Combinatorics and Physics

Chapter 1
Introduction to enumerative combinatorics, ordinary generating functions (summary of complements)

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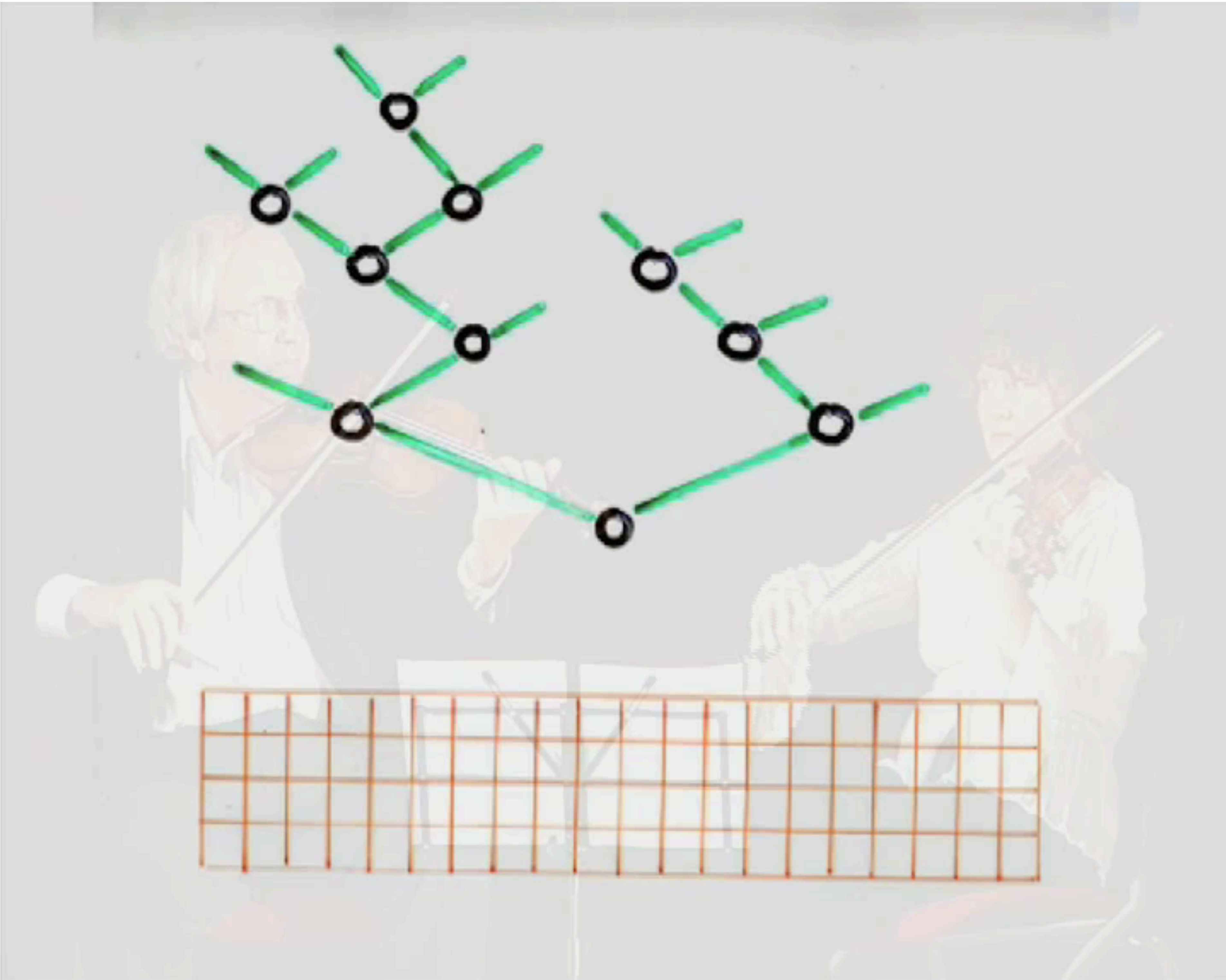
