Algebraic Combinatorics and interactions
The cellular Ansatz

Chapter 0
introduction
overview of the course

IIT-Bombay
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interplay between

- algebra
- combinatorics
- physics

website Xavier Viennot

main website   www.xavierviennot.org

secondary website: Courses   cours.xavierviennot.org
- course IIT Bombay 2013

Heisenberg
operators
U, D

$$UD = DU + 1$$

creation and annihilation opeators

quantum mechanics

normal ordering

$$UD = DU + I$$

$$UD \rightarrow DU \qquad\qquad UD \rightarrow I$$

$$
\begin{aligned}
UUDD &= UDUD + UD \\
&= DUUD + 2\,UD \\
&= DUDU + DU + 2\,(DU + I_d) \\
&= DDUU + 2\,DU \\
&= DDUU + 4\,DU + 2\,I_d
\end{aligned}
$$

$$UD = DU + I$$

Lemme. Tout mot $w \in \{U, D\}^*$ s'écrit

$$w = \sum_{i,j \geq 0} c_{i,j}(w) D^i U^j$$

every word w with letters U and D
can be written in a unique way ...

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

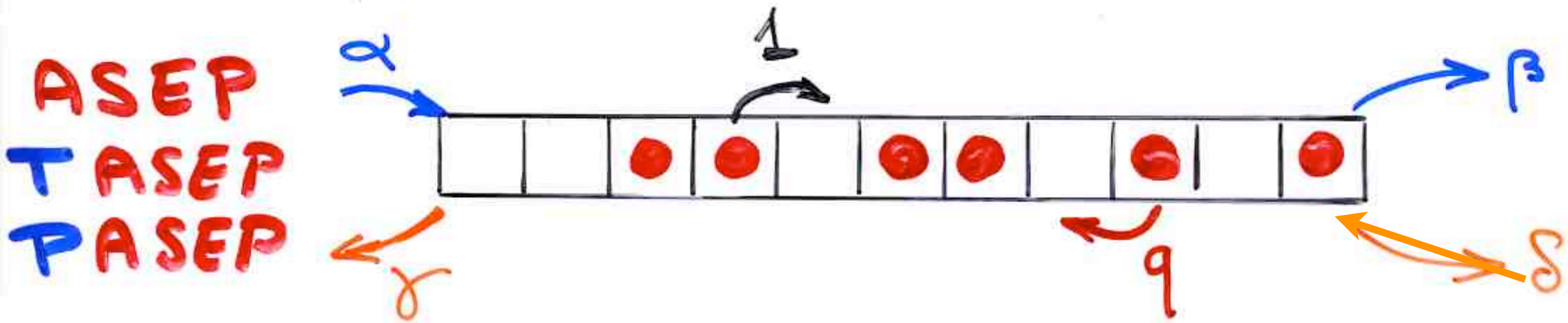
quadratic algebra Q defined by
generators and relations
$UD = DU + I$

$\updownarrow$

combinatorial objects:
here permutations

The PASEP algebra

$$DE = qED + E + D$$


ASEP
TASEP
PASEP
α
1
β
γ
q
δ

toy model in the physics of
dynamical systems far from equilibrium

The Matrix Ansatz

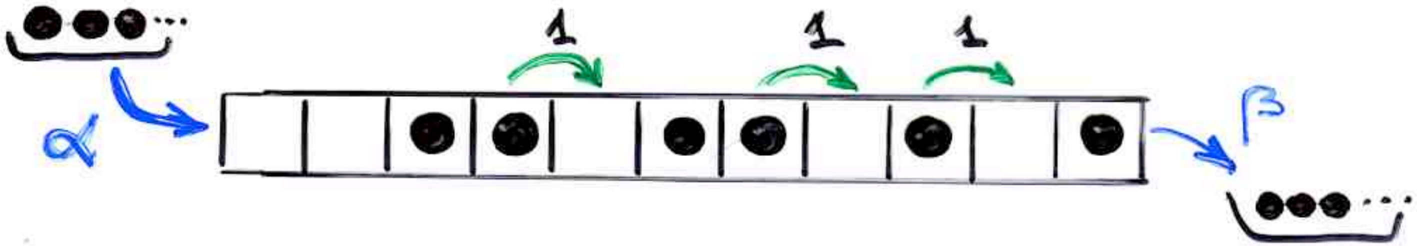
Derrida, Evans, Hakim, Pasquier (1993)

$$DE = qED + E + D$$

computation of the "stationary probabilities"

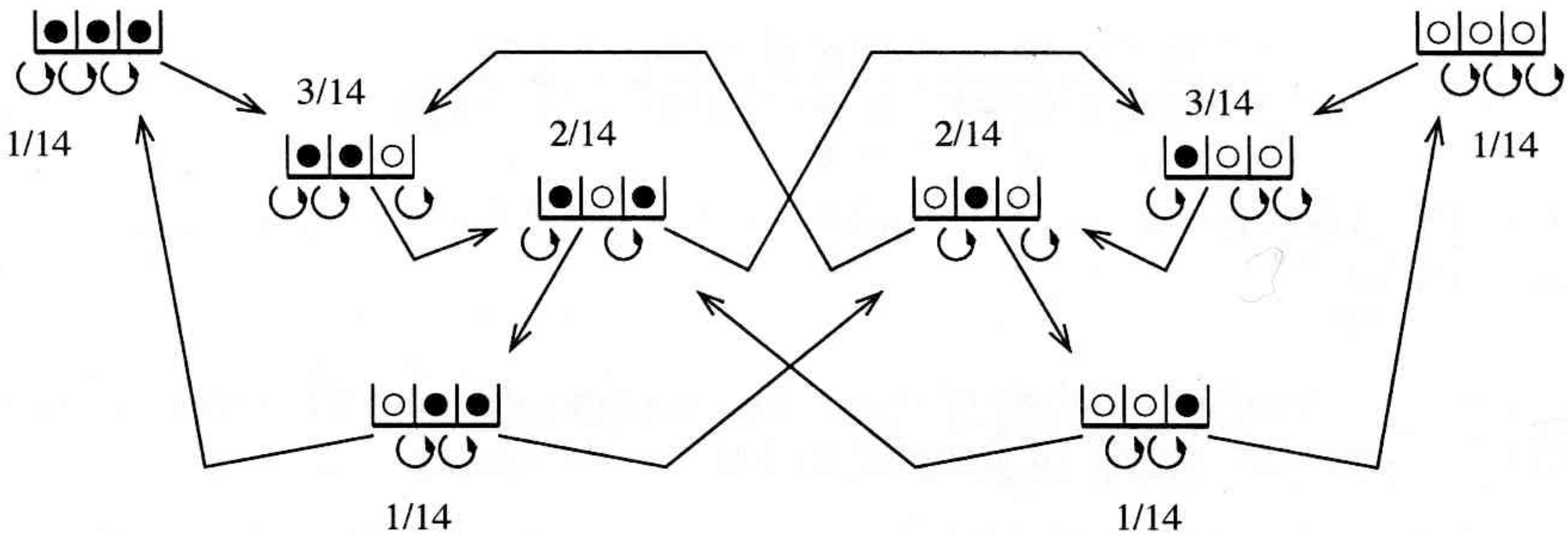
TASEP

"totally asymmetric exclusion process"



q=0

Markov chains



stationary
probabilities

$$DE = qED + E + D$$

$$D\,D\,E\,D\,E\,E\,D\,E$$

$$D\,DE\,(D\,E)\,E\,D\,E$$

$$DDE(E)EDE \quad + q\,DDE(ED)EDE \quad + \quad DDE(D)EDE$$

analog of the normal ordering

$$DE = qED + E + D$$

$$w(E, D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

word

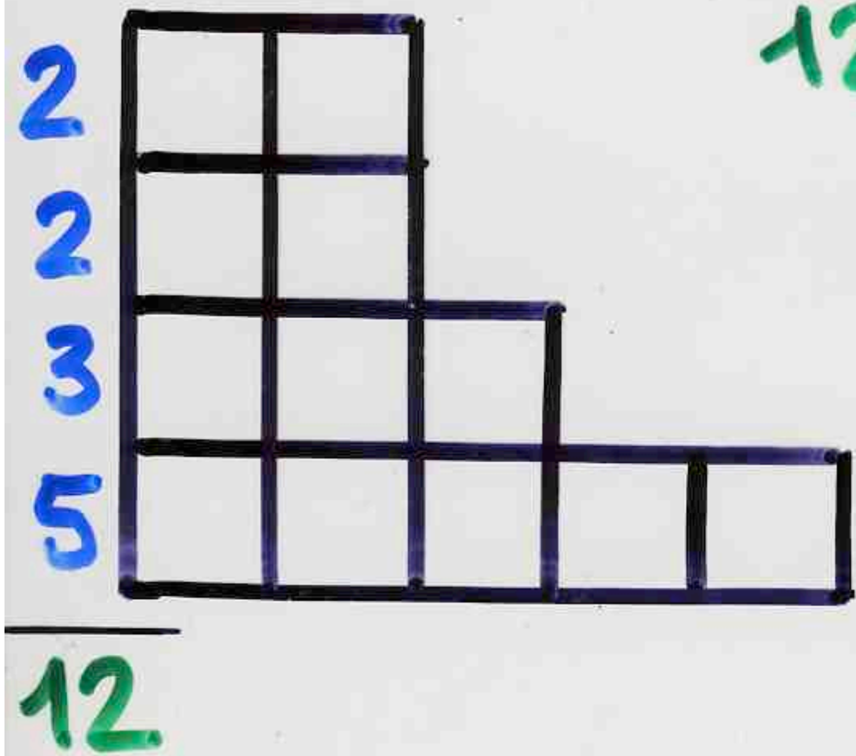
unique

quadratic algebra Q defined by
generators and relations
$DE=ED+E+D$

$\updownarrow$

combinatorial object T
here alternative tableaux

alternative tableaux



$12 = n = 5 + 3 + 2 + 2$

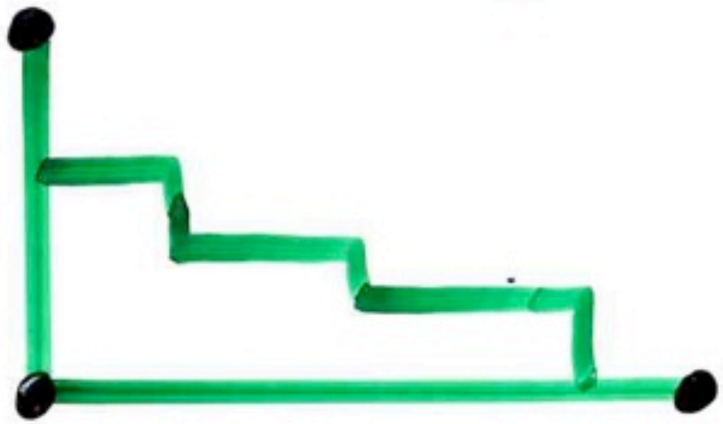
Ferrers

diagram

Partition of n

alternative tableau

- Ferrers diagram F



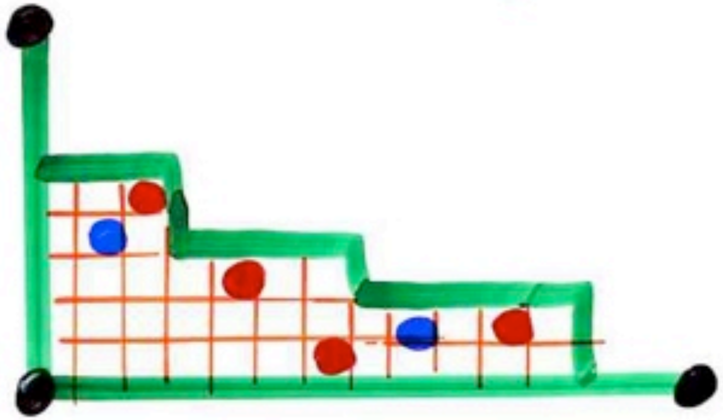
(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

alternative tableau

- Ferrers diagram **F**

(possibly empty rows or column)

