Combinatorics and Physics

Chapter 0
Introduction
Overview of the course
(part 4)

IIT-Madras
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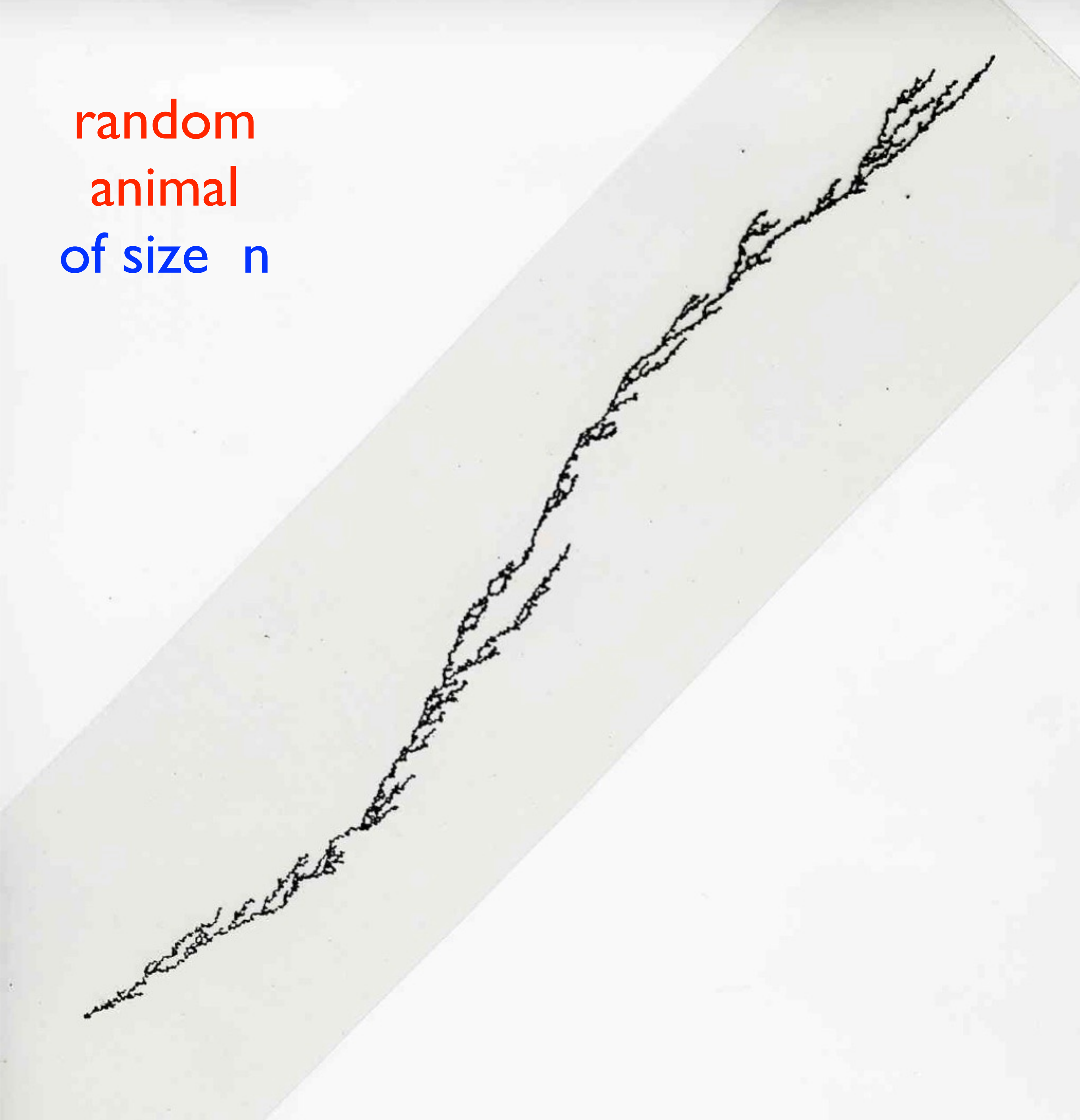
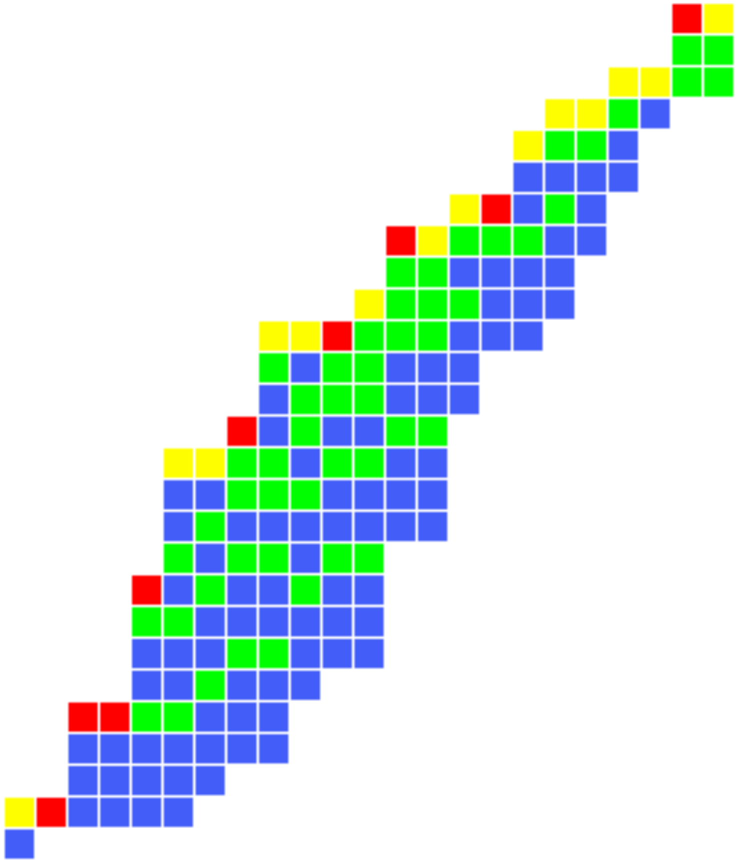
Xavier Viennot
CNRS, LaBRI, Bordeaux

