Combinatorics and Physics

Chapter 1
Introduction to enumerative combinatorics, ordinary generating functions

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generating function
in combinatorics

and

thermodynamic function
in statistical mechanics

phase transition

critical phenomena

Physics

exactly solved model

Baxter (1982)

Ising model

Onsager (1944)

Potts, ice model

Temperley-Lieb (1971)

Baxter (1982)

exactly solved models

Statistical physics

$$F(T) \cong \frac{1}{(T - T_c)^{\alpha}}$$

critical exponent

temperature

critical temperature

thermodynamic function

- $F(t) = \sum_{n \geq 0} a_n t^n$

number of ....

- $a_n \simeq \mu^n n^{-\theta}$

critical exponent

connective constant

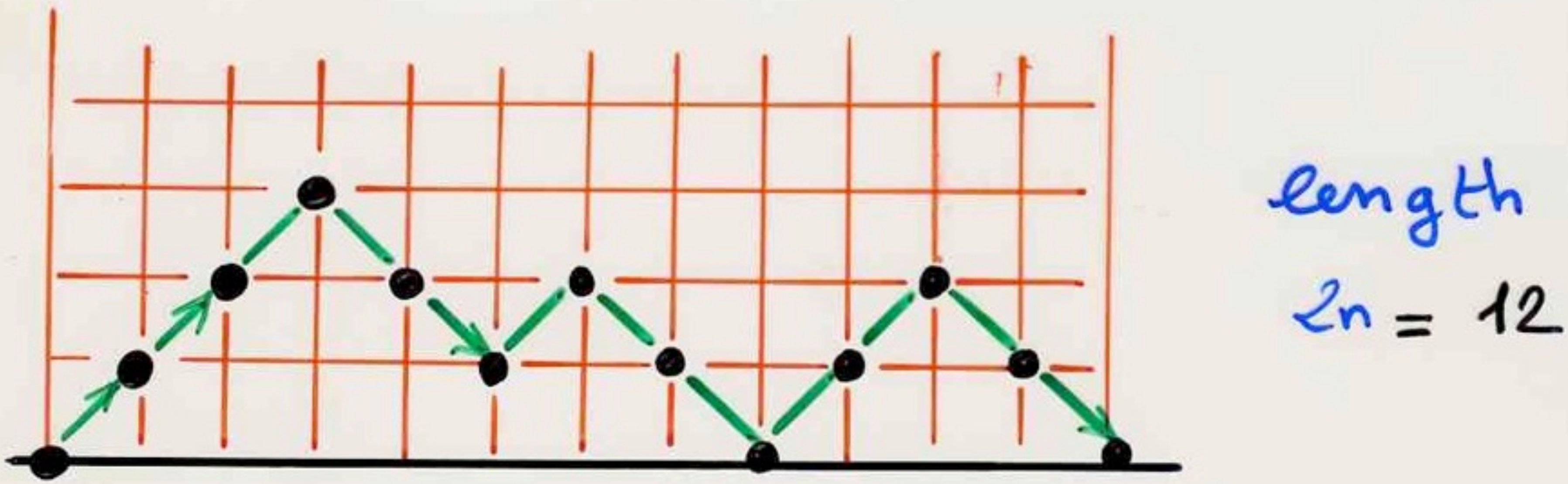
$\mu = \frac{1}{t_c}$

P.Flajolet

analytic combinatorics

Cambridge University Press

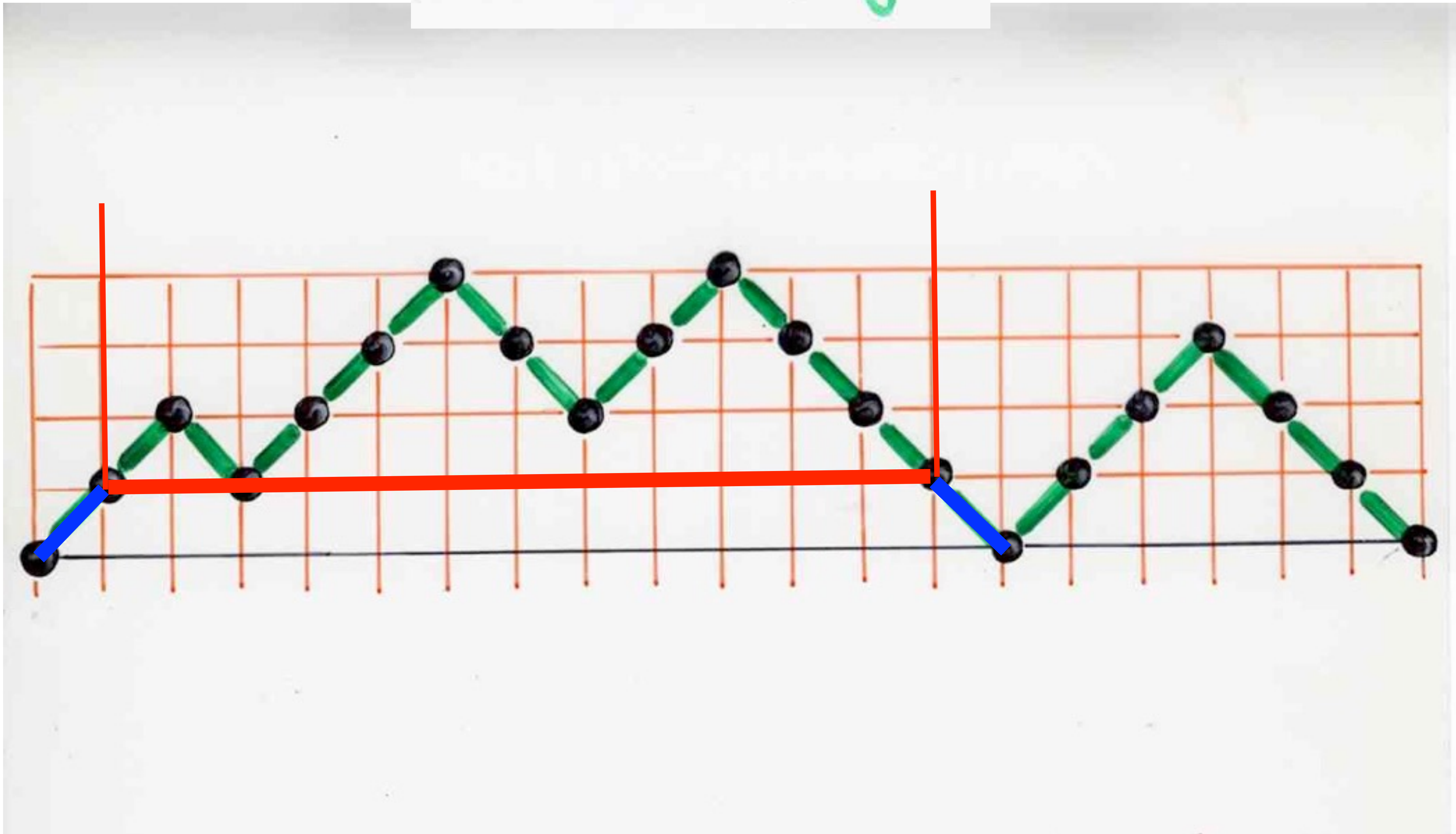
Dyck paths



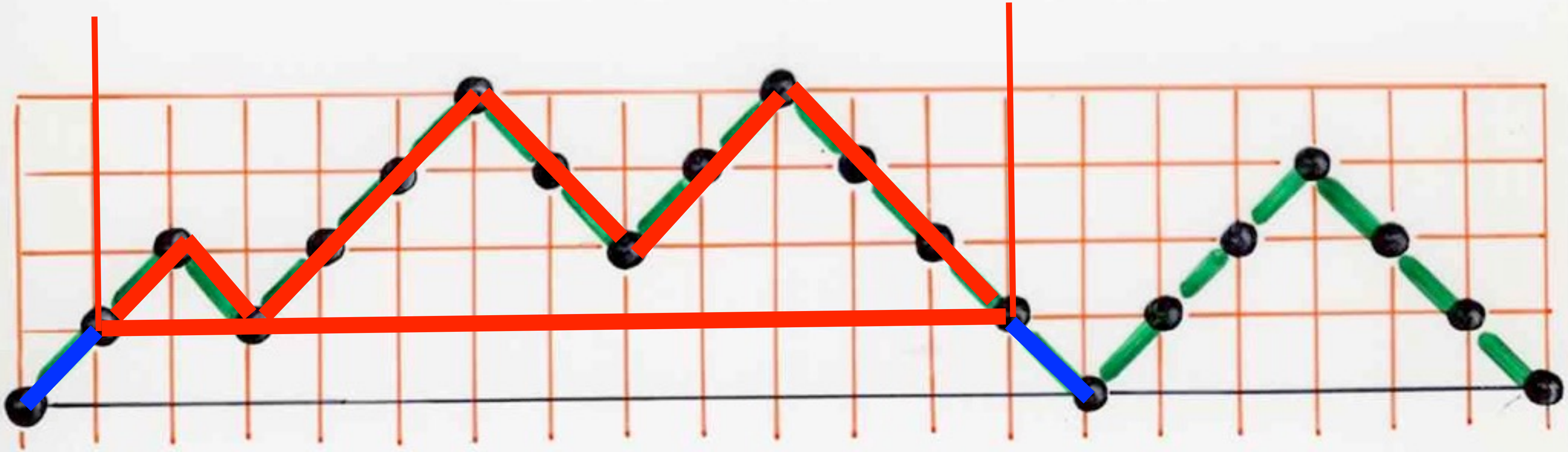
Dyck path

C_n = number of Dyck path of length $2n$

chemins de Dyck



chemins de Dyck

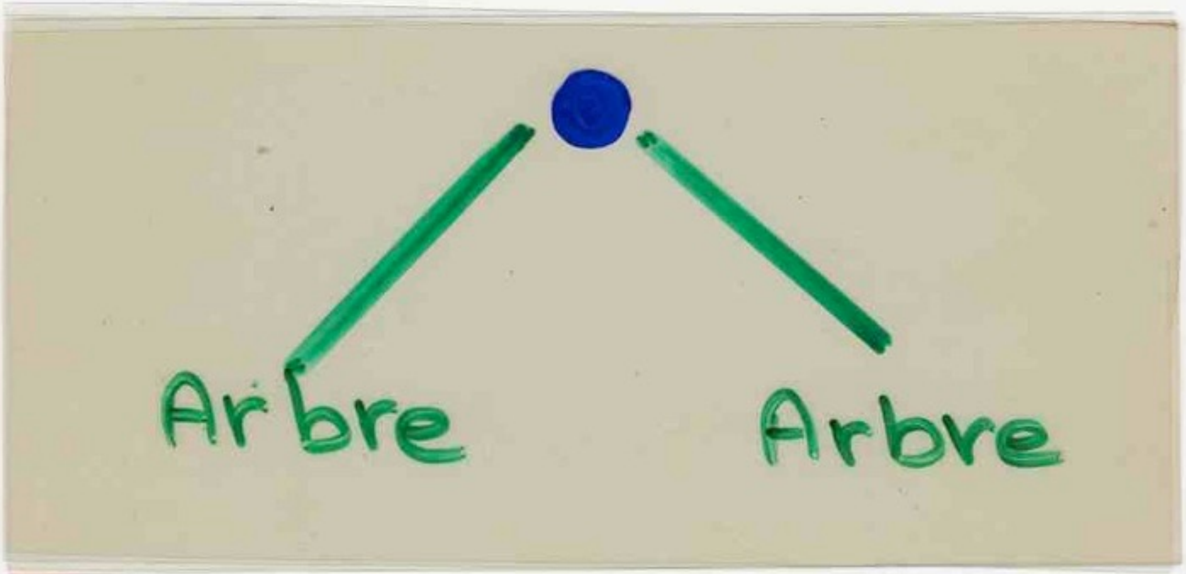