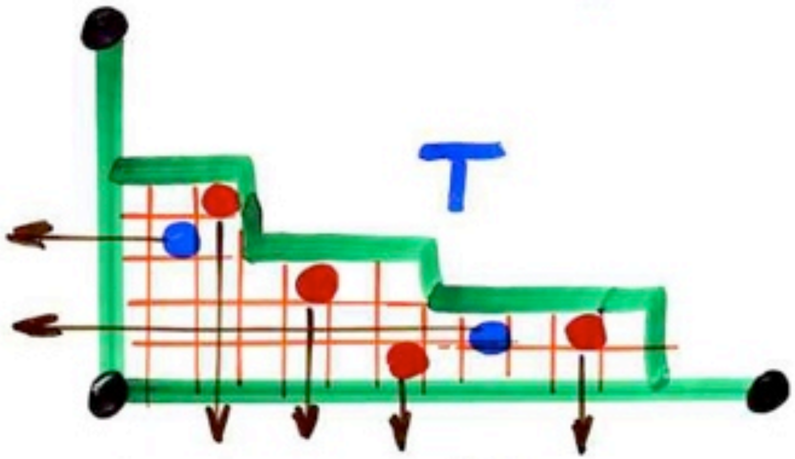


(nb of rows) + (nb of columns) $= n$

- some cells are coloured **red** or **blue**

alternative tableau T

- Ferrers diagram F



(possibly empty rows or column)

(nb of rows) + (nb of columns) $= n$

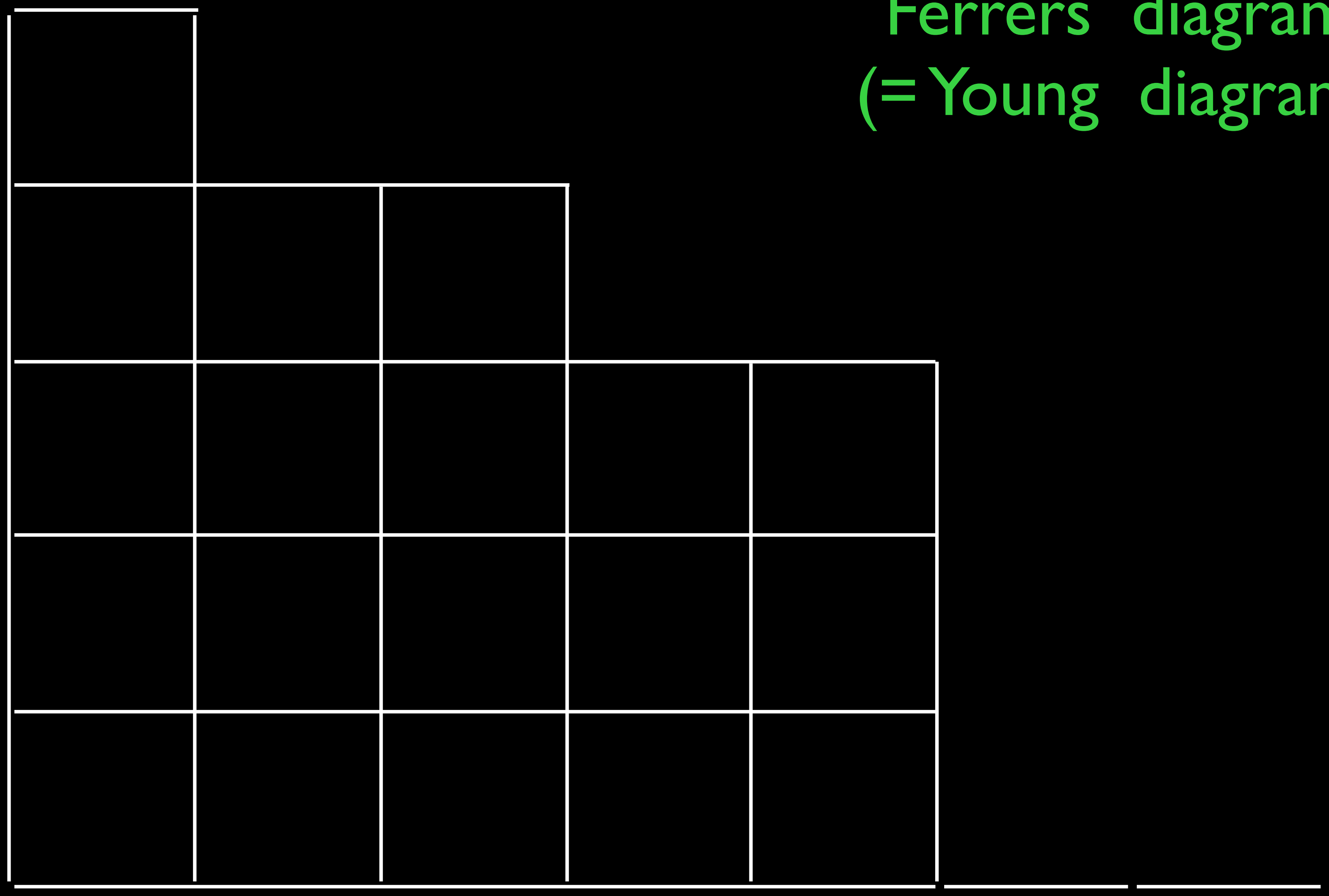
- some cells are coloured **red** or **blue**

- $\begin{cases} \text{no coloured cell at the left of } \boxed{\color{blue}\bullet} \\ \text{no coloured cell below } \boxed{\color{red}\bullet} \end{cases}$

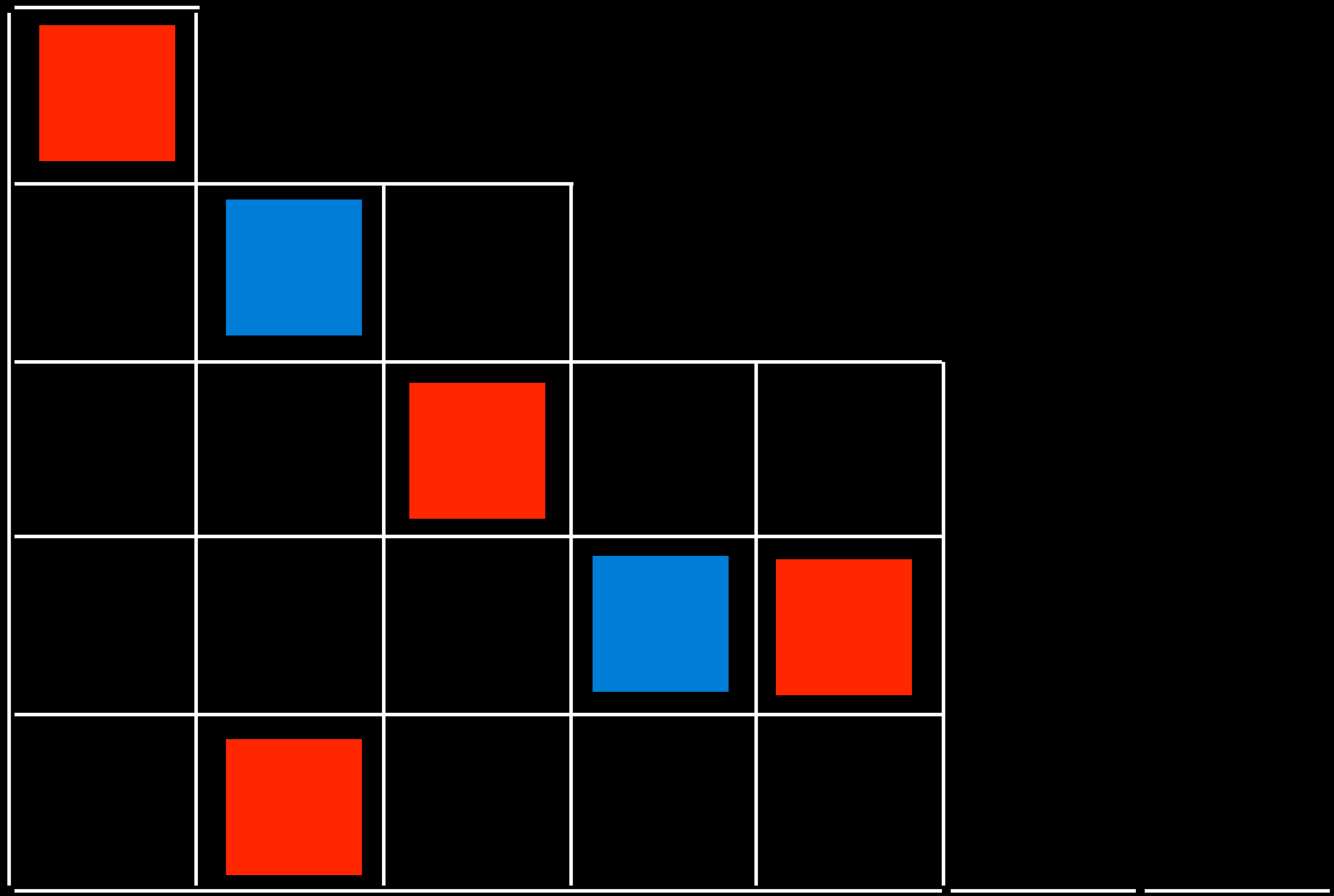
n size of T

alternative tableau

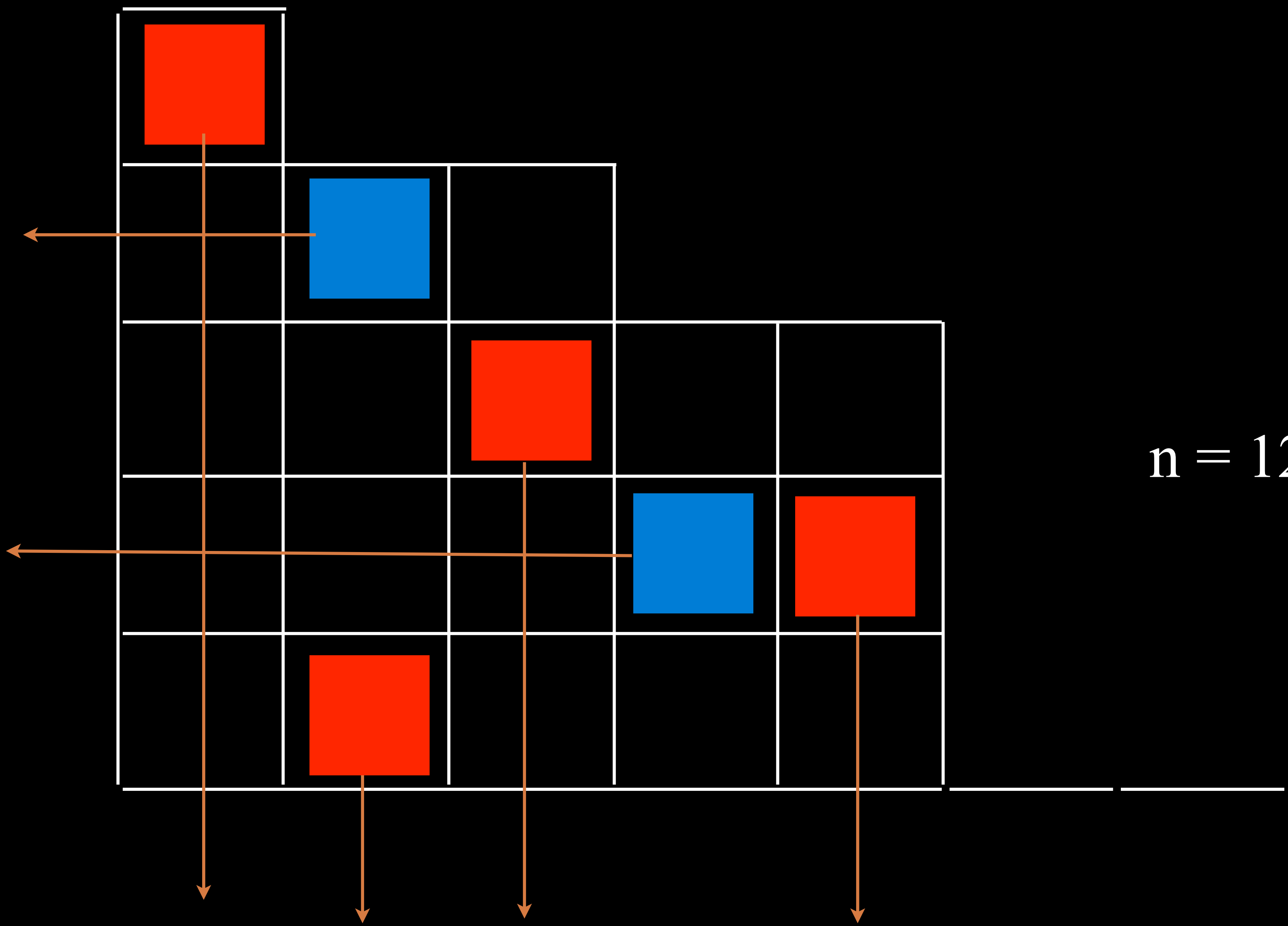
Ferrers diagram
(= Young diagram)