random combinatorial structures

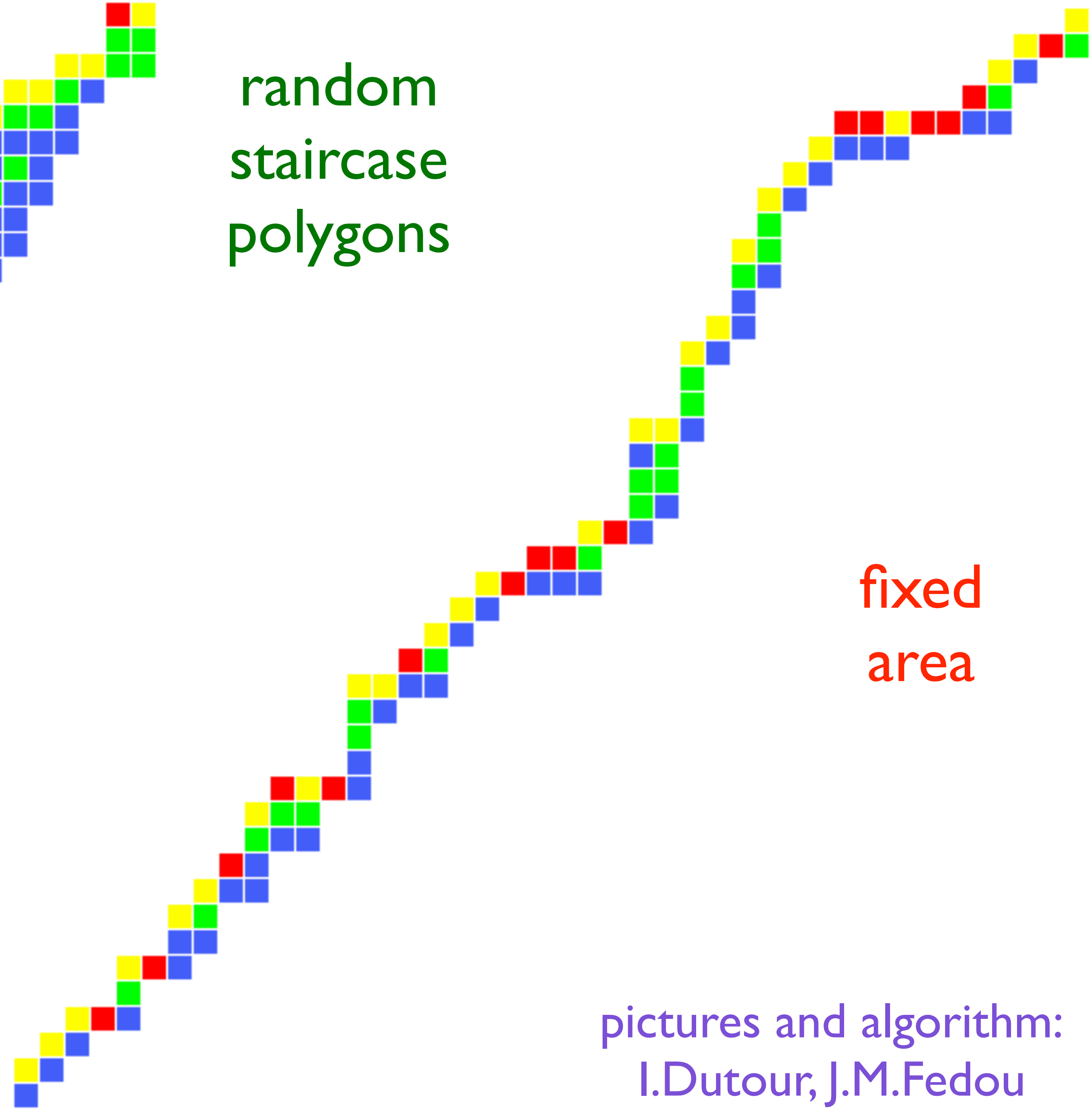
random
animal
of size n



random
staircase
polygons

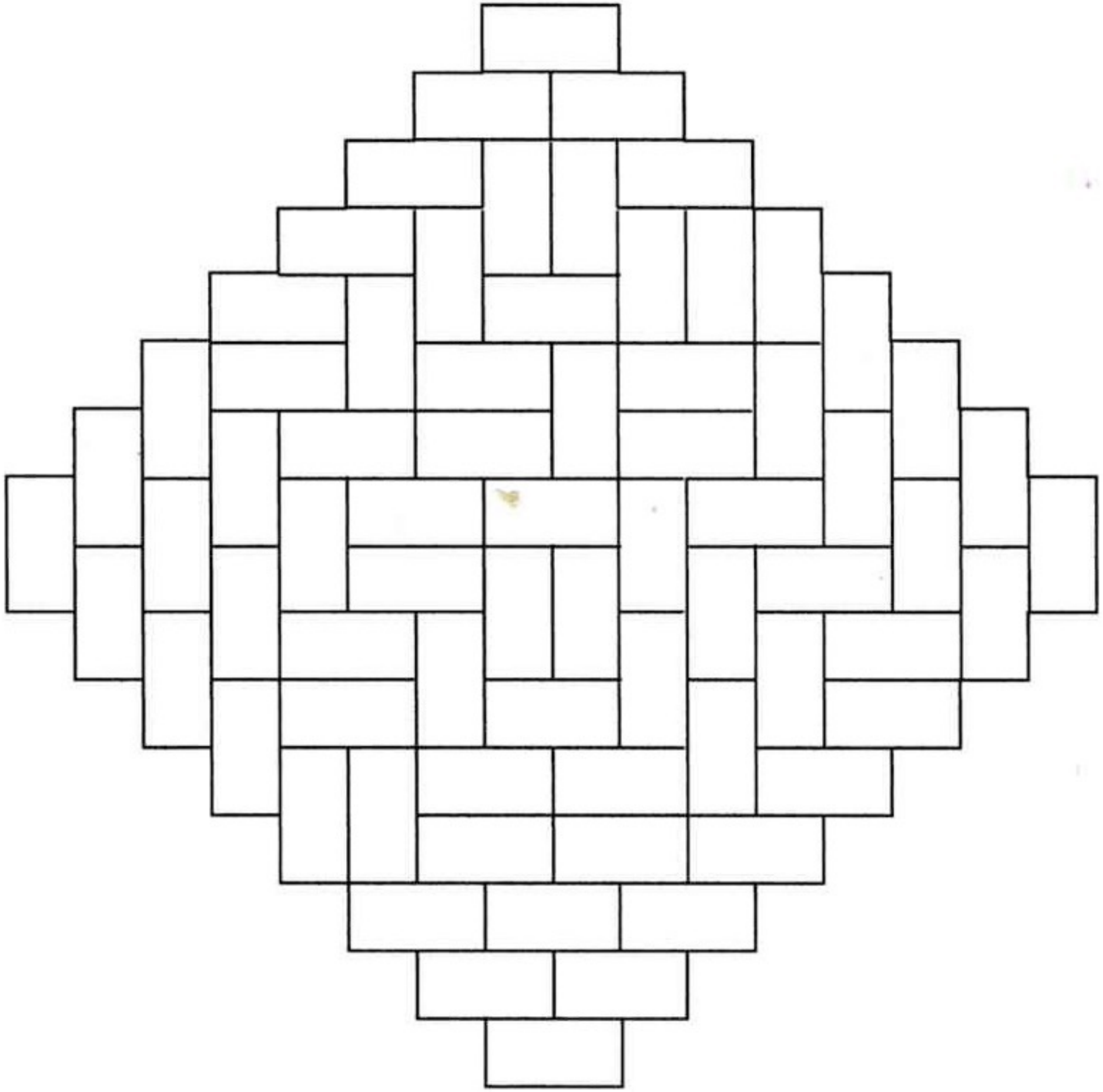


fixed
perimeter

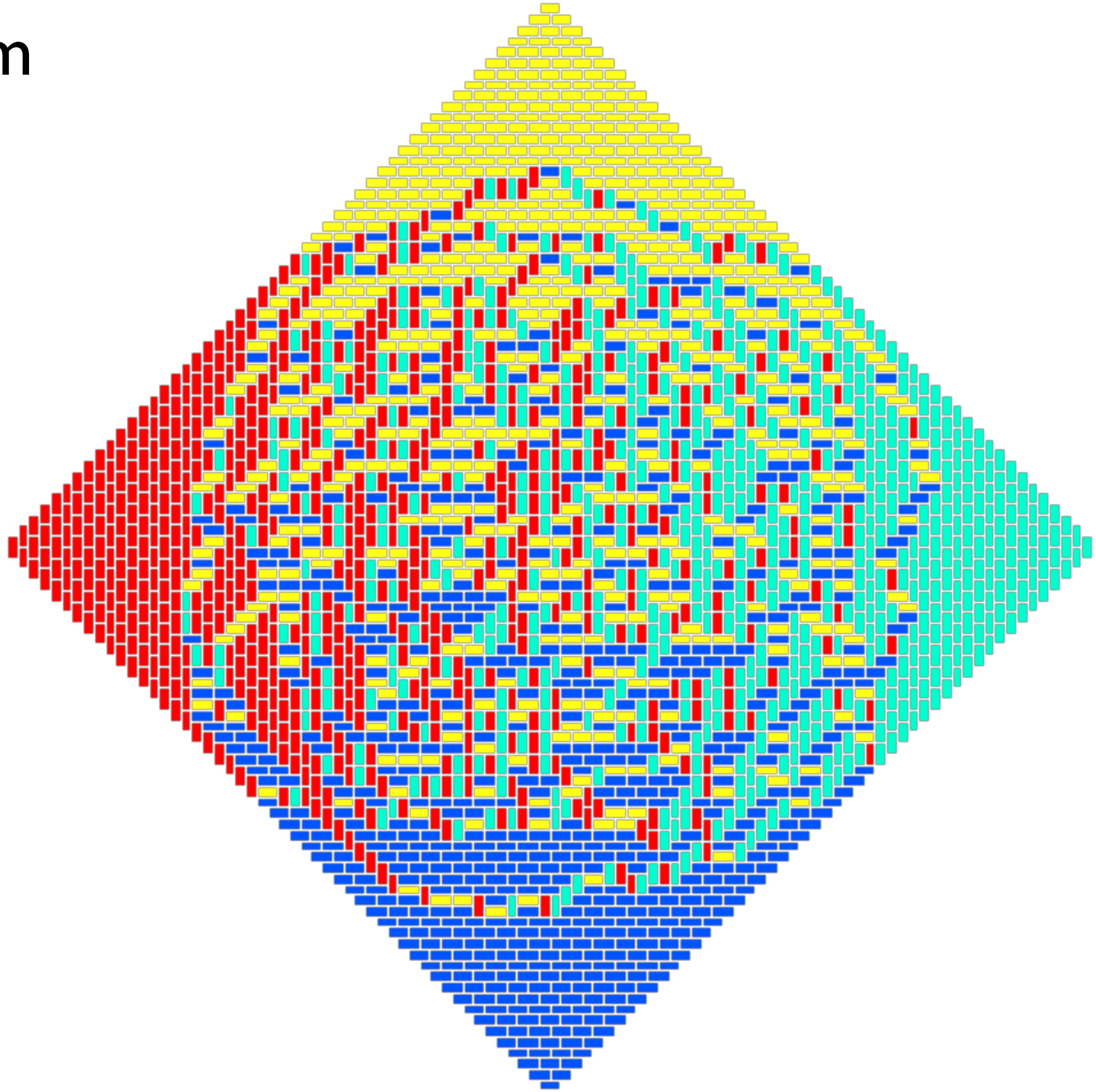


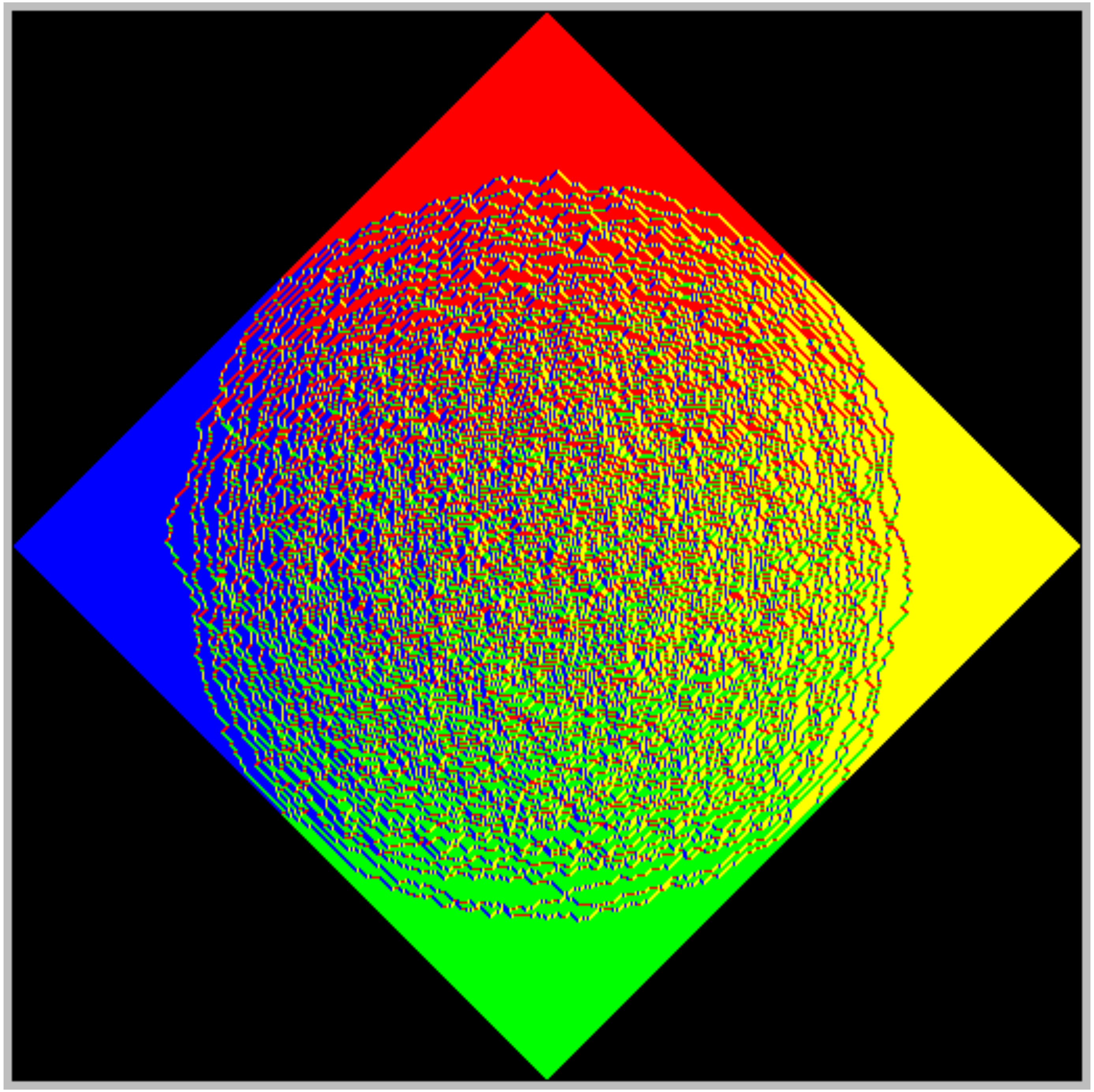
fixed
area

pictures and algorithm:
I.Dutour, J.M.Fedou



the arctic circle theorem

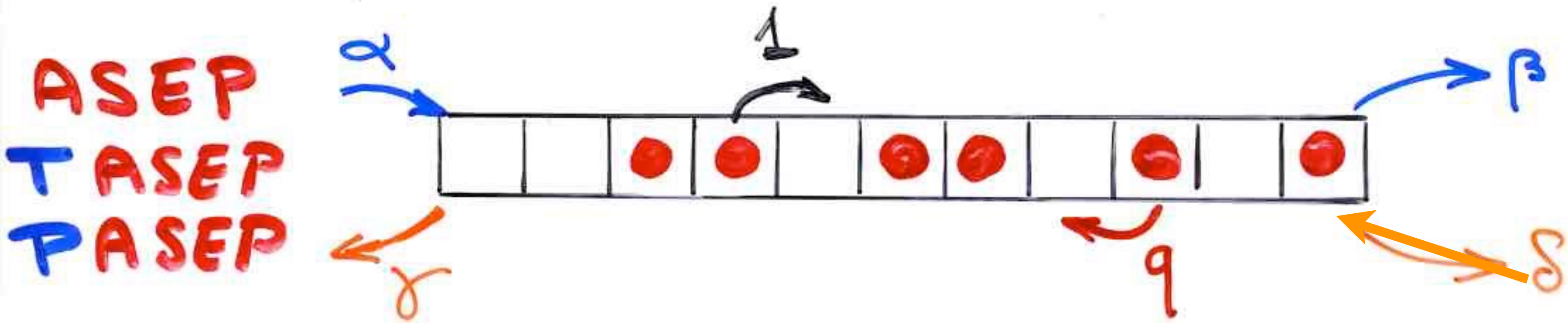




Dynamical systems

The PASEP model

(Partially asymmetric exclusion process)



ASEP
TASEP
PASEP
α
1
β
γ
q
δ

boundary induced phase transitions

molecular diffusion
linear array of enzymes
biopolymers
traffic flow
------
formation of shocks
---

non-equilibrium statistical mechanics

... relaxation → stationary state

states $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$

$\tau_i = \begin{cases} 1 & \text{site } i \text{ occupied} \\ 0 & \text{site } i \text{ empty} \end{cases}$

unique stationary state $\frac{d}{dt} P_n(\tau_1, \ldots, \tau_n) = 0$

Derrida, Evans, Hakim, Pasquier (1993)

- Orthogonal polynomials

→ Sasamoto (1999)

→ Blythe, Evans, Colaiori, Essler (2000)

q-Hermite polynomial

$\alpha, \beta, q \qquad \gamma = \delta = 1$

$$D = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}$$
$$E = \frac{1}{1-q} + \frac{1}{\sqrt{1-q}}\,\hat{a}^{+}$$
$$\hat{a}\,\hat{a}^{+} - q\,\hat{a}^{+}\,\hat{a} = 1$$

→ Uchiyama, Sasamoto, Wadati (2003)