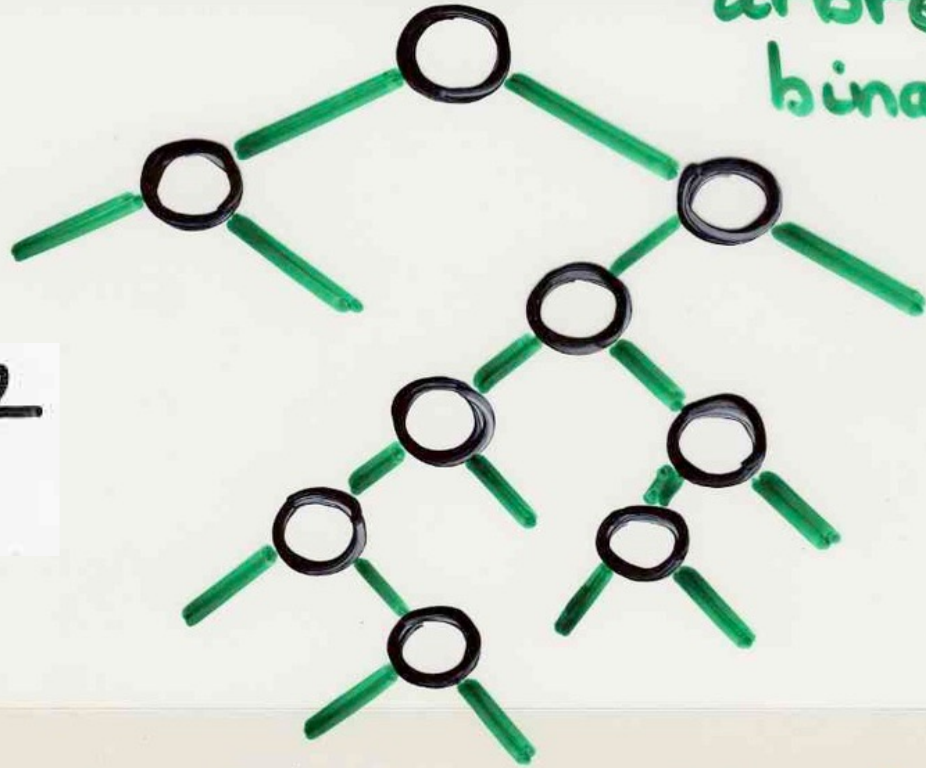
$$D = 1 + t\,D^2$$

Arbre

$$A = 1 + tA^2$$

=

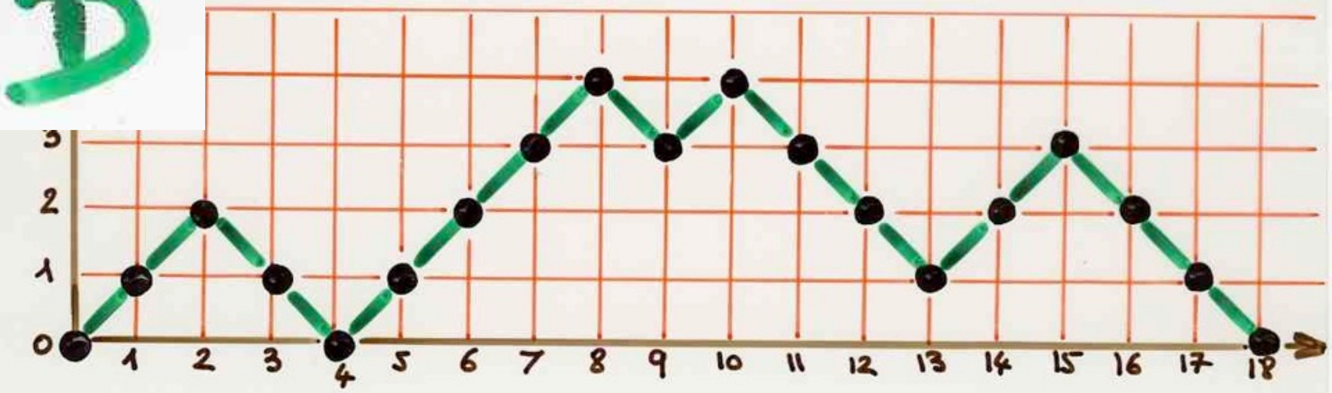




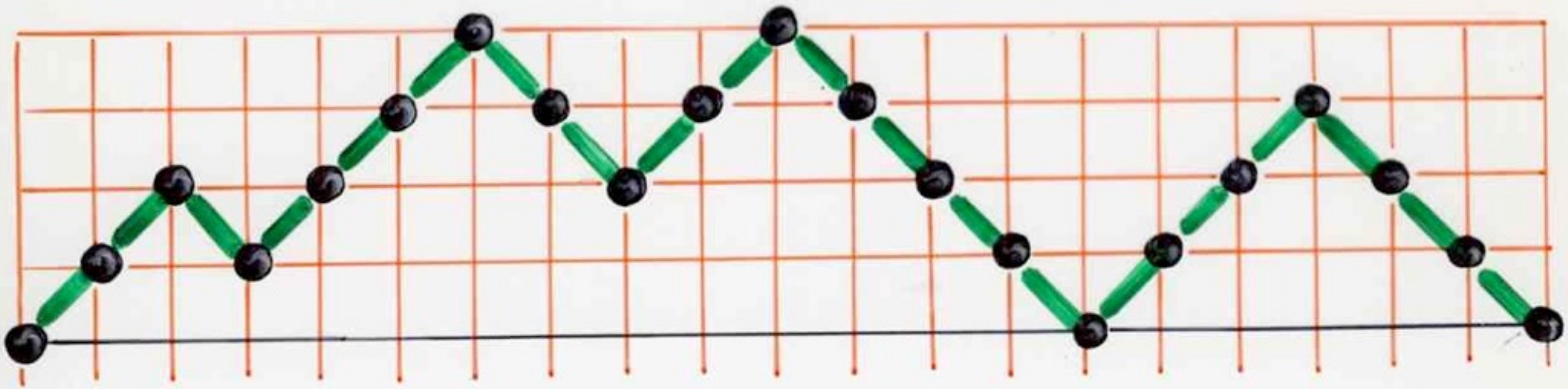
$$A = 1 + tA^2$$

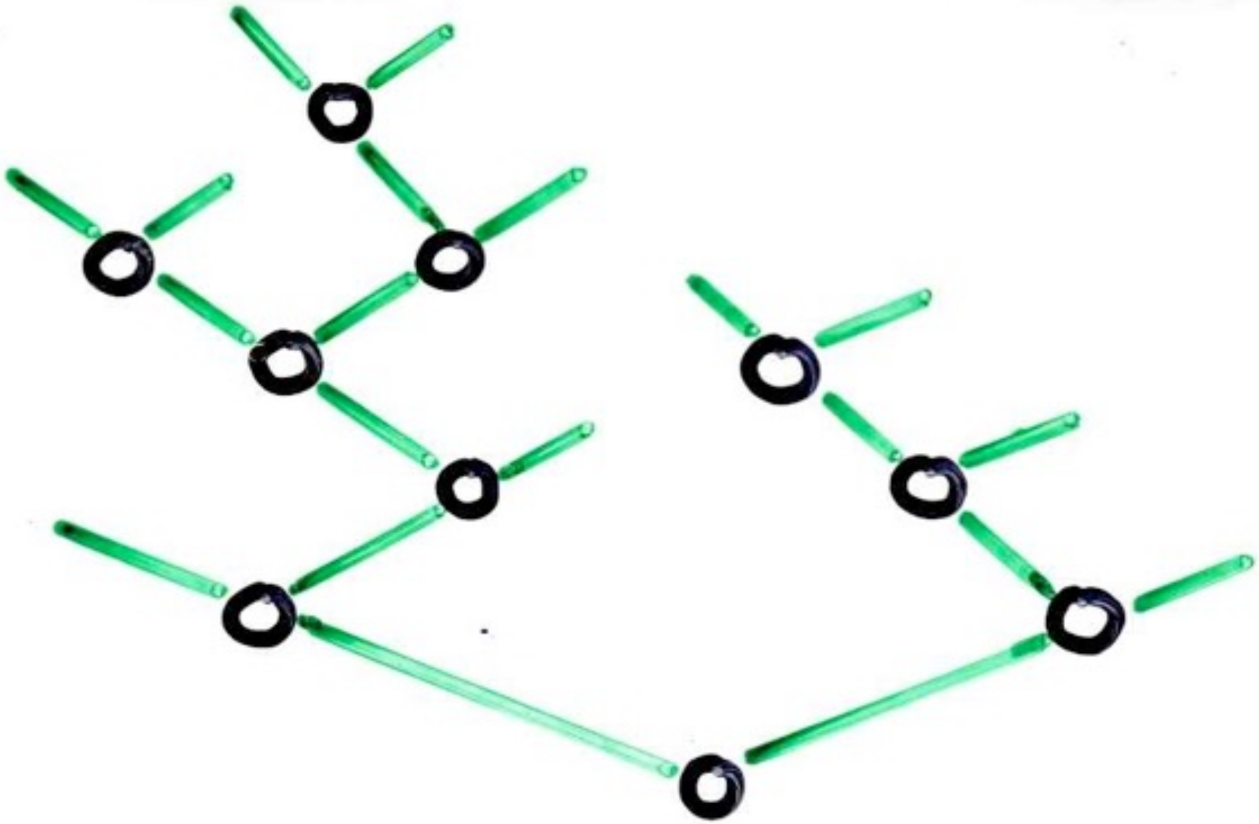
chemin de Dyck

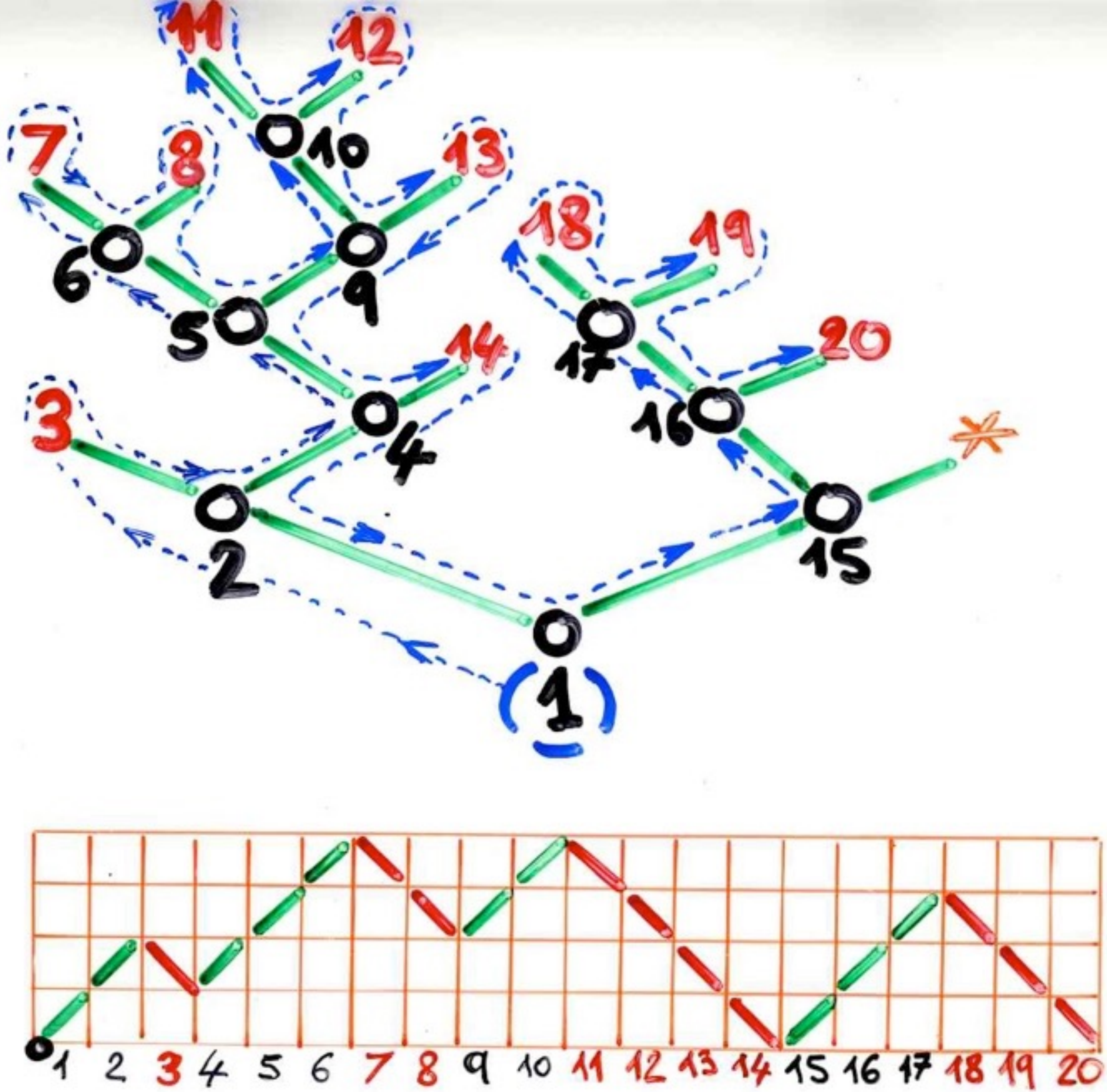
$$D = 1 + tD^2$$



from binary trees ...
to Dyck paths







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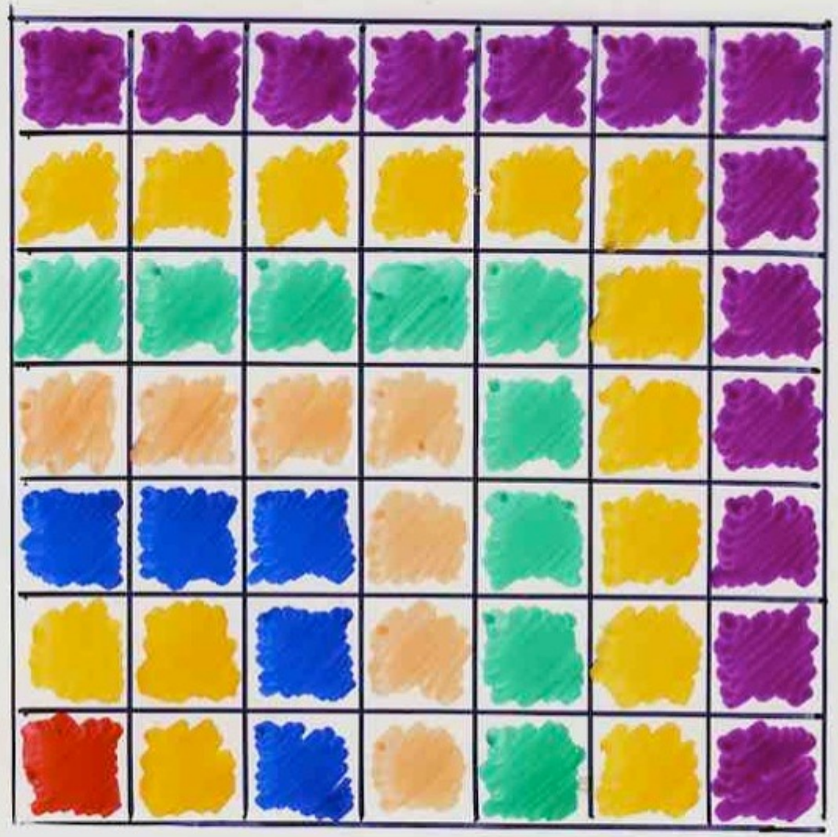
exercise 4:

give a bijective proof of Touchard identity

$$C_{n+1} = \sum_{0 \leqslant i \leqslant \lfloor n/2 \rfloor} \binom{n}{2i} C_i \, 2^{2n-i}$$