alternative tableau



alternative tableau



$n = 12$

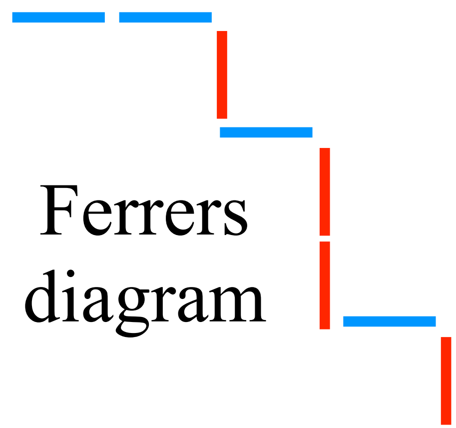
$$DE = qED + E + D$$

$$w(E,D) = \sum_{T} q^{k(T)} E^{i(T)} D^{j(T)}$$

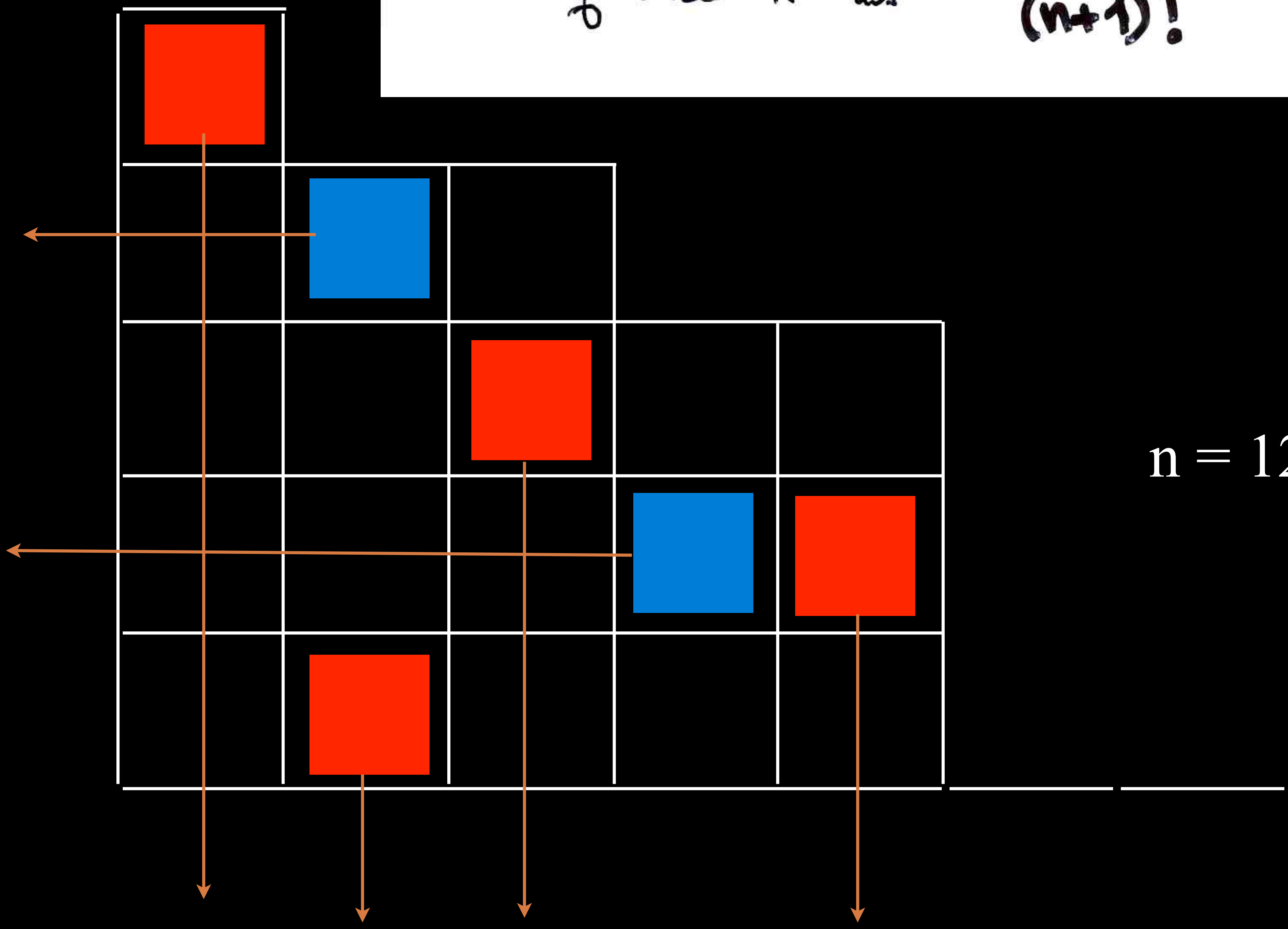
word

unique

w = D D E D E E D E $\longrightarrow$ Ferrers diagram



Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

Physics

combinatorial
objects
on a 2d lattice

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

towers placements

permutations

alternative tableaux

quadratic algebra Q

commutations
rewriting rules

alternating sign matrices (ASM) and a quadratic algebra

Def - **ASM** alternating sign matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(i) entries: $0, 1, -1$

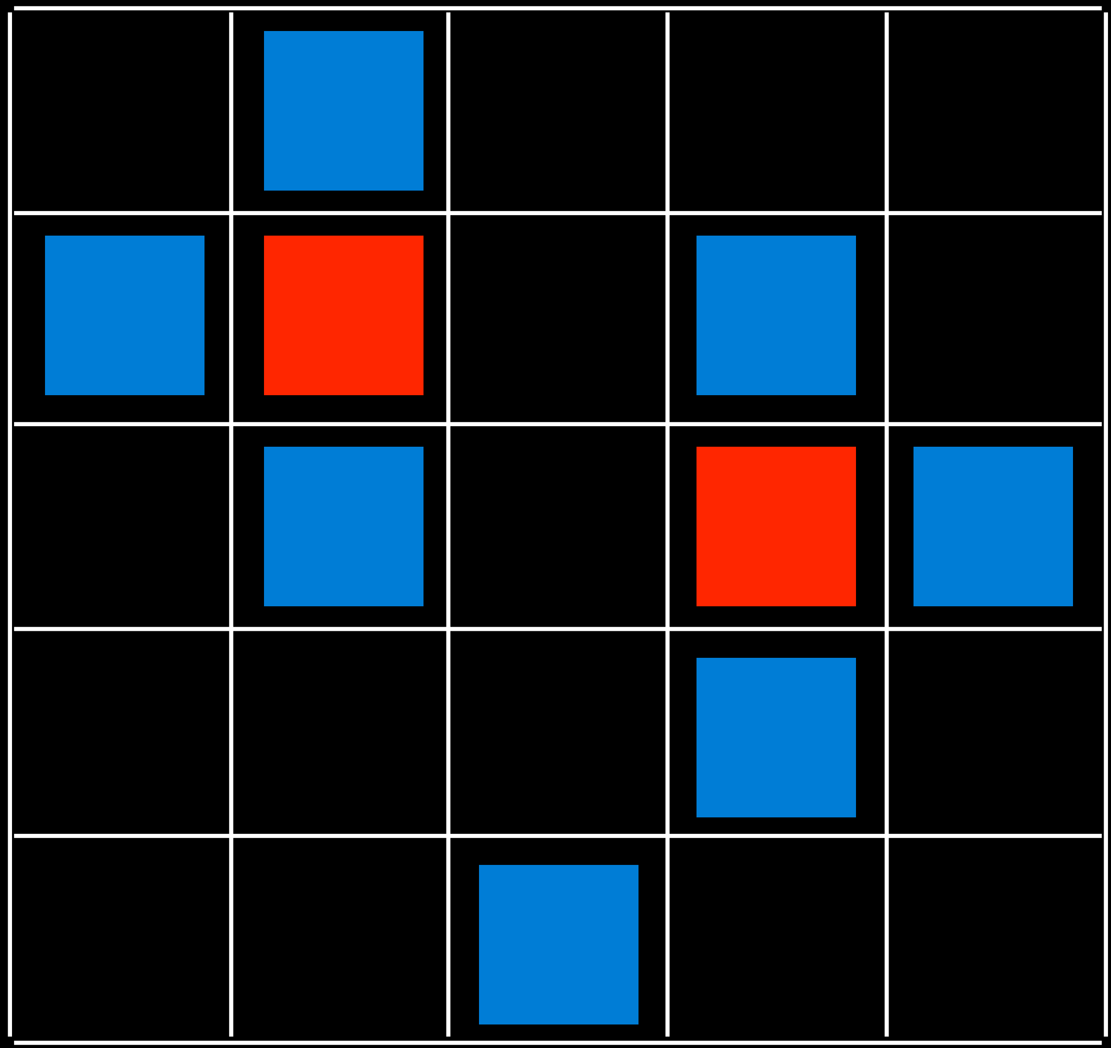
(ii) sum of entries in each $\begin{cases} \text{row} \\ \text{column} \end{cases} = 1$

(iii) non-zero entries alternate in each $\begin{cases} \text{row} \\ \text{column} \end{cases}$

ASM

Alternating sign matrices

$$
\begin{matrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & -1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & -1 & 1 \\
\cdot & \cdot & 1 & -1 & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{matrix}
$$



Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 6$$

permutations

$$1, 2, 7, 42, 429, \ldots$$

$$1,\ 2,\ 7,\ 42,\ 429,\ \ldots.$$

$$\frac{1!\ 4!\ \cdots\ (3n-2)!}{n!\ (n+1)\ \cdots\ (n+n-1)!}$$

alternating sign matrices (ex-) conjecture

Mills, Robbins, Rumsey (1982)

Physics

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

$DE = qED + E + D$

PASEP

quadratic algebra Q

commutations

rewriting rules

combinatorial

objects

on a 2d lattice

towers placements

permutations

alternative tableaux

Q-tableaux

ex: ASM,

(alternating sign matrices)

