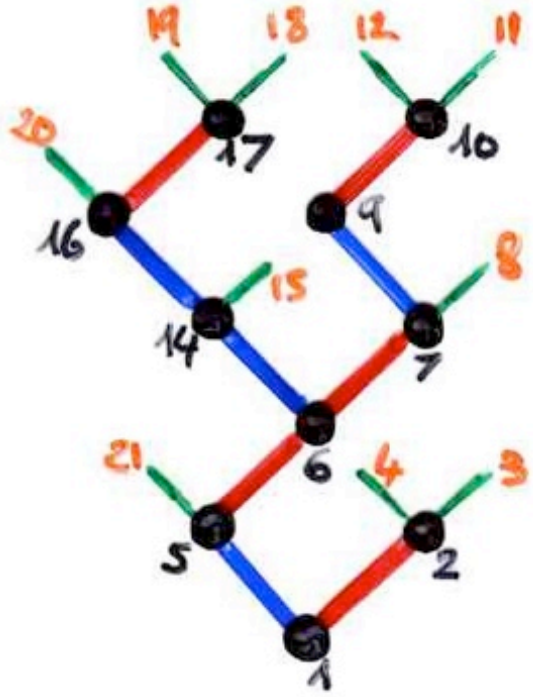
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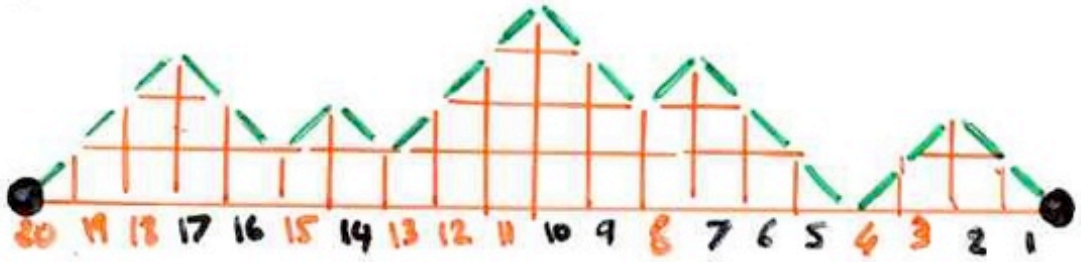
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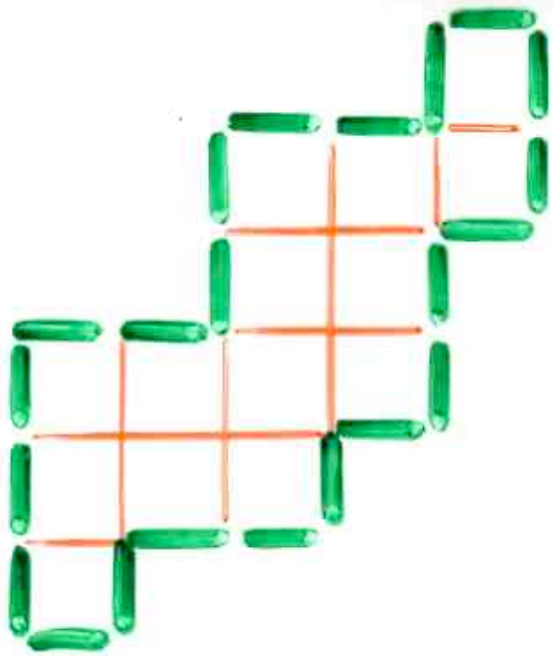