bijective proof of an identity

The "bijective paradigm"



$$n^2 = 1 + 3 + \ldots + (2n-1)$$

$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

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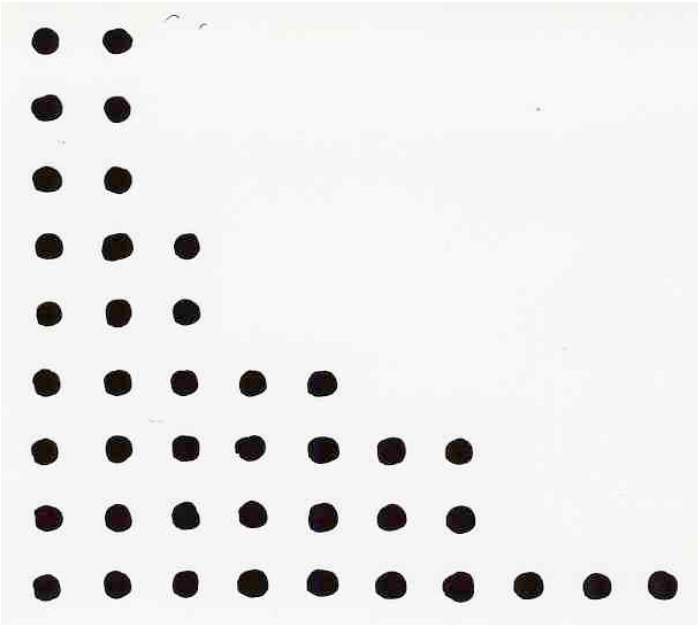
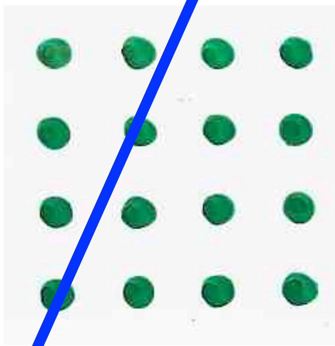


$$= \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

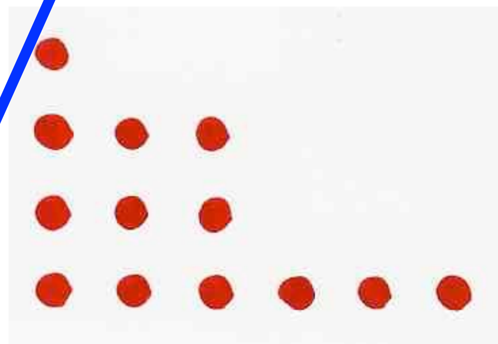
$$\sum_{m\geq 1} \frac{q^{m^2}}{[(1-q)(1-q^2)\cdots(1-q^m)]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$