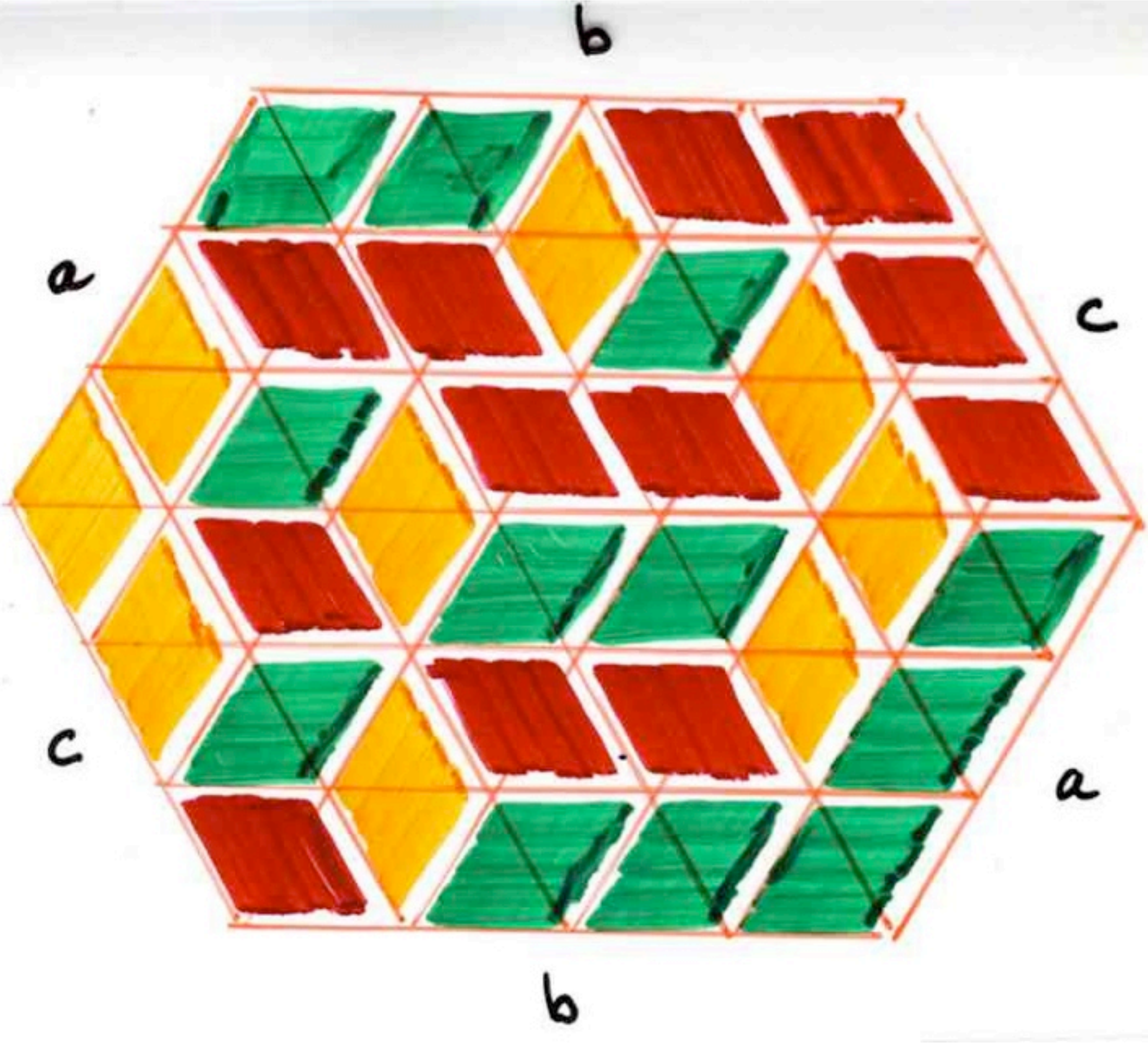
tilings

non-crossing paths

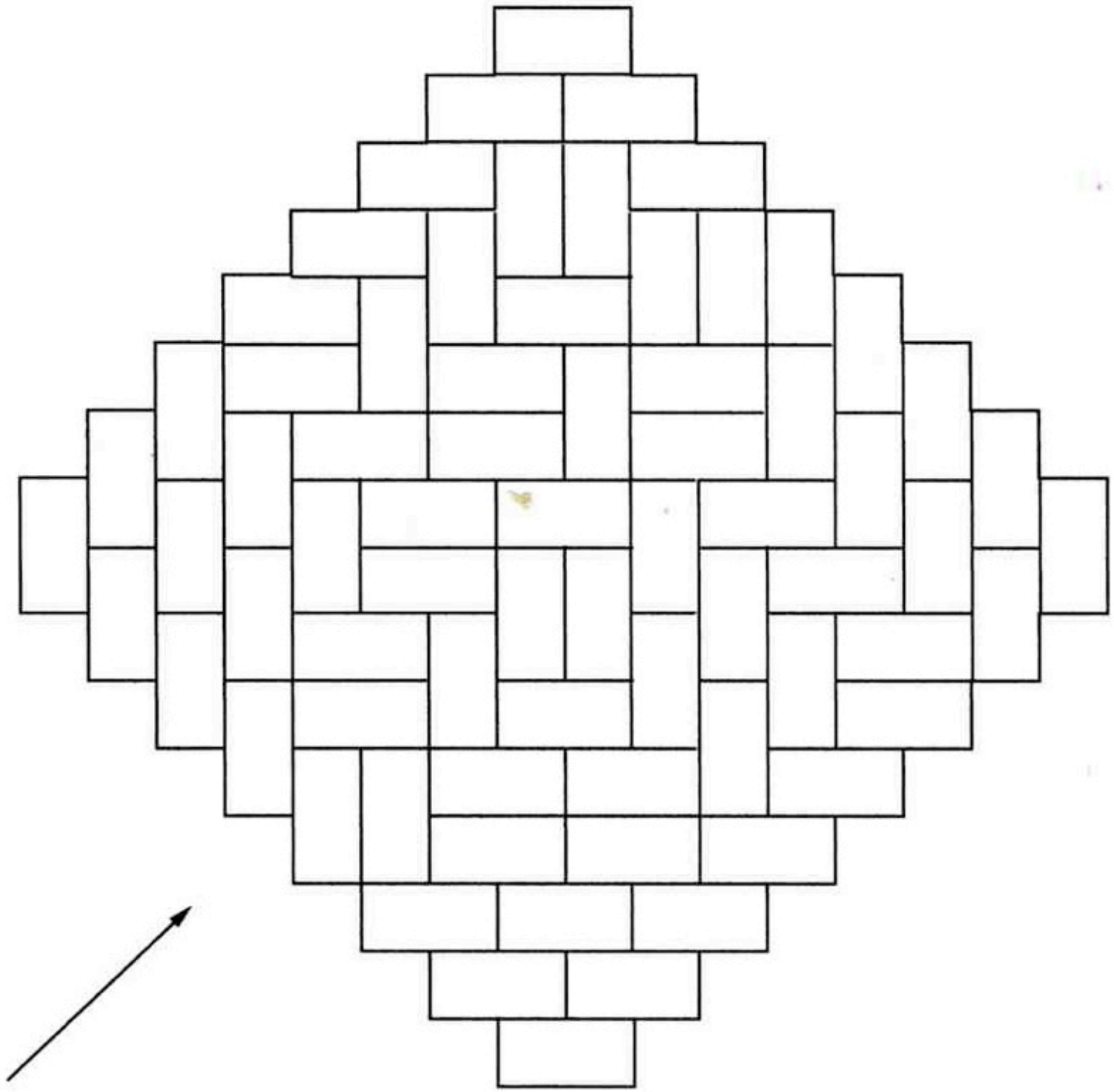
quadratic algebra Q (of a certain type)

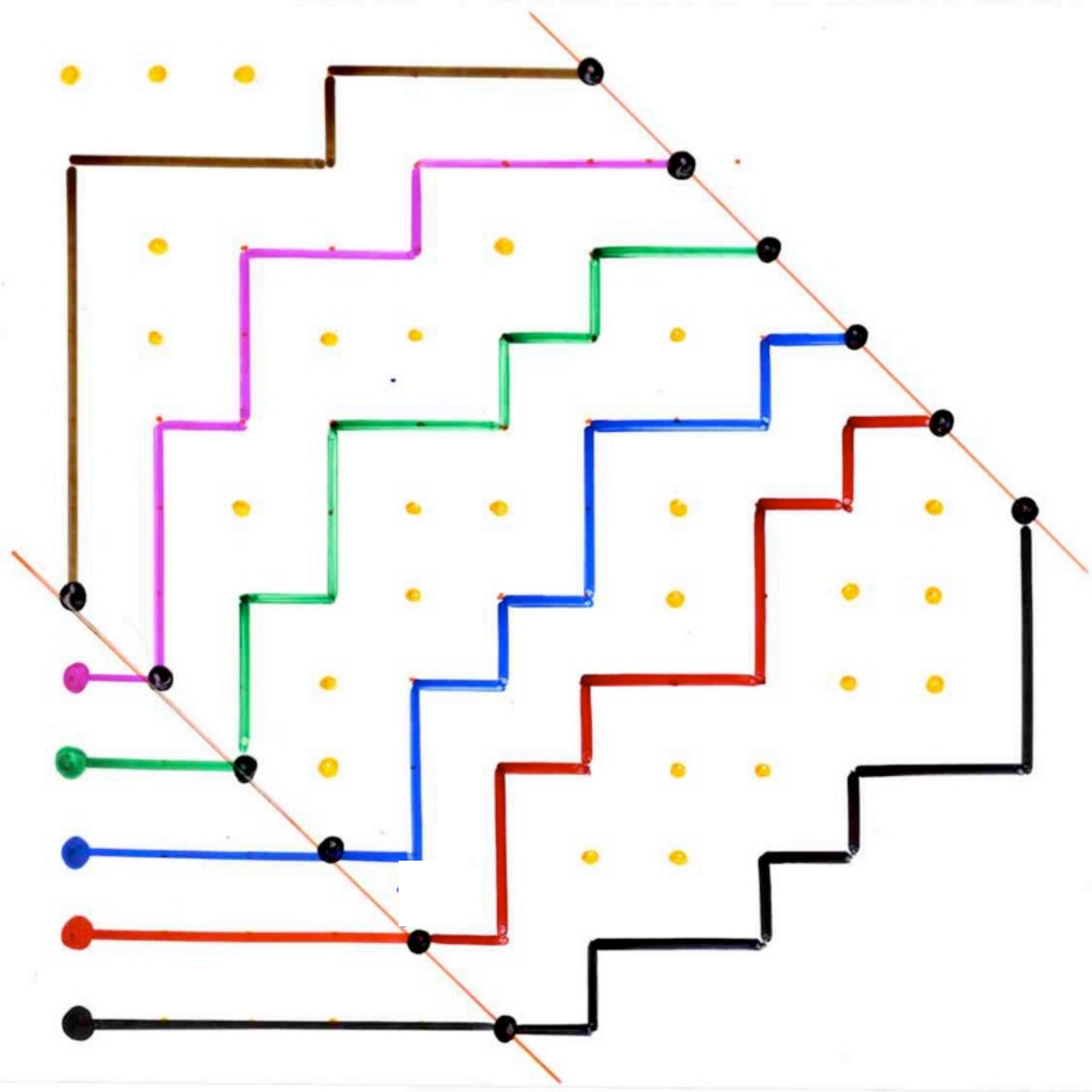
$$\downarrow$$

Q- tableaux


b
a
c
c
a
b

Aztec
tiling





representation of the quadratic algebra Q
with combinatorial operators

for Q: $UD = DU + I$

The Robinson-Shensted-Knuth correspondence

RSK

representation
by operators

combinatorial
objects
on a 2d lattice

Physics

bijections

"normal ordering"