$\alpha, \beta, \gamma, \delta, \qquad q$

Askey-Wilson polynomials

A matrix ansatz

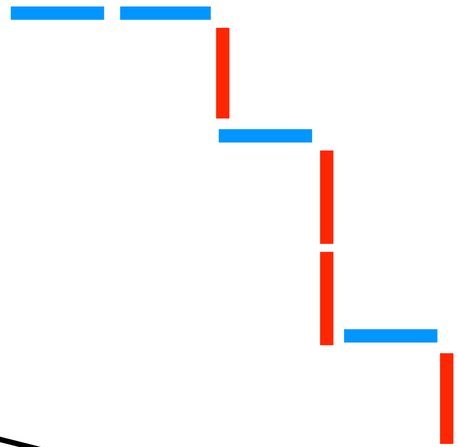
The PASEP algebra

$$DE = qED + E + D$$

$$DE = qED + E + D$$

D D E D E E D E

D DE (D E) E D E



DDE(E)EDE + qDDE(ED)EDE + DDE(D)EDE

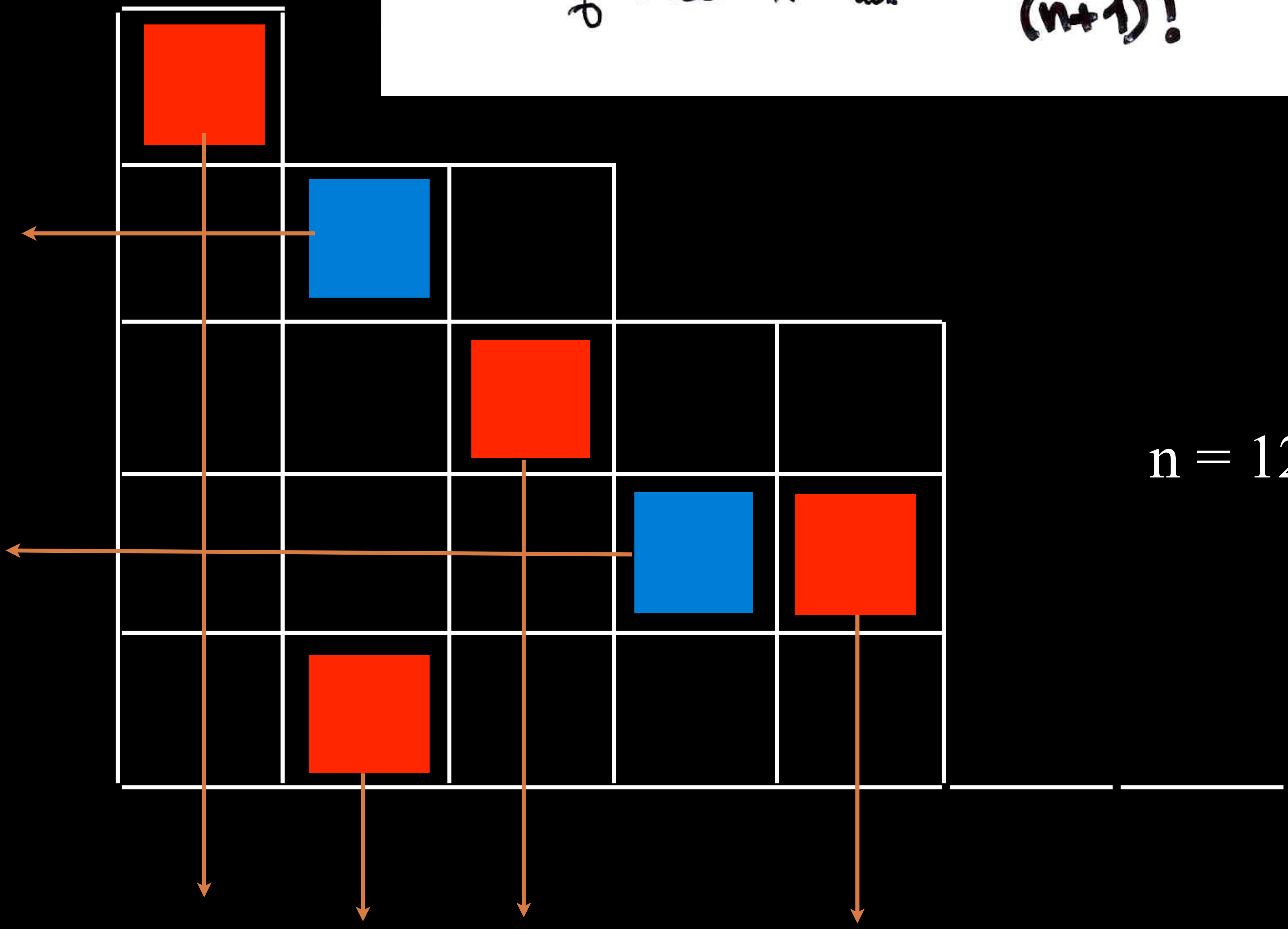
q-analog

$$DE = qED + E + D$$

$$w(E,D) = \sum_{T \text{ alternative tableau with profile } w} q^{k(T)} E^{i(T)} D^{j(T)}$$