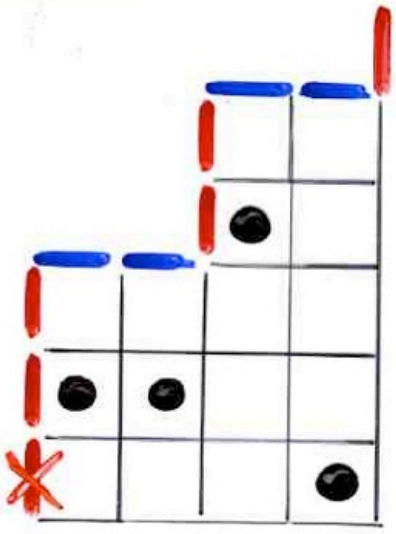

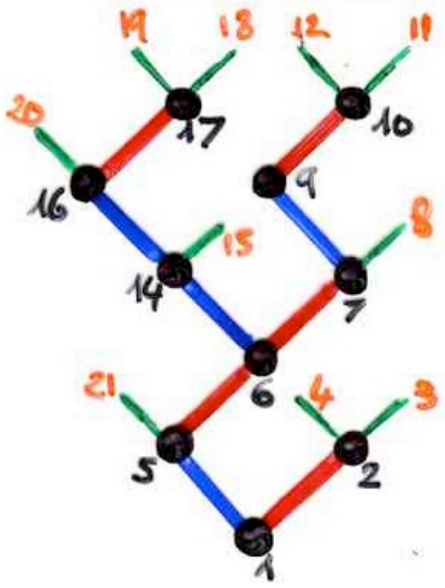
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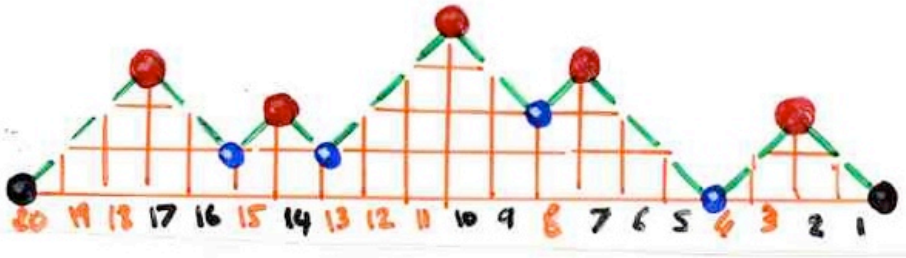
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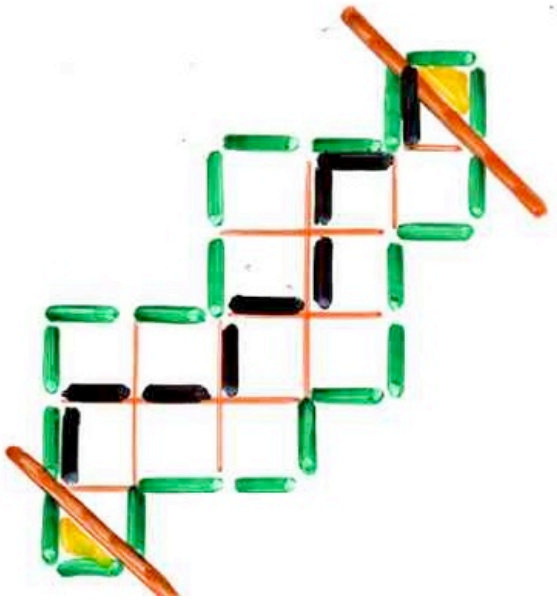

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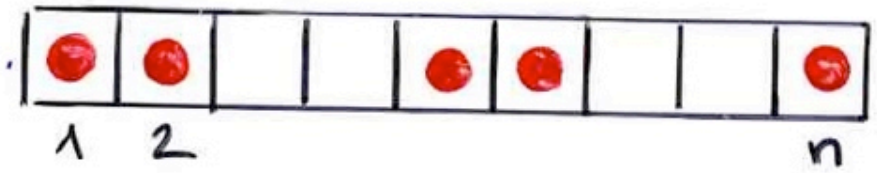

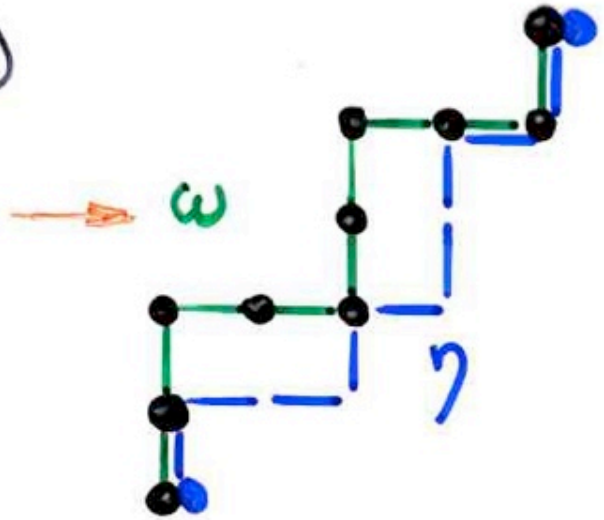

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Shapiro, Zeilberger, 1982

some references:

on X.G.V. website (page «publication», papers for downloading

[92] J.C. Aval and X.G. Viennot, The product of trees in the Loday-Ronco algebra through Catalan tableaux, SLC (Séminaire Lotharingien de Combinatoire) 63 (2010) B63h, 8pp.

[88] X.G. Viennot, "Catalan tableaux and the asymmetric exclusion process", in Proc. FPSAC'07 (Formal Power Series and Algebraic Combinatorics), Tienjin, Chine, 2007, 12 pp. arXiv:0905.3081 [math.CO]