$UD = DU + Id$

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PASEP

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alternative tableaux

RSK

$\longleftrightarrow$

pairs of Tableaux Young

quadratic algebra Q

commutations
rewriting rules

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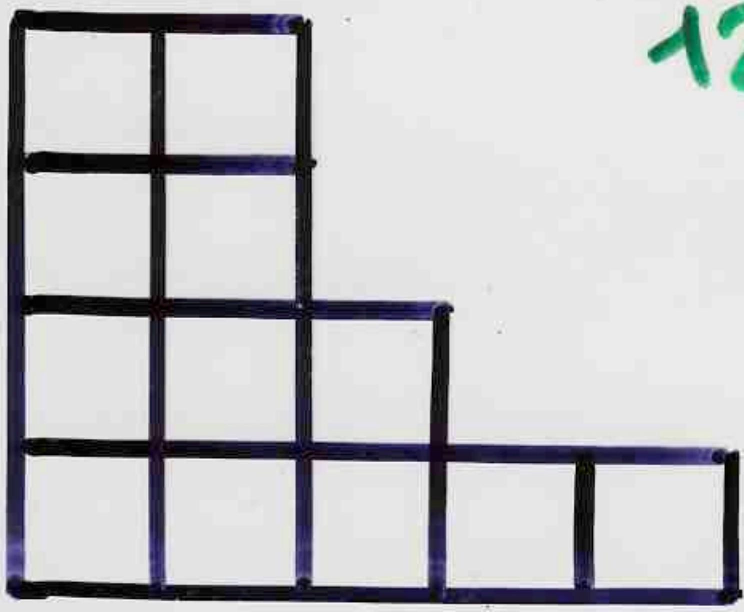
$12 = n = 5 + 3 + 2 + 2$

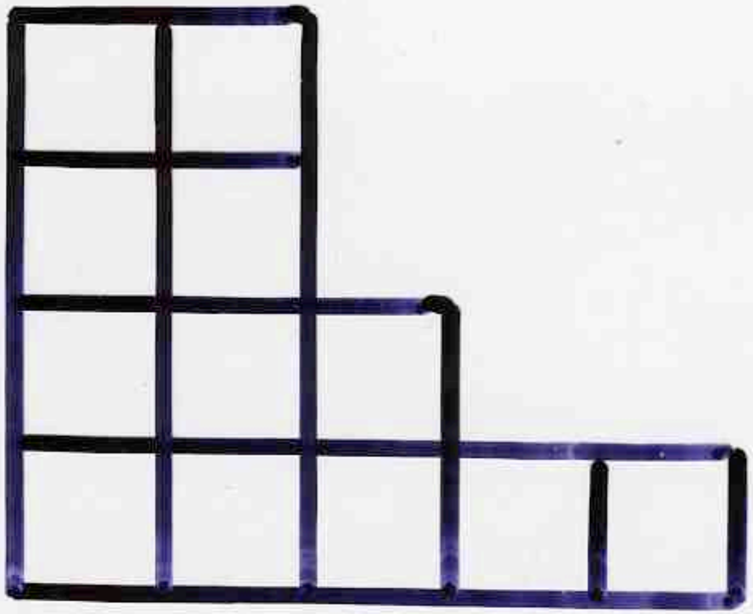


2
2
3
5

12

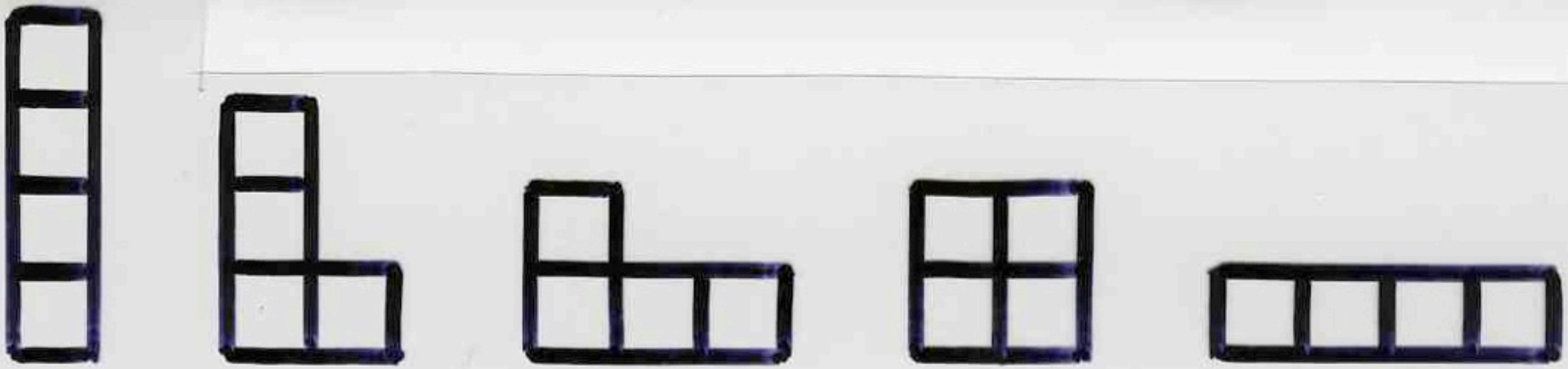
Ferrers
diagram

Partition of n

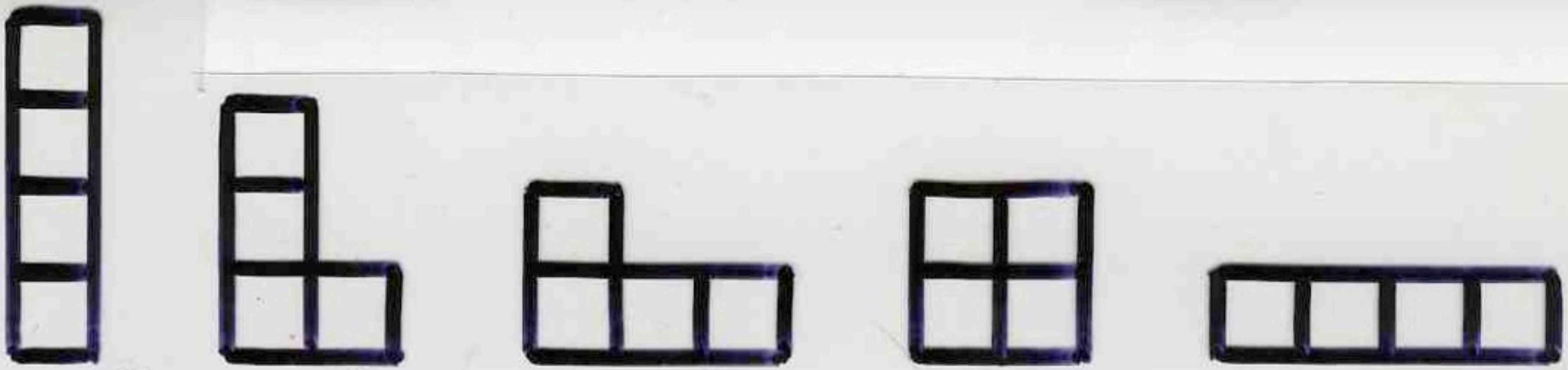


7	12			
6	10			
3	5	9		
1	2	4	8	11

Young tableau



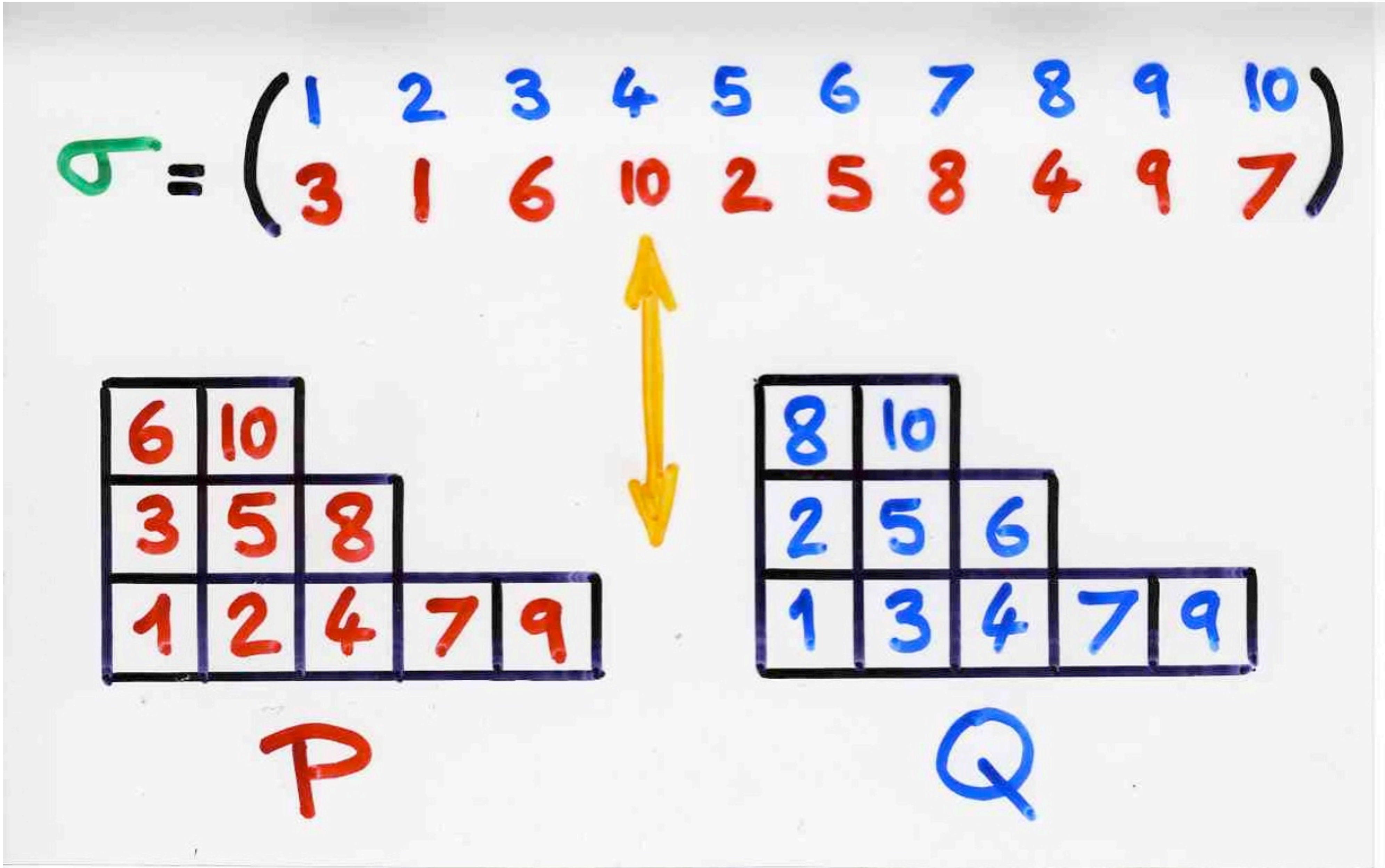
1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$



The Robinson-Schensted correspondence between permutations and pair of (standard) Young tableaux with the same shape