alternative tableau

Prop. The number of alternative tableaux of size n is $(n+1)!$



$n = 12$

ex: n=2

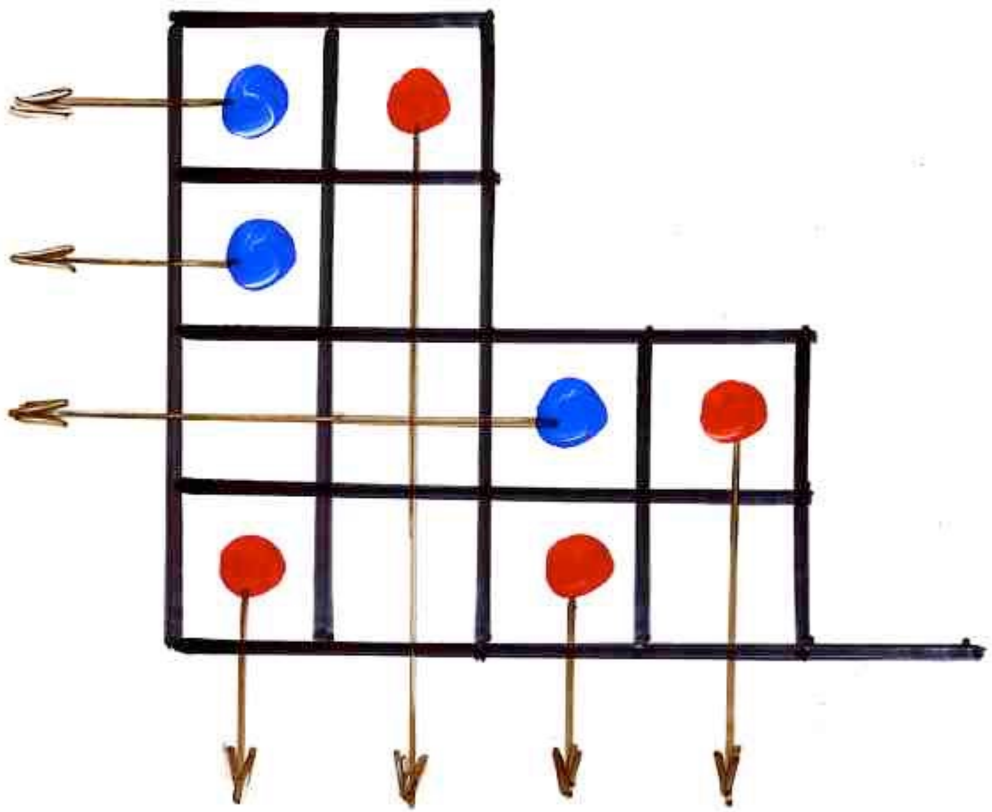
moments of

q- Laguerre polynomials

Def Catalan alternative tableau T

alt. tab. without cells $\boxtimes$

i.e: every empty cell is below a red cell or on the left of a blue cell



combinatorial theory for
(formal) orthogonal polynomials

combinatorial theory for
(analytic) continued fractions

Orthogonal polynomials

Def. $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

orthogonal iff $\exists\ f : \mathbb{K}[x] \longrightarrow \mathbb{K}$

linear functional

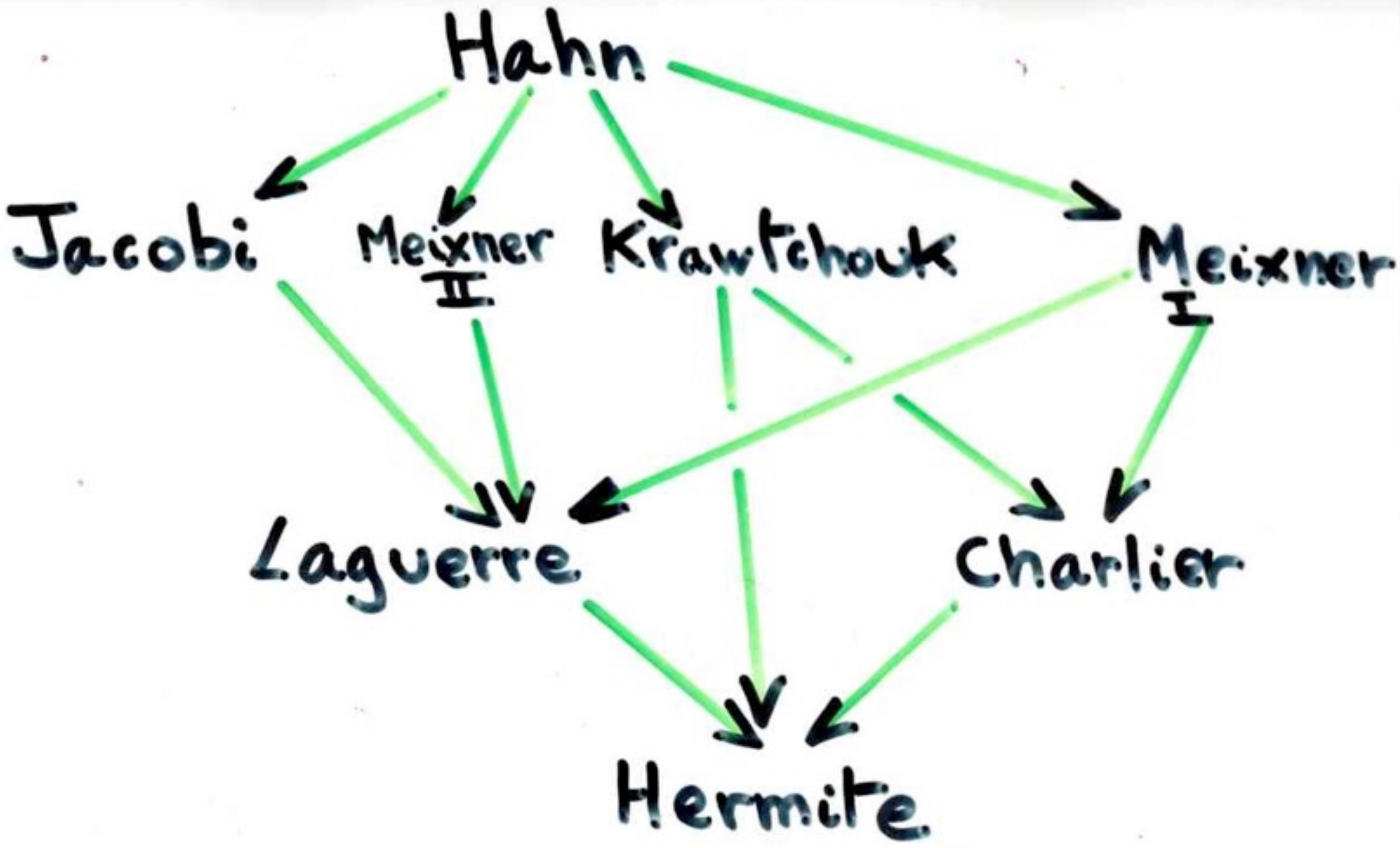