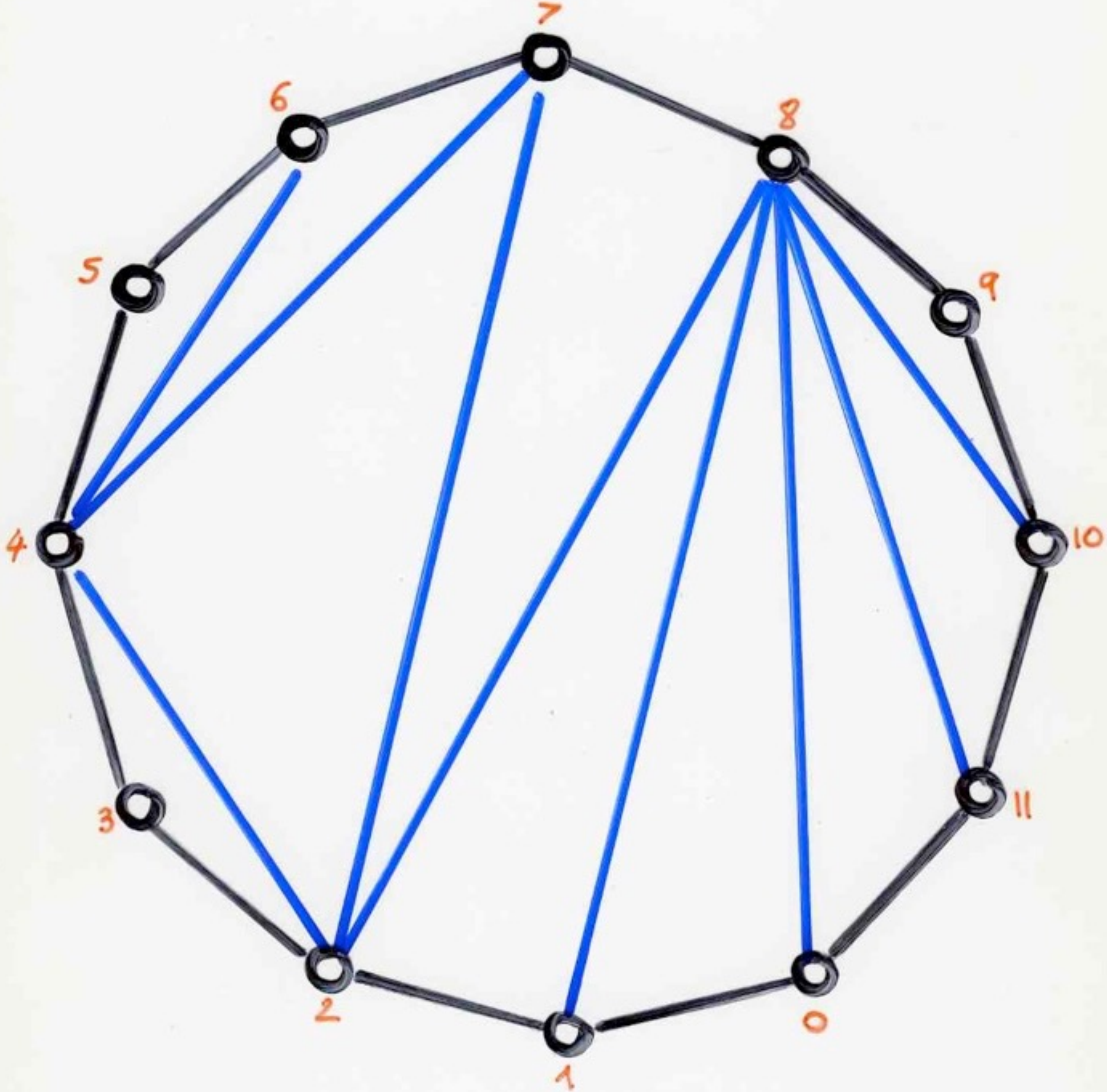
from binary trees
to Dyck paths



from triangulations

to