The Robinson-Shensted-Knuth correspondence

RSK

related to

the representation theory of finite groups

symmetric group of permutations

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

φ représentation irréductible

$$n! = \sum_{\lambda} f_\lambda^2$$

ordre groupe fini $\mathfrak{S}_n$

λ représentations irréductibles

degré

The Robinson-Shensted-Knuth correspondence

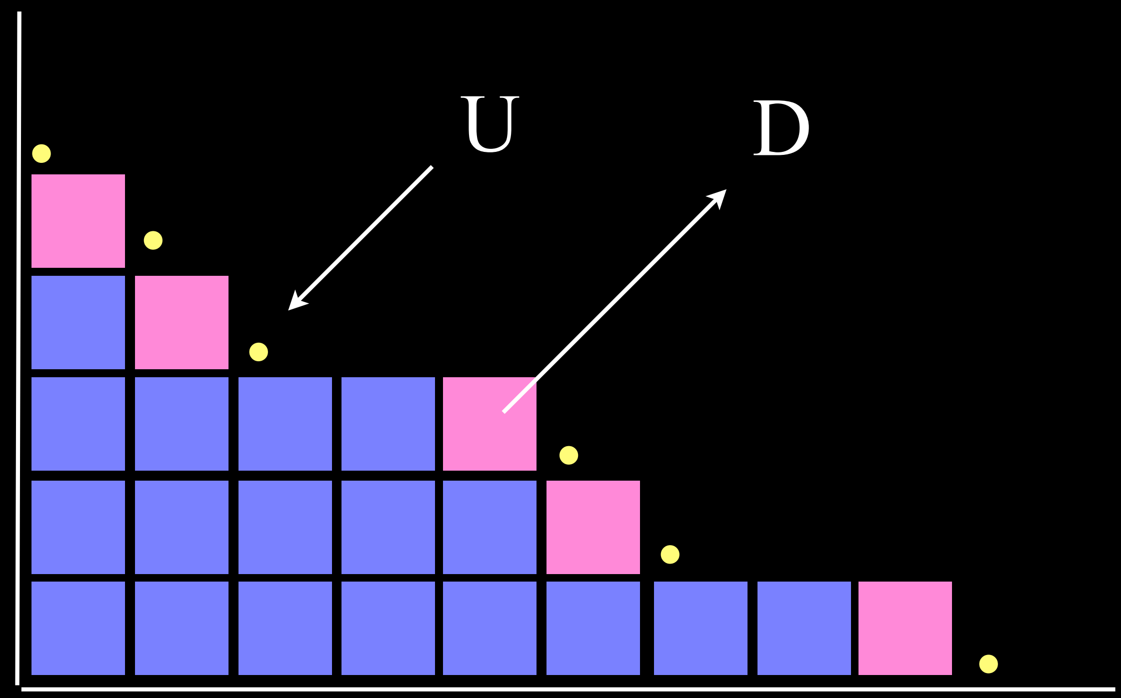
RSK

- Schensted's insertions
- geometric version with "shadow lines" (X.V.)
- Schützenberger "jeu de taquin"
- Fomin "local rules" or "growth diagrams"

for Q: $UD=DU+I$

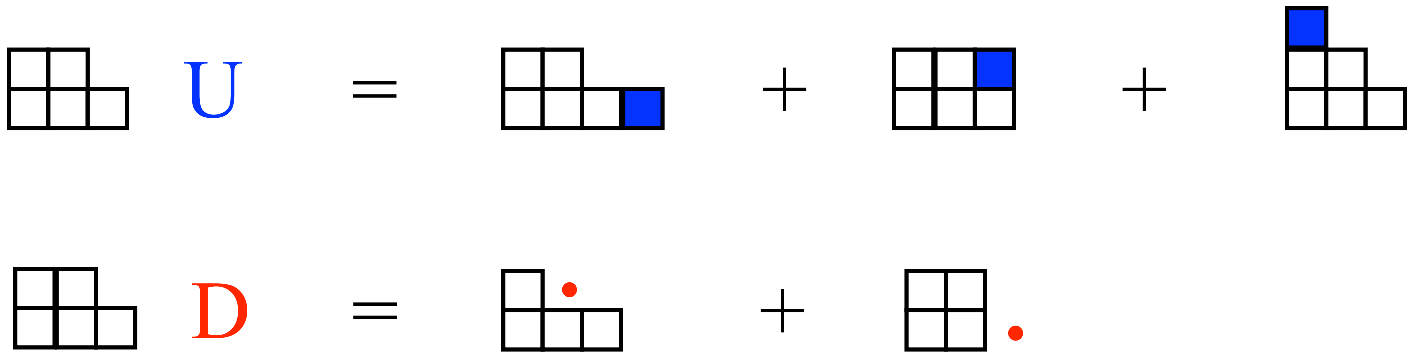
representation of the quadratic algebra Q
with combinatorial operators

Operators U and D

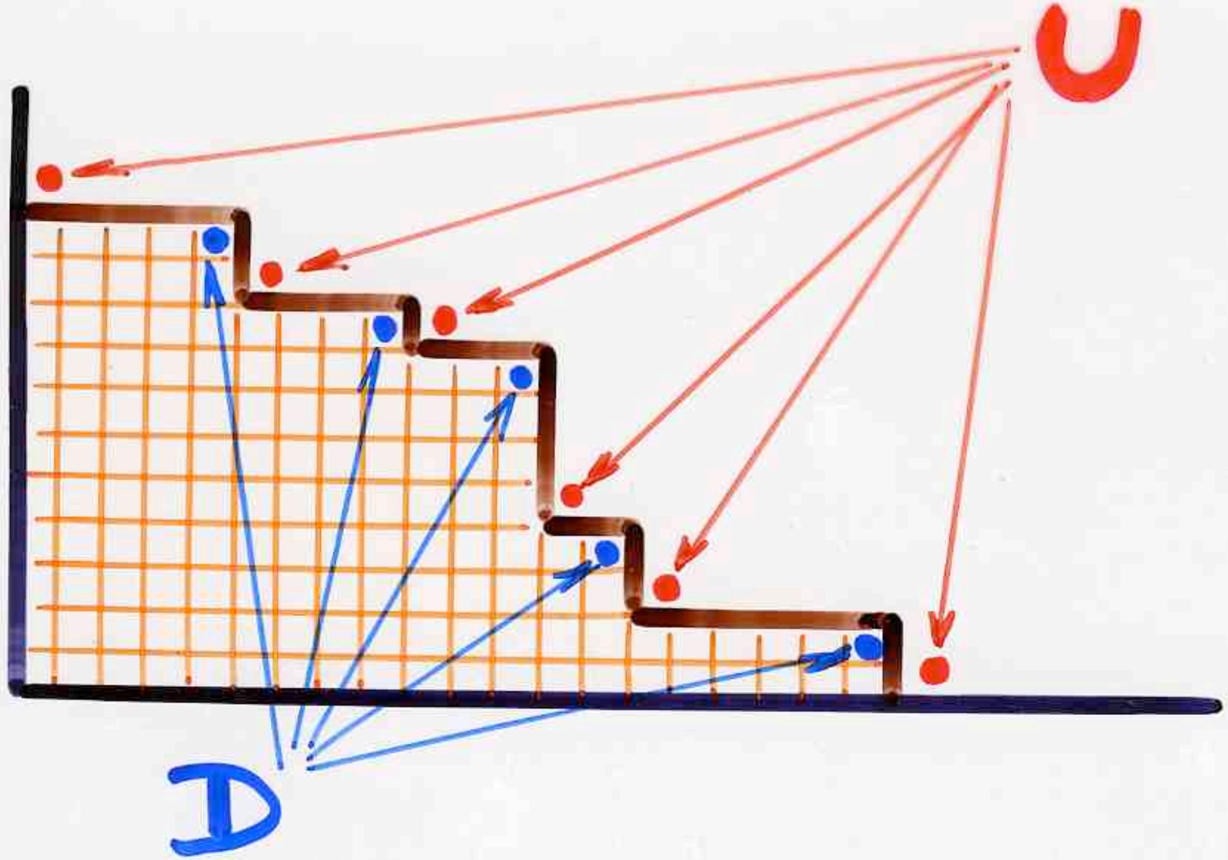


adding
or deleting
a cell in
a Ferrers
diagram

Young lattice


U
D

$$UD = DU + I$$



for the PASEP algebra

$$DE = qED + E + D$$

representation with operators
related to the combinatorial theory
of orthogonal polynomials

representation
by operators

combinatorial
objects
on a 2d lattice

Physics

data structures
"histories"

bijections

orthogonal
polynomials

"normal ordering"

$UD = DU + Id$

Weyl-Heisenberg

towers placements

permutations $\longleftrightarrow$ pairs of Tableaux Young

RSK

$DE = qED + E + D$

PASEP

alternative tableaux $\longleftrightarrow$ permutations

Laguerre histories

quadratic algebra Q

Q-tableaux
ex: ASM,
(alternating sign matrices)
tilings
non-crossing paths

commutations
rewriting rules

Combinatorial theory of

(formal) orthogonal polynomials

n! moments of laguerre polynomials

bijection permutations --- Laguerre histories

(certain weighted paths)