$$\begin{cases} \text{(i)} & \deg(P_n(x)) = n & (\forall n \geq 0) \\ \text{(ii)} & f(P_k P_l) = 0 & \text{for } k \neq l \geq 0 \\ \text{(iii)} & f(P_k^2) \neq 0 & \text{for } k \geq 0 \end{cases}$$

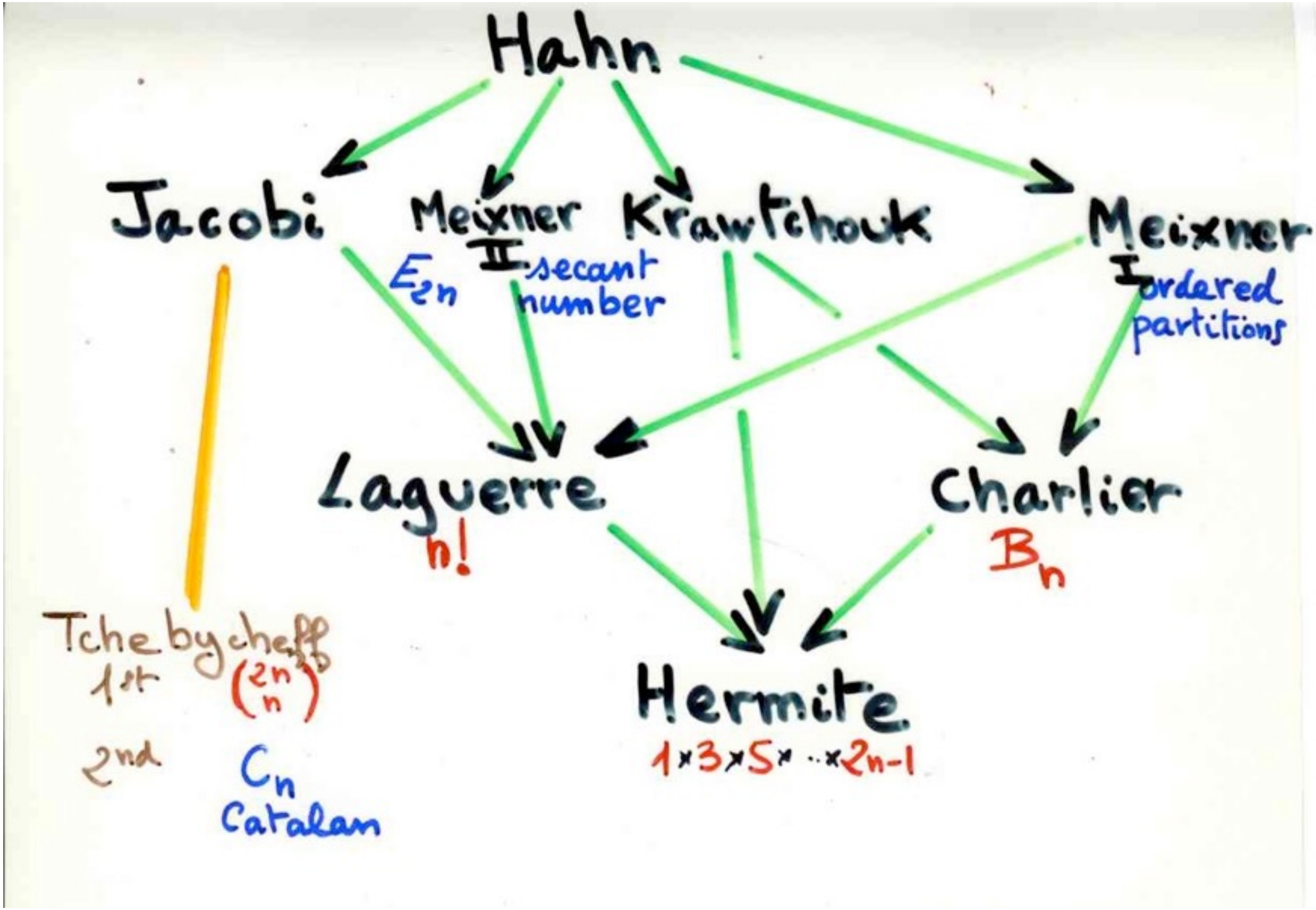
$$\mathcal{f}(x^n) = \mu_n \qquad (n \geqslant 0)$$

moments

$$\mathcal{f}(PQ) = \int_a^b P(x)\,Q(x)\,d\mu$$

mesure


Hahn
Jacobi
Meixner II
Krawtchouk
Meixner I
Laguerre
Charlier
Hermite



Hahn
Jacobi
Meixner II
E_{2n}
secant number
Krawtchouk
Meixner I
ordered partitions
Laguerre
$n!$
Charlier
B_n
Tchebycheff
1st
$\binom{2n}{n}$
2nd
C_n
Catalan
Hermite
$1\times3\times5\times\cdots\times2n-1$

- introduction to enumerative and bijective combinatorics
- non-crossing paths, tilings, determinants and Young tableaux. The LGV Lemma.