$$q^{m^2}$$



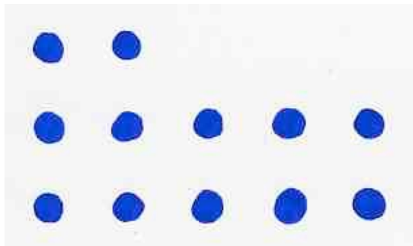
square
m X m

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

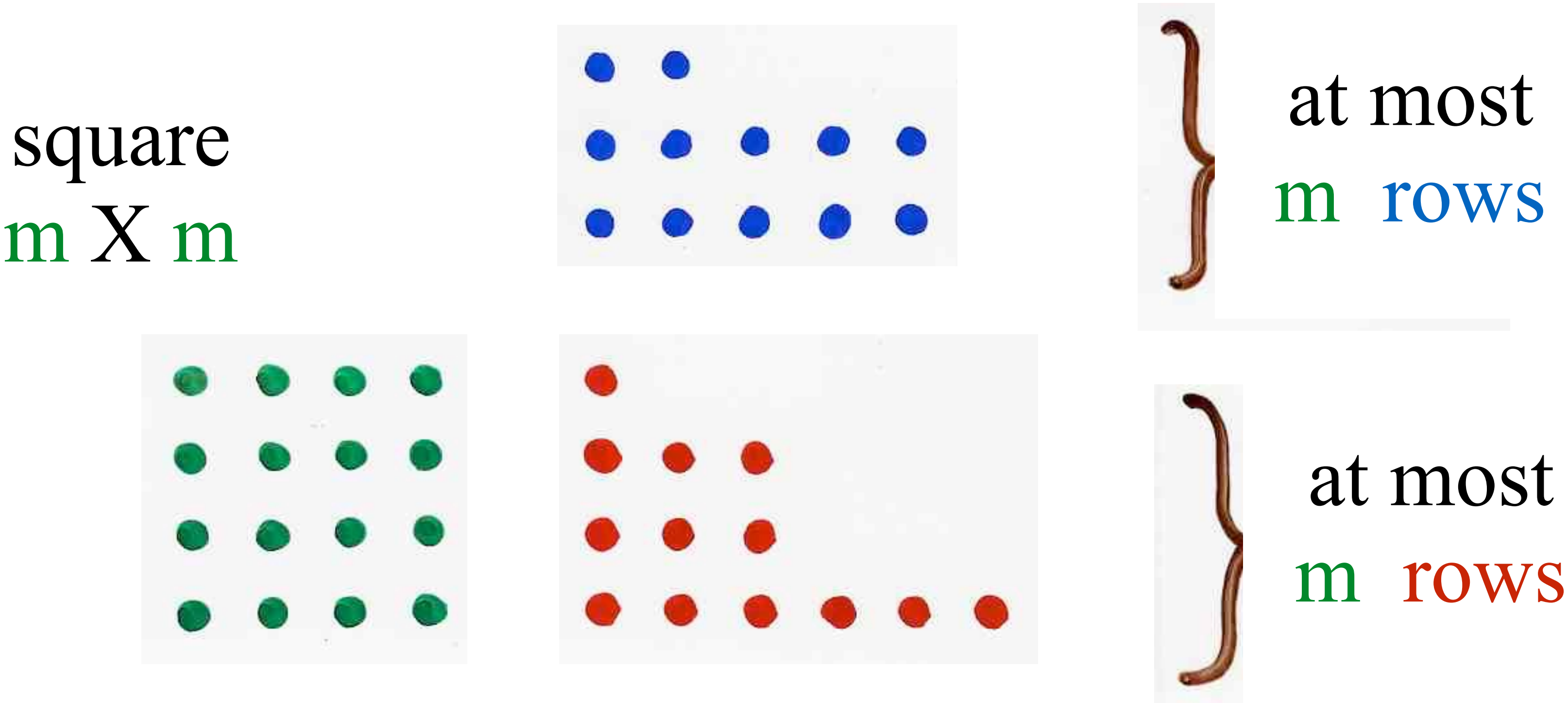


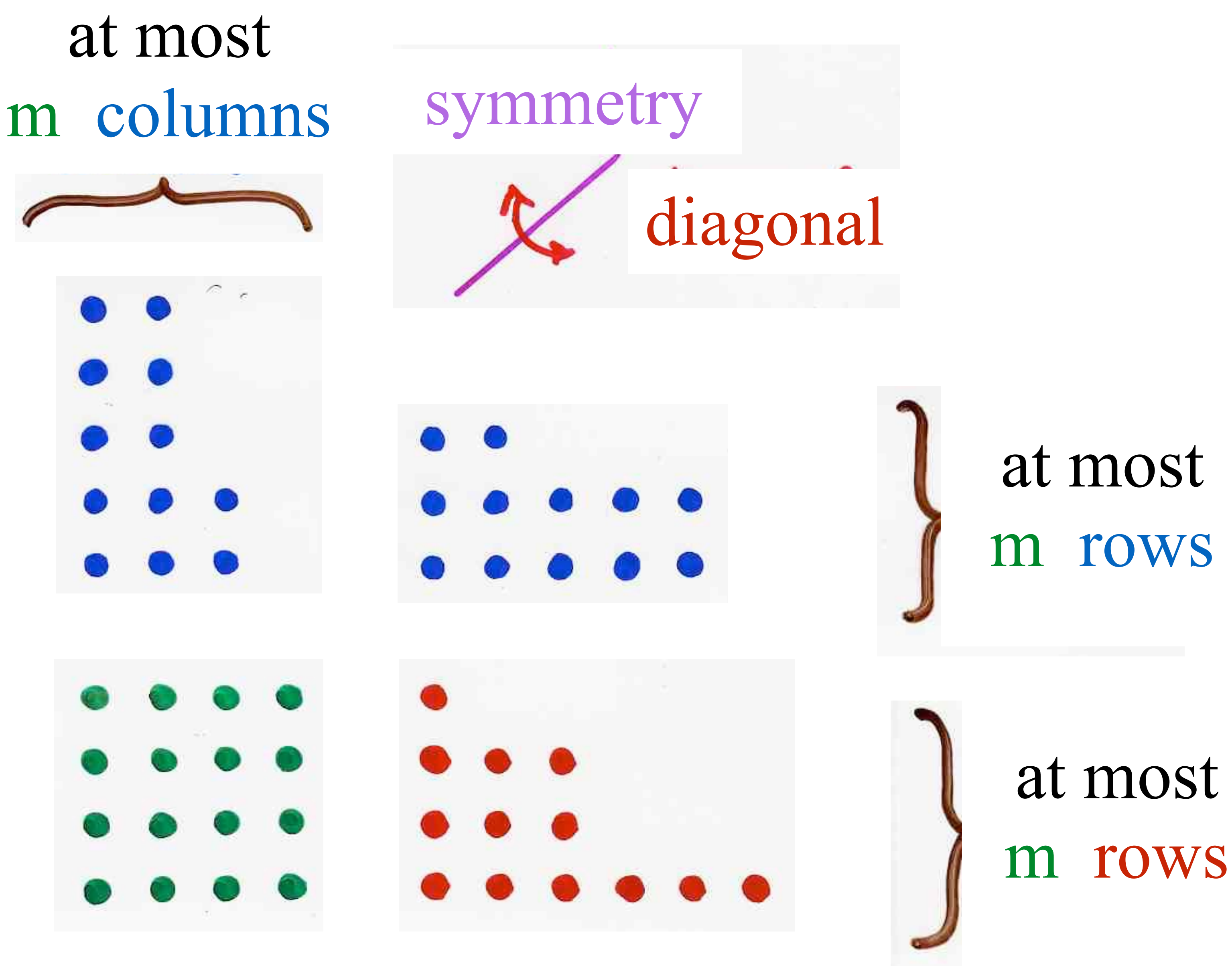
at most
m rows

$$\frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

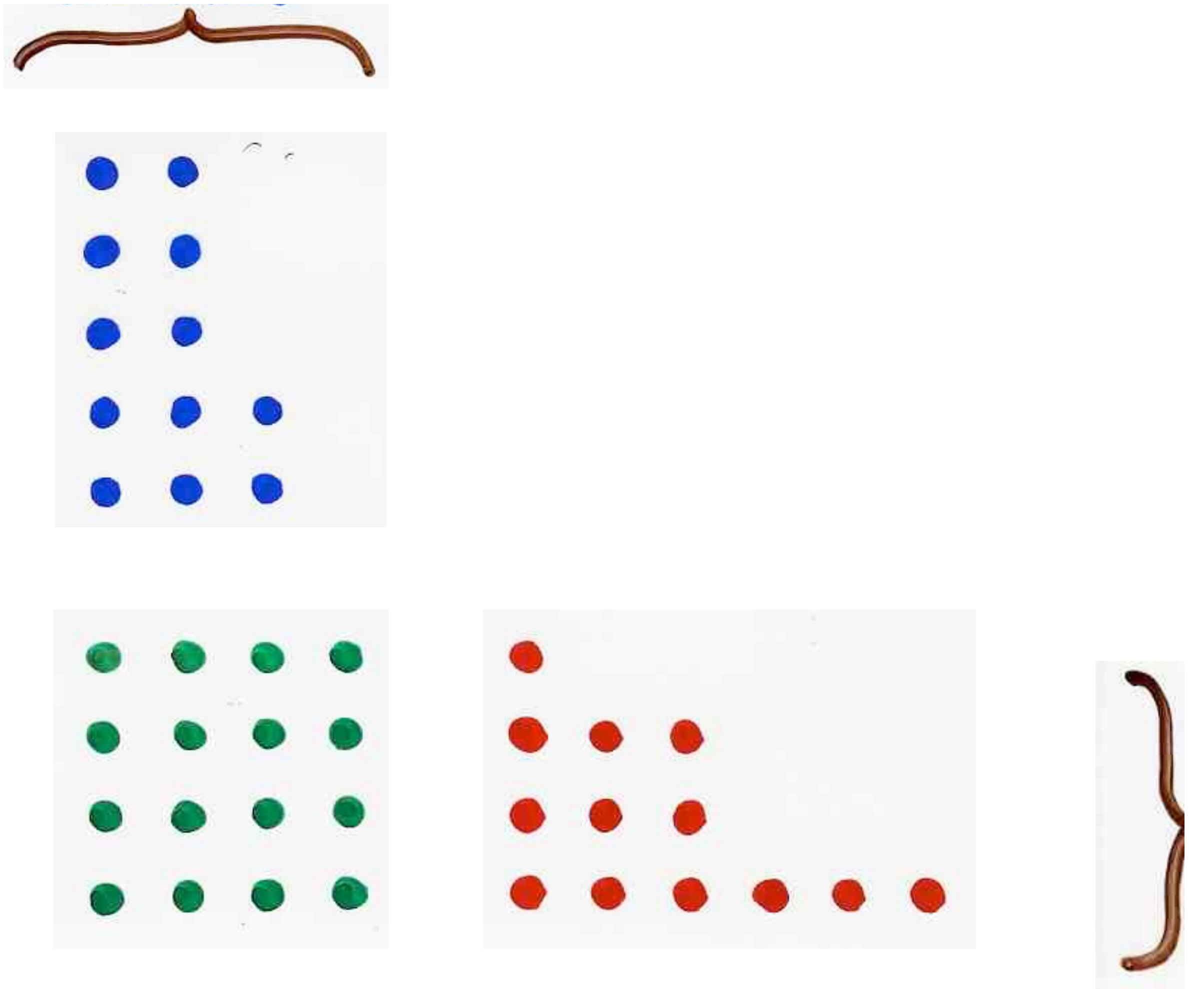


at most
m rows

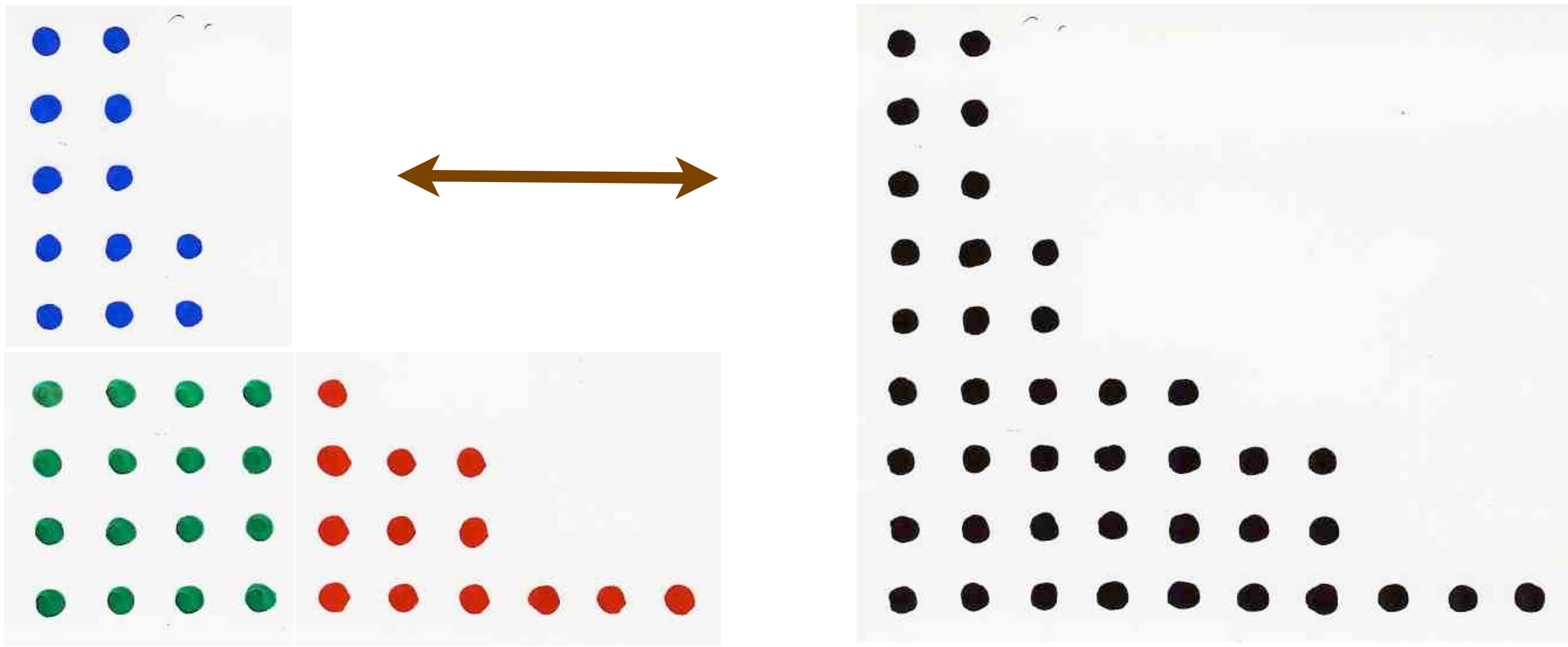

square
m X m
at most
m rows