bijection alernative tableaux --- Laguerre histories

stationary probabilities for the PASEP, q-Laguerre

other tableaux: permutation tableaux
tree-like tableaux
staircase tableaux

TASEP $q=0$ $DE=E+D$

Catalan alternative tableaux

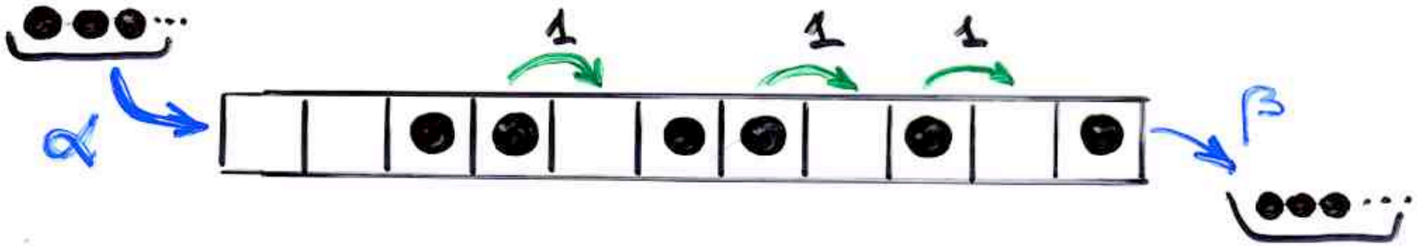
bijection with binary trees

relation with the Loday-Ronco Hopf algebra on binary trees

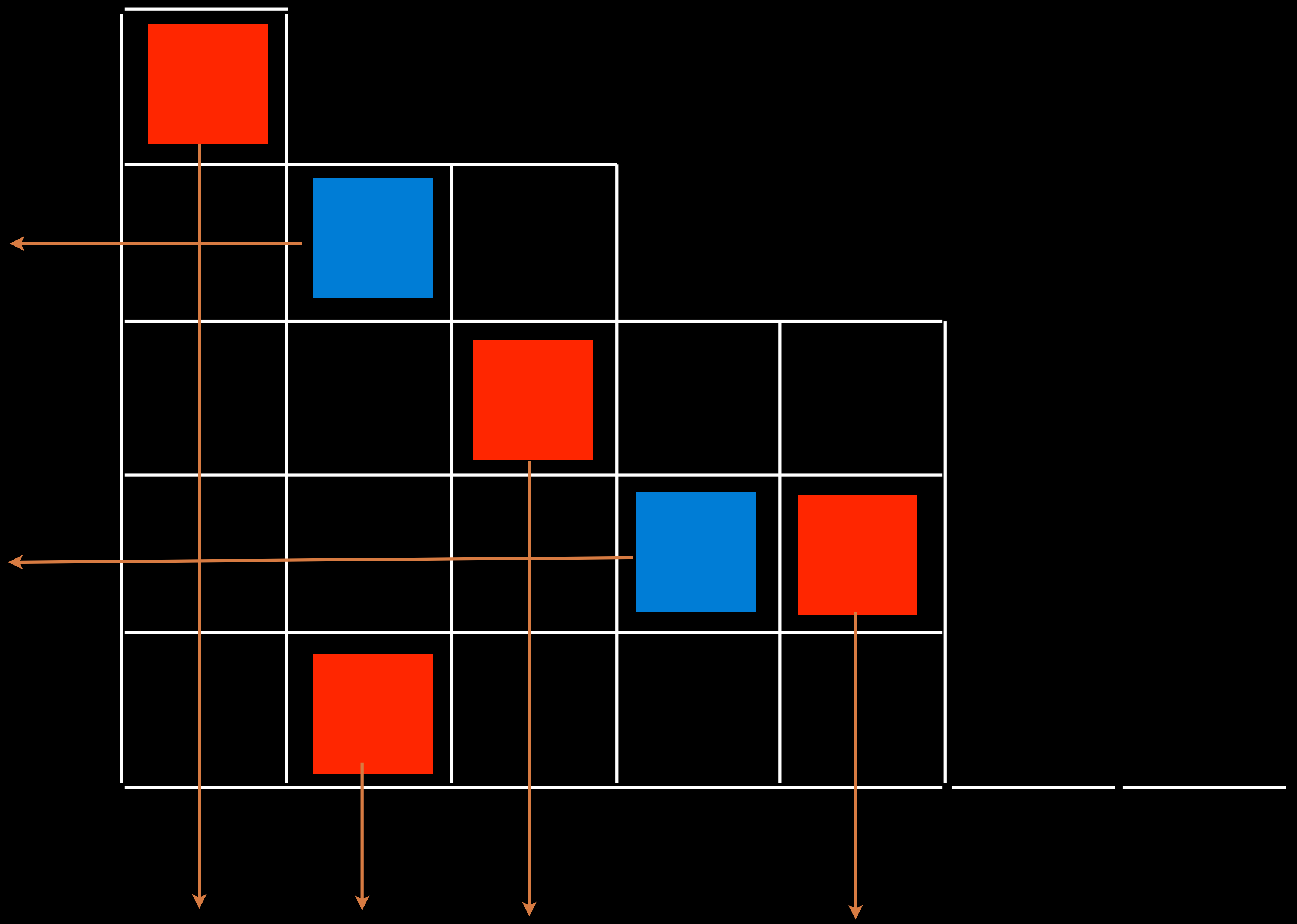
Catalan alternative tableaux

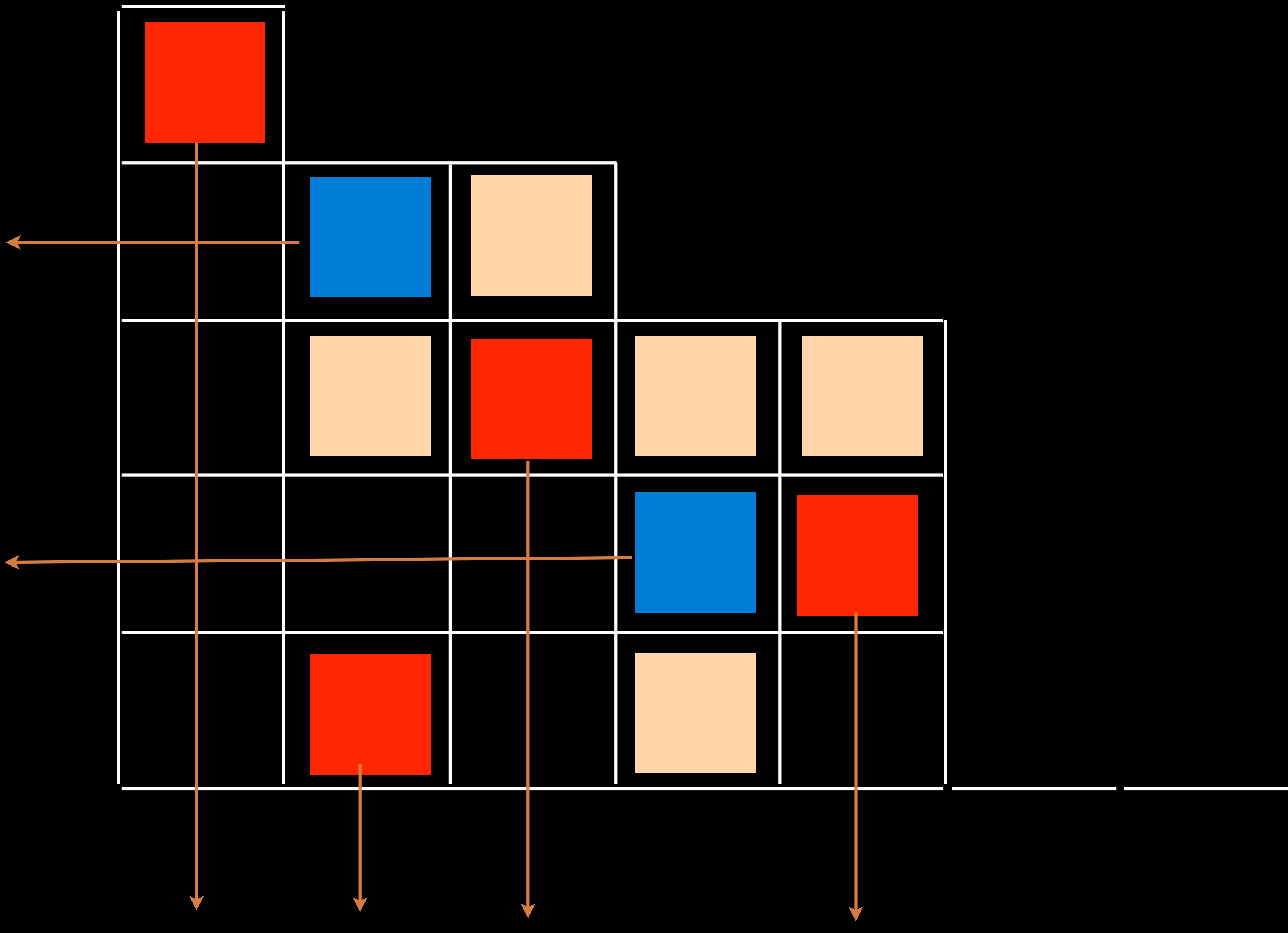
TASEP

"totally asymmetric exclusion process"



q=0

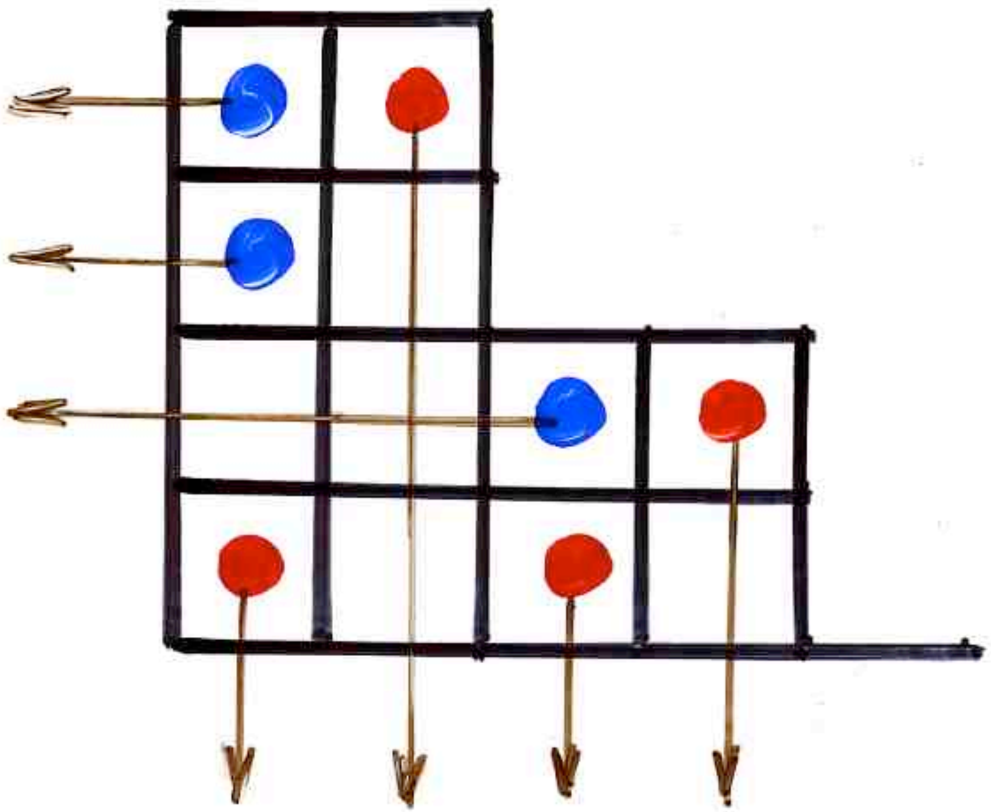




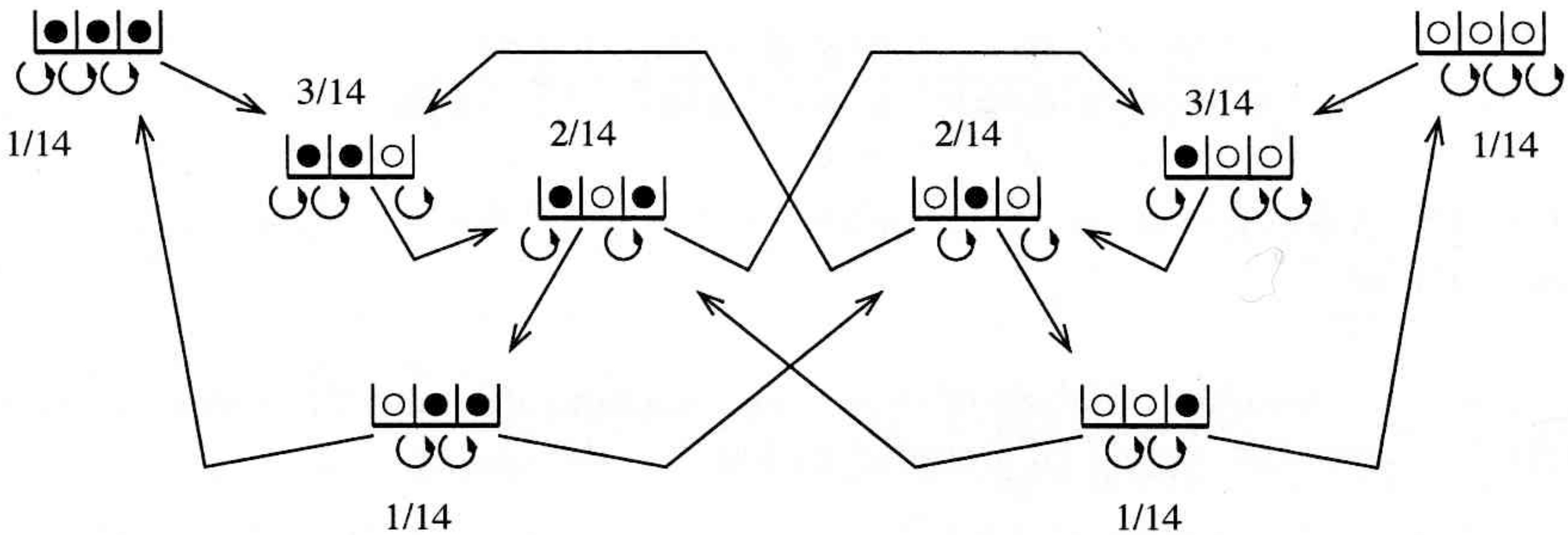
Def Catalan alternative tableau T

alt. tab. without cells $\boxed{\times}$

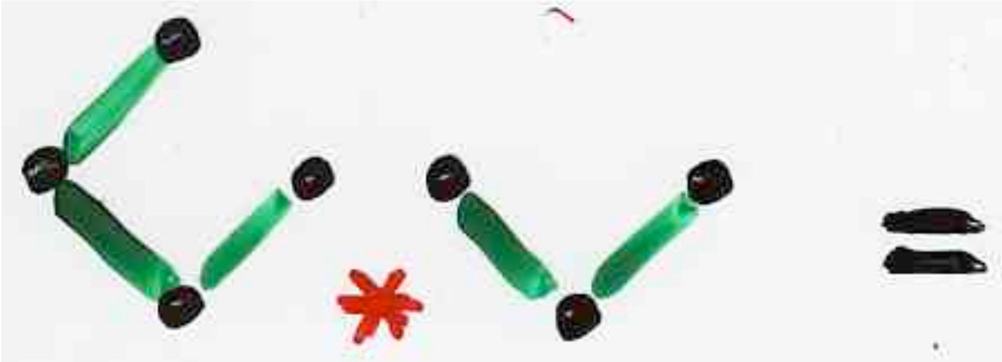
i.e: every empty cell is below a red cell or on the left of a blue cell