- introduction to the theory of heaps of pieces: the 3 basics lemma
- heaps of pieces and statistical mechanics: directed animals, gas models, q-Bessel functions in physics
- heaps of pieces and 2D Lorentzian quantum gravity

- combinatorics of the PASEP), relation with orthogonal polynomials algebraic combinatorics:

Young tableaux

and

representation of the symmetric group

G fini

$$|G| = \sum_{\varphi} \deg^2(\varphi)$$

représentation
irréductible

$$n! = \sum_{\lambda} f_\lambda^2$$

ordre
groupe fini
$\mathfrak{S}_n$

représentations
irréductibles

degré

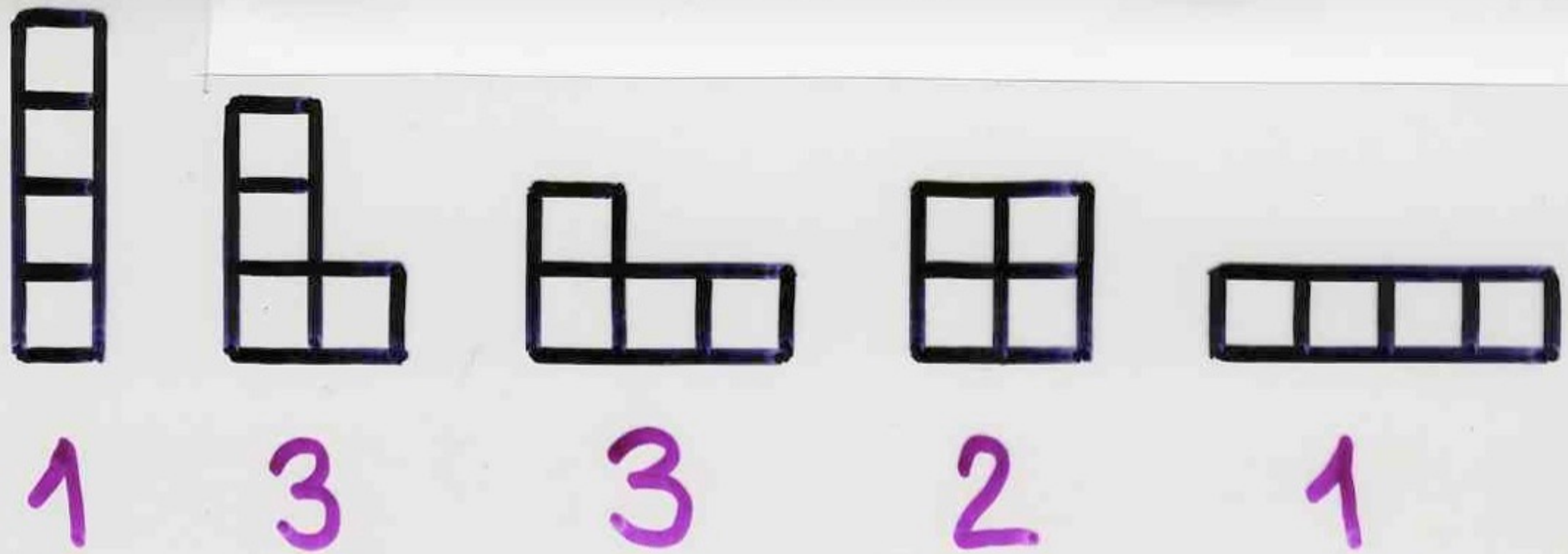
nombre de permutations

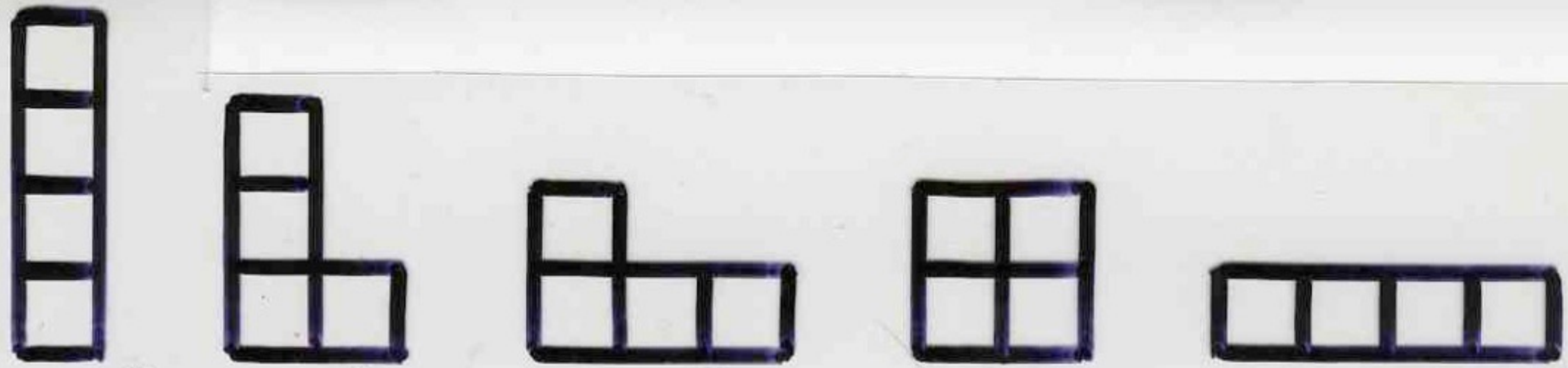
$$n! = \sum_{\lambda} \left(f_{\lambda} \right)^2$$