binary trees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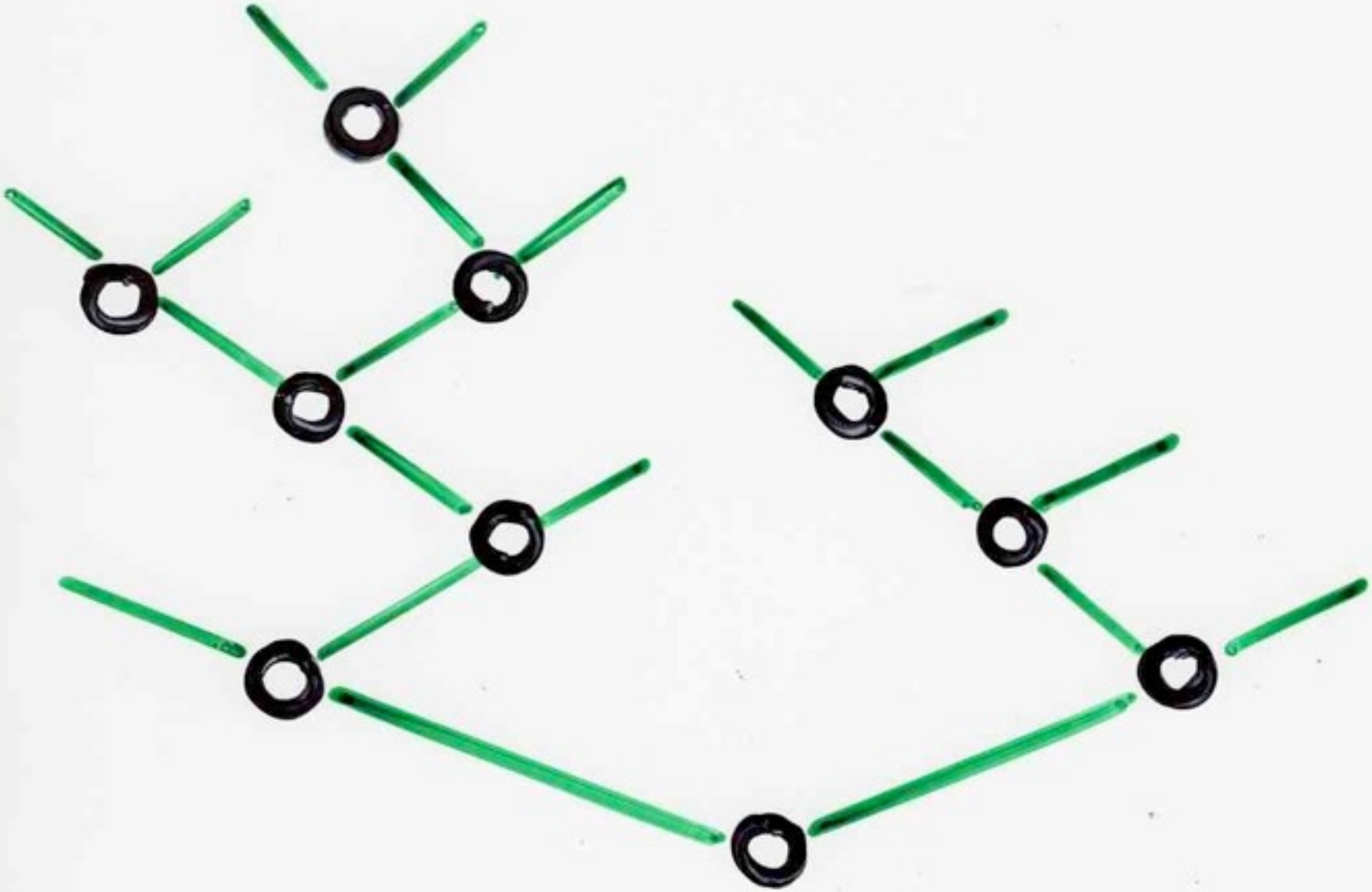