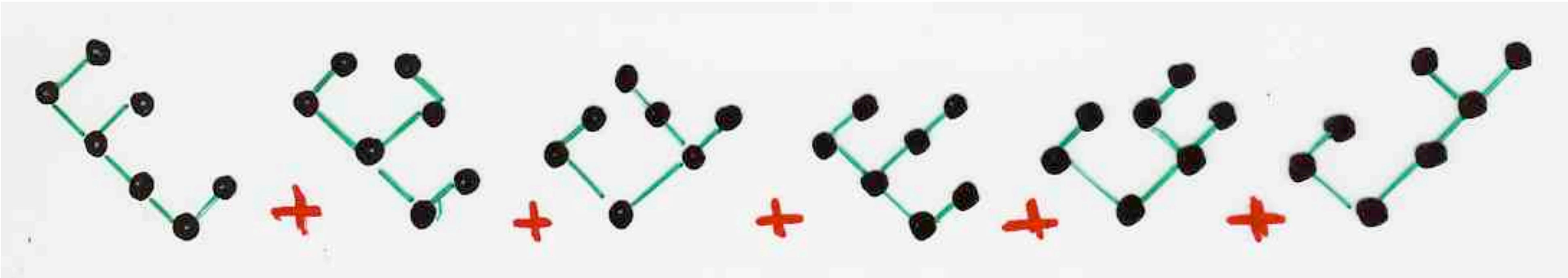


Markov chains



stationary
probabilities





product of binary trees
in the Loday-Ronco algebra

The cellular Ansatz

quadratic algebra Q

- Q- tableaux

- from a representation of Q, construction of bijections

"The cellular Ansatz"

Physics

"normal ordering"

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quadratic algebra Q

commutations

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combinatorial objects on a 2d lattice

towers placements

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ex: ASM,

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tilings

non-crossing paths

8-vertex

representation by operators

data structures "histories"

bijections

orthogonal polynomials

RSK

$\longleftrightarrow$ pairs of Tableaux Young

$\longleftrightarrow$ permutations

Laguerre histories

?

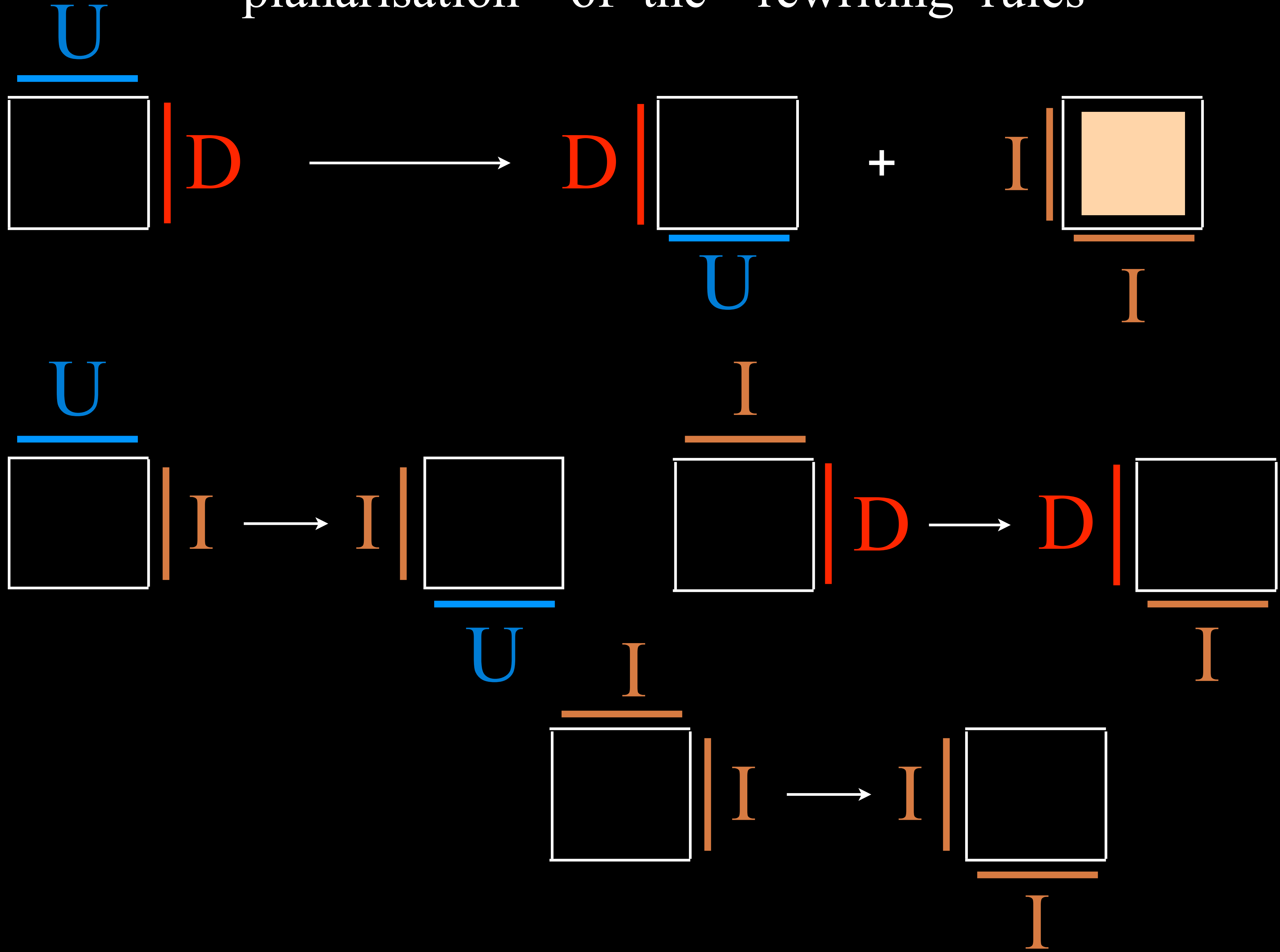
Koszul algebras

duality ?

why the name "cellular Ansatz" ?

"planarization" on a grid of the rewriting rules

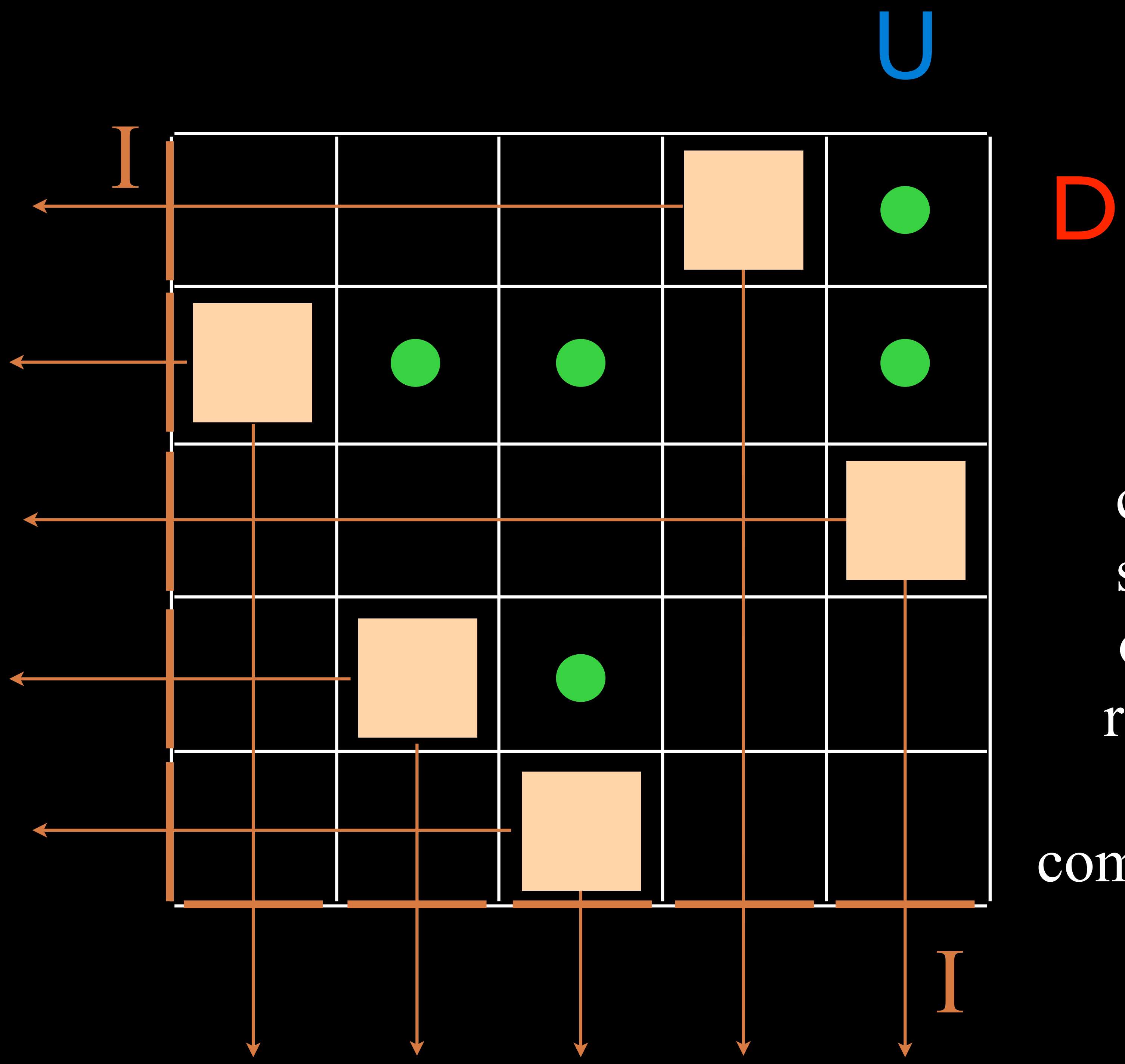
“planarisation” of the “rewriting rules”



$$\begin{cases} UD = DU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

$$\begin{cases} UD \to DU \\ U I_v \to I_v U \\ I_h D \to D I_h \\ I_h I_v \to I_v I_h \end{cases} \qquad UD \to I_v I_h$$

rewriting rules



see the
complete
sequence
of planar
rewritings
on
complementary
slides

The cellular Ansatz

quadratic algebra Q (of a certain type)

(1) "planarization" on a grid of the rewriting rules

Q- tableaux planar automata

(2) "planarization" on a grid of the bijection constructed from the representation of the algebra Q

"The cellular Ansatz"

combinatorial objects on a 2d lattice

representation by operators

Physics

data structures "histories"

bijections

orthogonal polynomials

"normal ordering"

UD = DU + Id

Weyl-Heisenberg

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PASEP

towers placements

permutations $\longleftrightarrow$ RSK pairs of Tableaux Young

alternative tableaux $\longleftrightarrow$ permutations

Laguerre histories

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?

commutations

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planarization

planar automata

Koszul algebras

duality ?

The cellular Ansatz

quadratic algebra Q (of a certain type)

(1) "planarization" on a grid of the rewriting rules

Q- tableaux planar automata

(2) "planarization" on a grid of the bijection constructed from the representation of the algebra Q

(3) how to guess a representation: demultiplication of the commuation relations