forme

n cases

nombre de tableaux de Young de forme λ

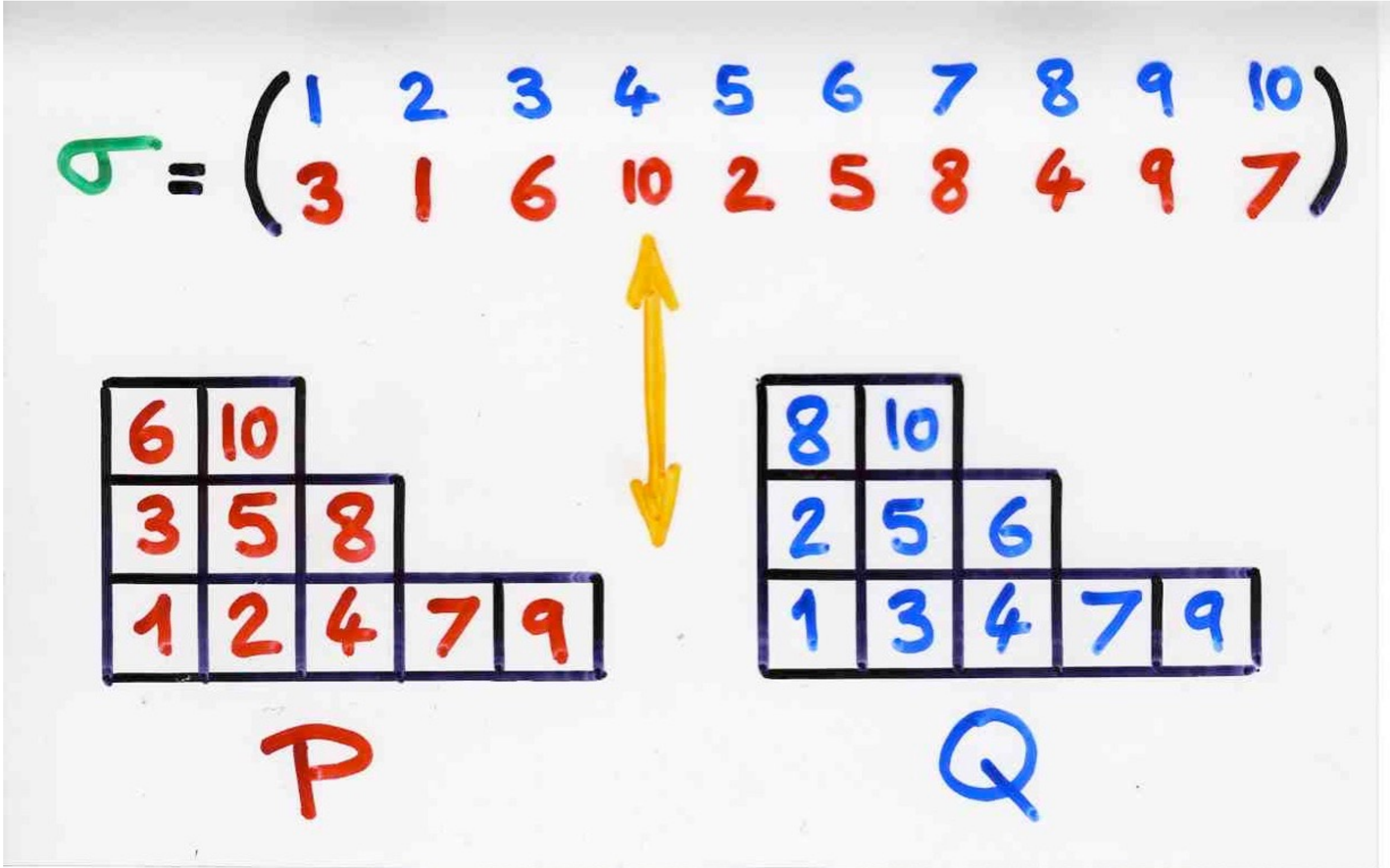

1 3 3 2 1



$$1^2 + 3^2 + 3^2 + 2^2 + 1^2$$

$$= 1 + 9 + 9 + 4 + 1$$

$$= 24 = 4!$$



The Robinson-Schensted correspondence (RSK) between permutations and pair of (standard) Young tableaux with the same shape

Cellular Ansatz

PASEP algebra

$$DE = qED + E + D$$

RSK

$$UD = qDU + I$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 10 & 2 & 5 & 8 & 4 & 9 & 7 \end{pmatrix}$$

6	10			
3	5	8		
1	2	4	7	9

P

8	10			
2	5	6		
1	3	4	7	9

Q

- operator algebra

$$DE - qED = D + E$$

$\longrightarrow$ $n!$

$$UD - DU = 1 \qquad n!$$

$\longrightarrow$ Robinson-Schensted

Fomin

normal ordering

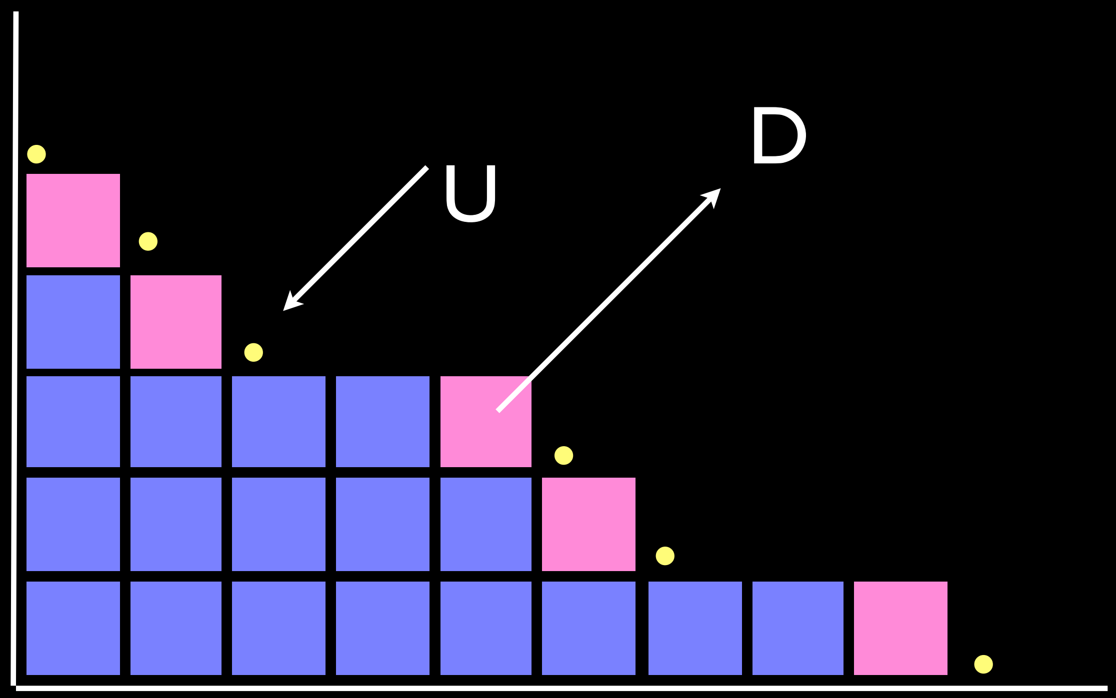
$$UUU \cdots \underline{UD} D \cdots D \longrightarrow$$

$$U^n D^n = \sum_{0 \le i \le n} c_{n,i} \, D^i U^i$$

normal ordering

$$c_{n,0} = n!$$

Operators U and D



Young lattice

Cellular Ansatz

Combinatorial representation of the PASEP algebra

PASEP algebra

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