at most
m rows


at most
m columns
symmetry
diagonal
at most
m rows
at most
m rows

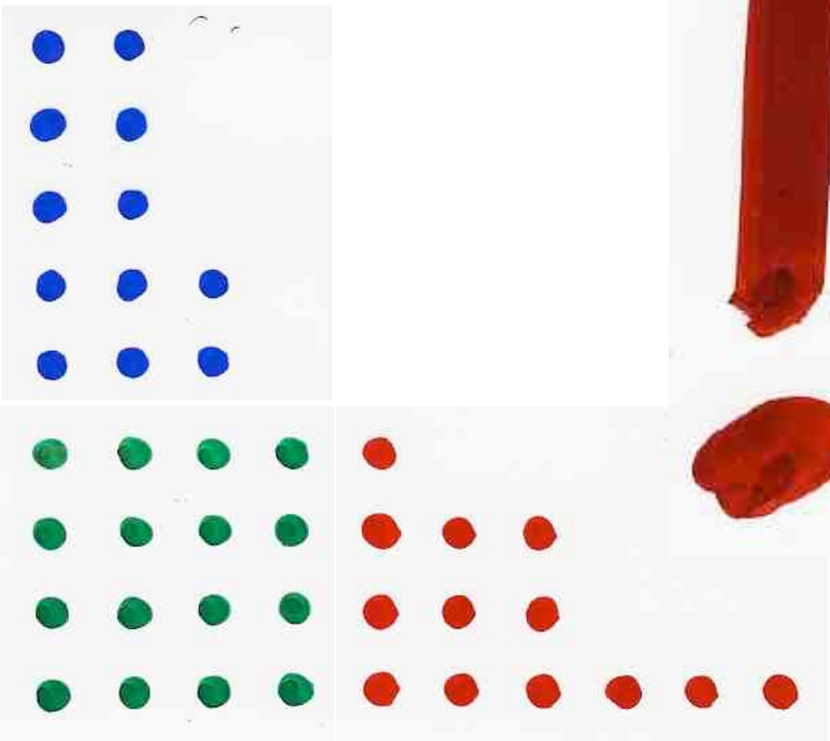
at most
m columns



at most
m rows



$$\sum_{m\geq 1} \frac{q^{m^2}}{\left[(1-q)(1-q^2)\cdots(1-q^m)\right]^2} = \prod_{i\geq 1} \frac{1}{(1-q^i)}$$



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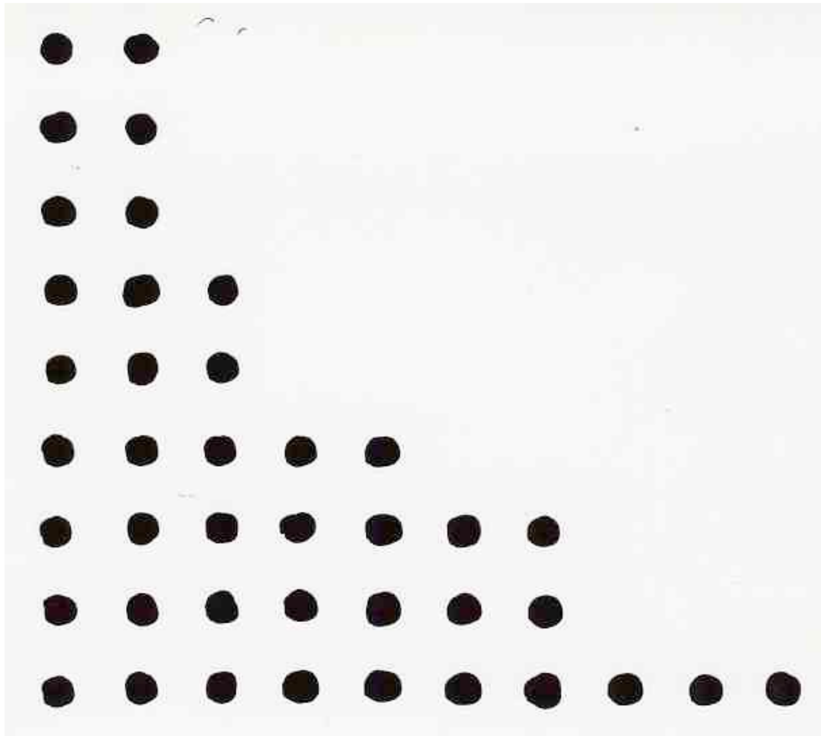
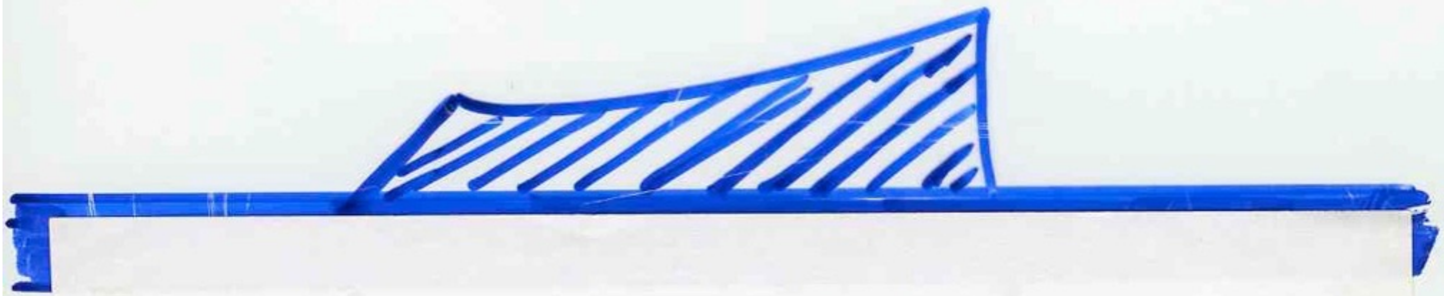


drawing calculus

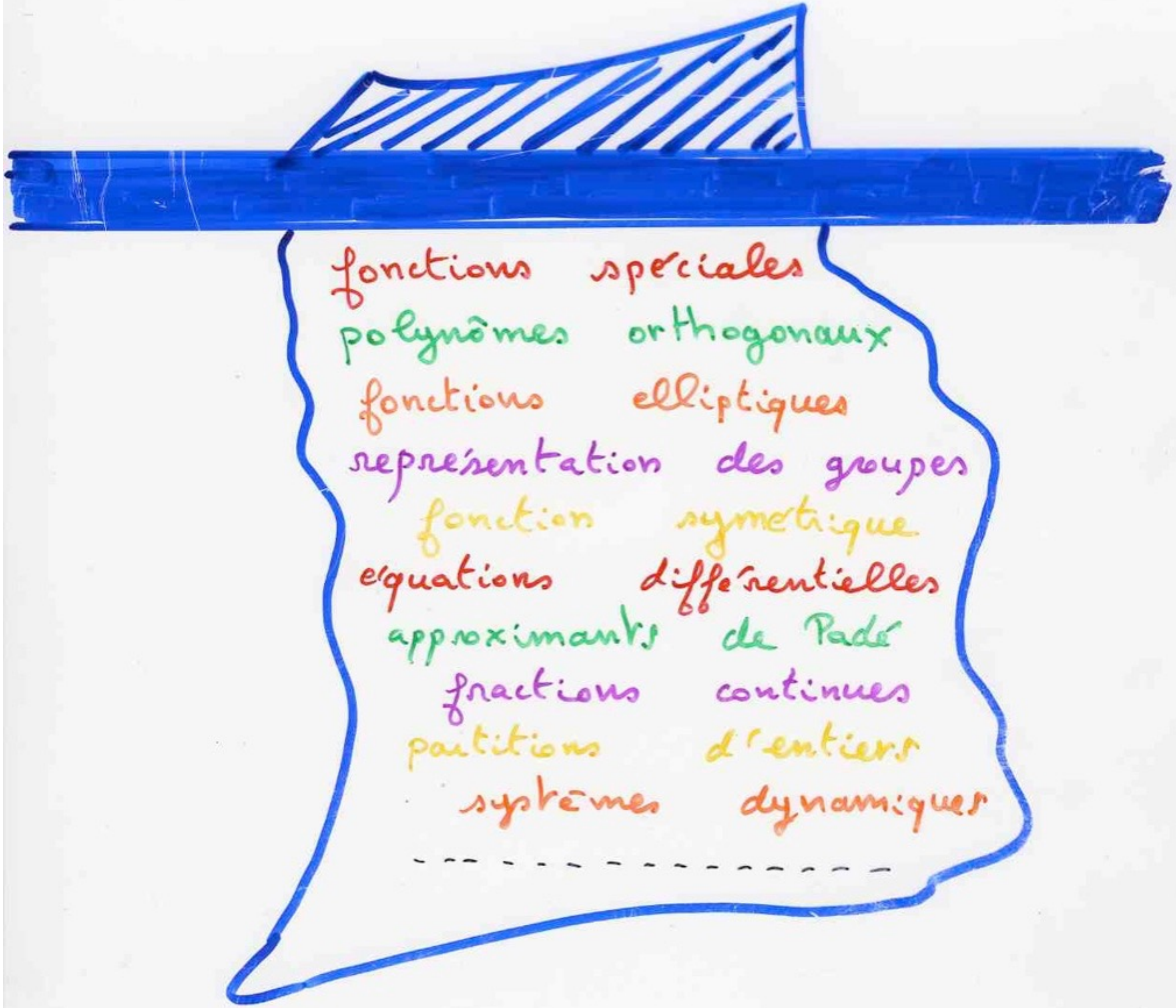
...

computing drawings

better understanding



The bijective paradigm


fonctions spéciales
polynômes orthogonaux
fonctions elliptiques
représentation des groupes
fonction symetrique
équations différentielles
approximants de Padé
fractions continues
partitions d'entiers
systèmes dynamiques
- - - - - - - - - - - - - - -

rational generating functions

Rational generating function

$$\sum_{n \geq 0} a_n t^n = \frac{N(t)}{D(t)}$$

$N(t)$
$D(t)$ polynomials in t

Path (or walk)

$$\omega = (s_0, s_1, \dots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

length n

(s_i, s_{i+1}) elementary step

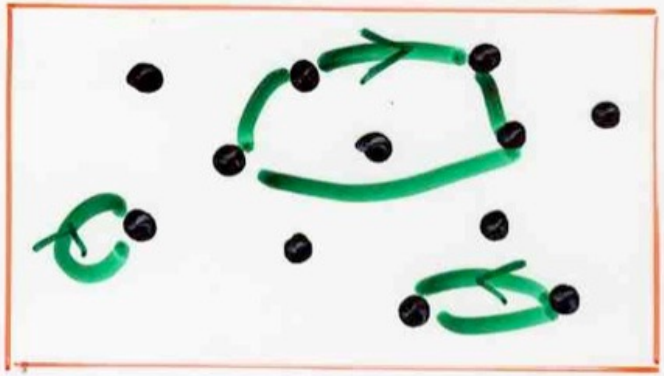
valuation (weight)

$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

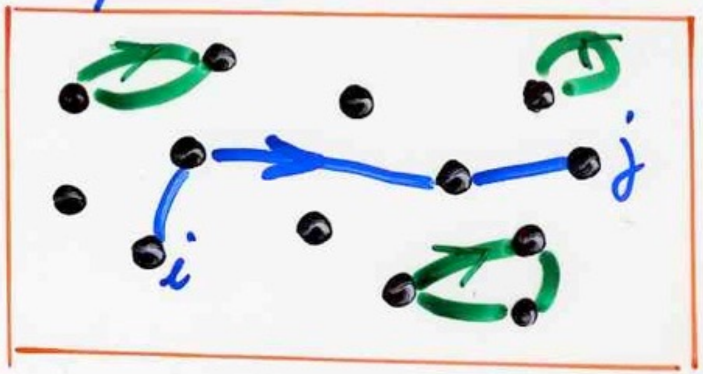
$$v : S \times S \to \mathbb{K}[x]$$

Prop. $$\sum_{\substack{\omega \\ i \leadsto j}} v(\omega) = \frac{N_{ij}}{D}$$

$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint} \\ \text{cycles}}} (-1)^r \, v(\gamma_1) \cdots v(\gamma_r)$$



$$N_{ij} = \sum_{\{\eta;\, \gamma_1, \ , \gamma_r\}} (-1)^r \, v(\eta) \, v(\gamma_1) \cdots v(\gamma_r)$$



Lemma $\quad S = \{1, 2, \dots, n\}$

$$v(i,j) = a_{ij} \qquad A = (a_{ij})_{1 \le i,j \le n}$$

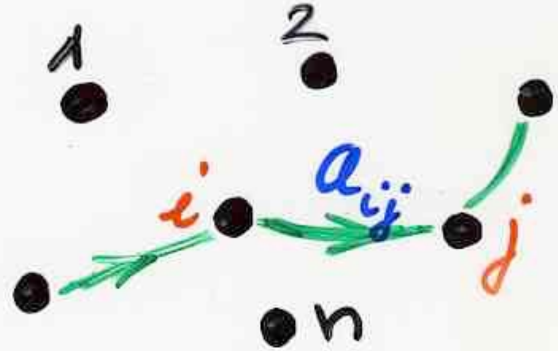
matrix

$$(I - A)^{-1}_{ij} = \sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega)$$

$$(I_n - A)^{-1} = \frac{\mathrm{cof}_{ji}(I_n - A)}{\det(I_n - A)}$$

$$I_n + A + A^2 + \dots + A^{\overset{ij}{n}} + \dots$$

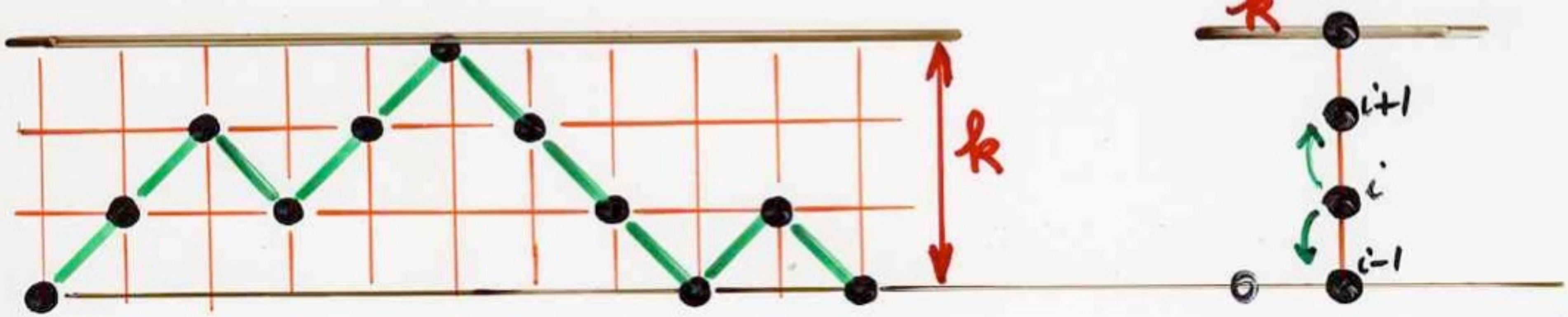
$$A = (a_{ij})$$



ex : Fibonacci numbers

$$\sum_n F_n t^n = \frac{1}{1-t-t^2}$$

ex: Dyck path bounded at height k

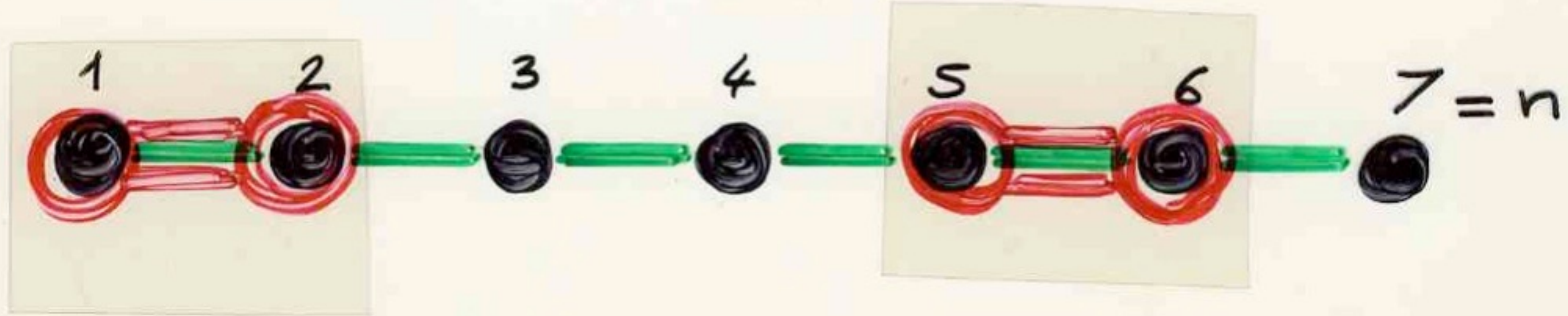


$$\sum_{\omega} t^{|\omega|/2} = \frac{F_k(t)}{F_{k+1}(t)}$$

Dyck paths
bounded k

$$A = (a_{ij}) = \begin{pmatrix} 0 & t & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

matching of the “segment” graph

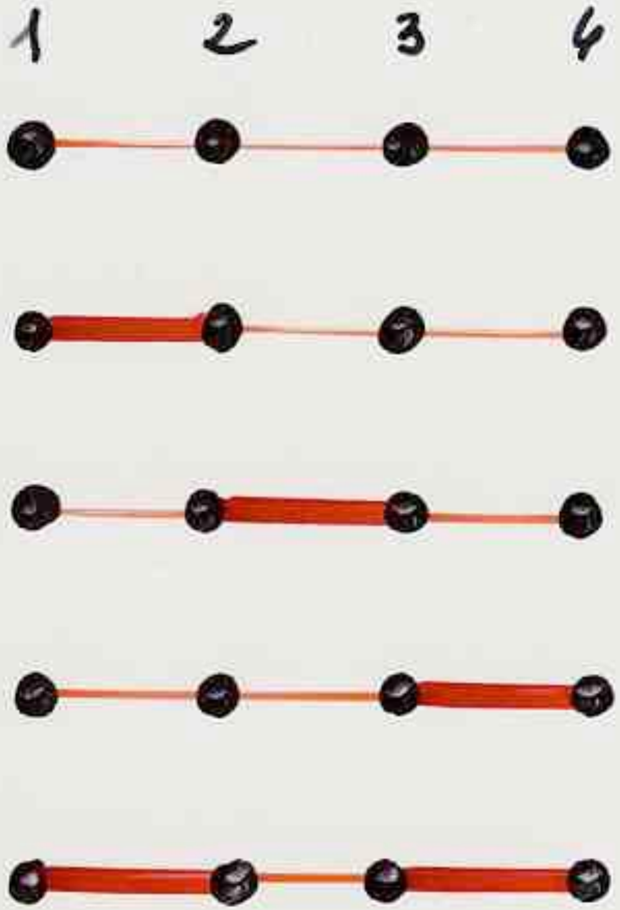


$a_{n,k}$ = number of matchings of $\{1, 2, \dots, n\}$ with k dimers

$$P_n(x) = \sum_{k=1}^{n} a_{n,k}\, x^k$$

$$F_n(x) = P_n(-x)$$

Fibonacci polynomial



1

$-t$

$-t$

$-t$

$+t^2$

$$F_4(t) = 1 - 3t + t^2$$

Fibonacci polynomials

$F_0 = F_1 = 1$

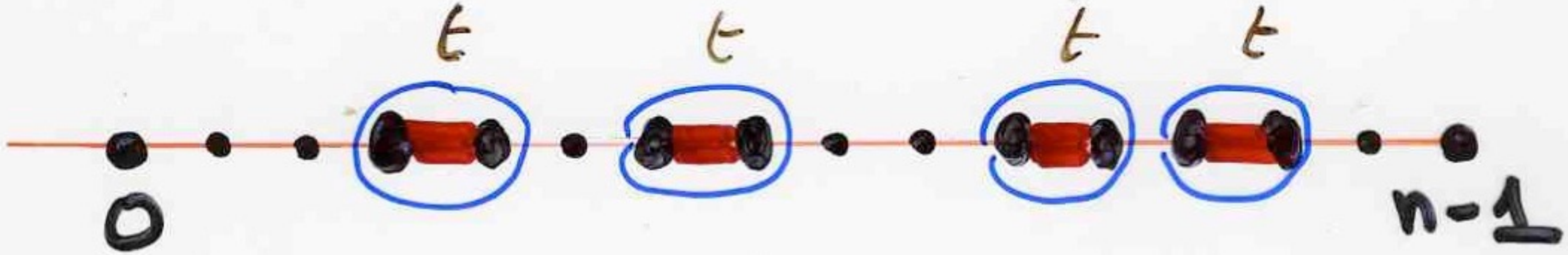
$F_n = F_{n-1} - t F_{n-2}$

1

$1 - t$

$1 - 2t$

$1 - 3t + t^2$



$a_{n,k}$ = number of matchings of $\{1, 2, \ldots, n\}$ with k dimers

exercise

$$a_{n,k} = \binom{n-k}{k}.$$

Tchebycheff polynomials

$$\sin((n+1)\theta) = \sin\theta \, U_n(\cos\theta)$$

$$U_n(x) = \bar{F}_n(2x)$$

$$\bar{F}_n(x) = \sum_{k=1}^{n} (-1)^k a_{n,k} \, x^{n-2k}$$

$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

x^3

$-x$

$-x$

$$F_3(x) = x^3 - 2x$$

$$\sin(4\theta) = \sin\theta \left[8\cos^3\theta - 4\cos\theta \right]$$

complements:

- Tchebycheff

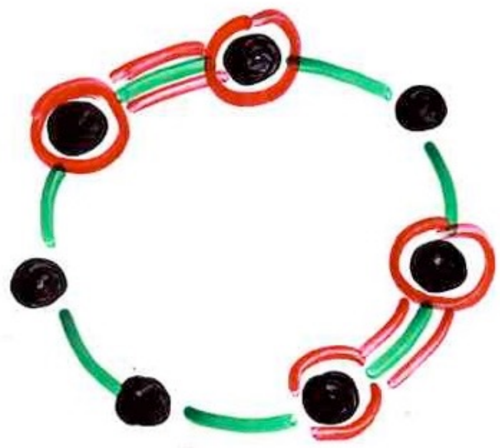
$U_n(x)$ degré n

Polynômes orthogonaux

$$\frac{2}{\pi}\int_{-1}^{+1} U_n(x)\,U_m(x)\,(1-x^2)^{1/2}\,dx = \begin{cases} 0 & \text{si } n \neq m \\ 1 & \text{sinon} \end{cases}$$

complements:

$$\cos(n\theta) = T_n(\cos\theta)$$



binary trees

generating power series

power series algebra

operations on combinatorial objects

formalisation

example: integers partitions

bijective combinatorics

from binary trees ... to Dyck paths

rational generating power series

The "bijective paradigm"