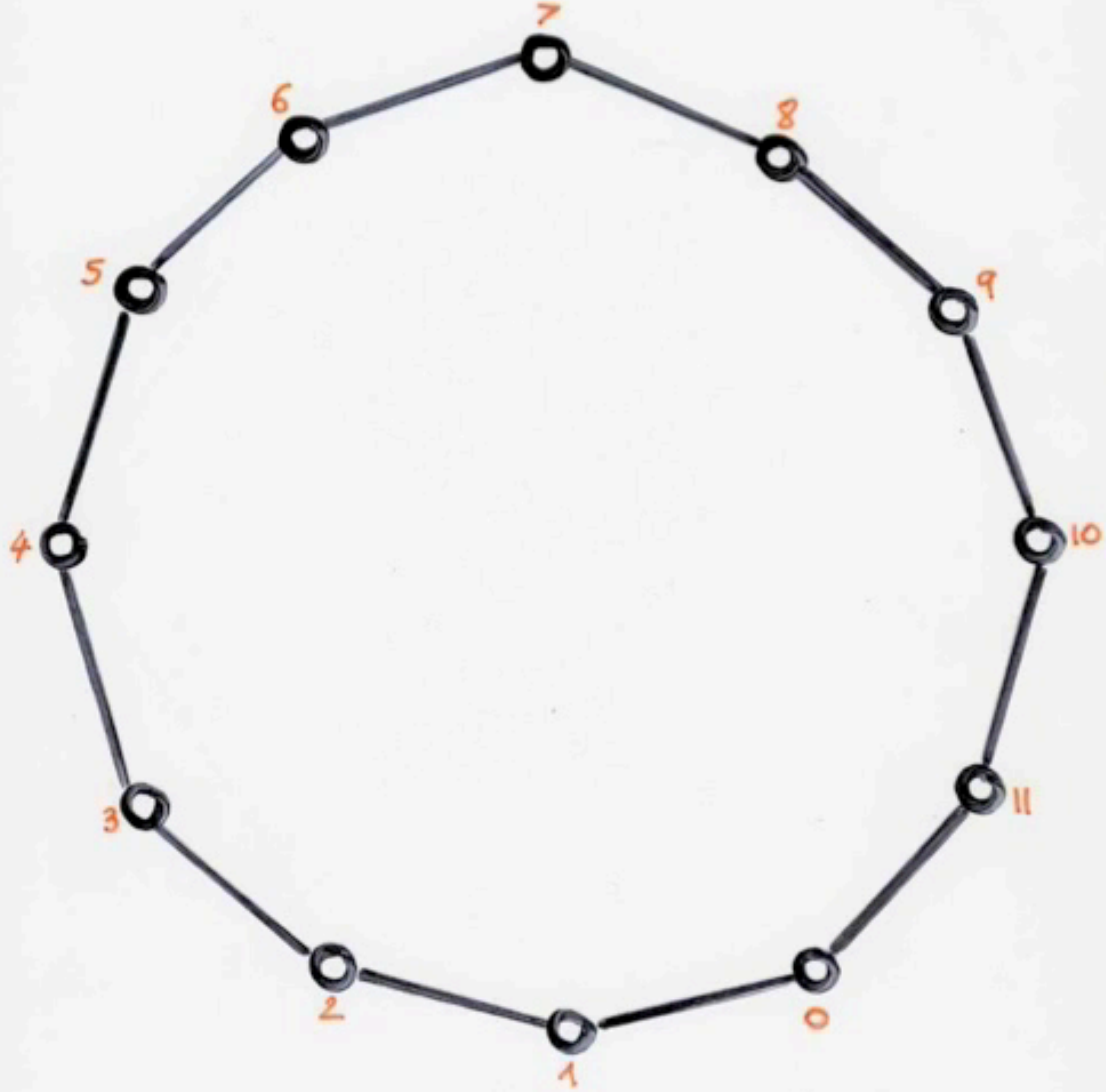
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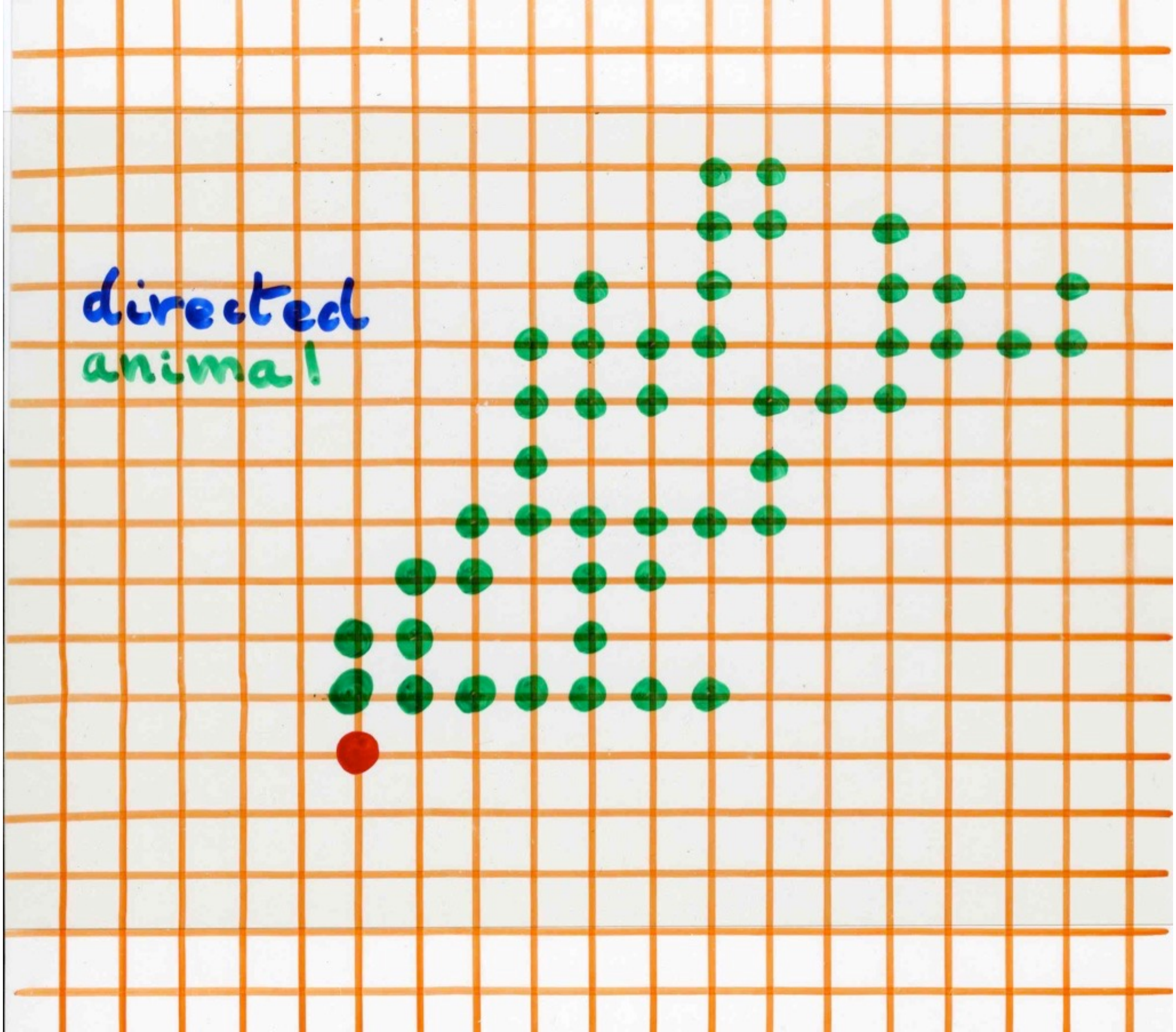
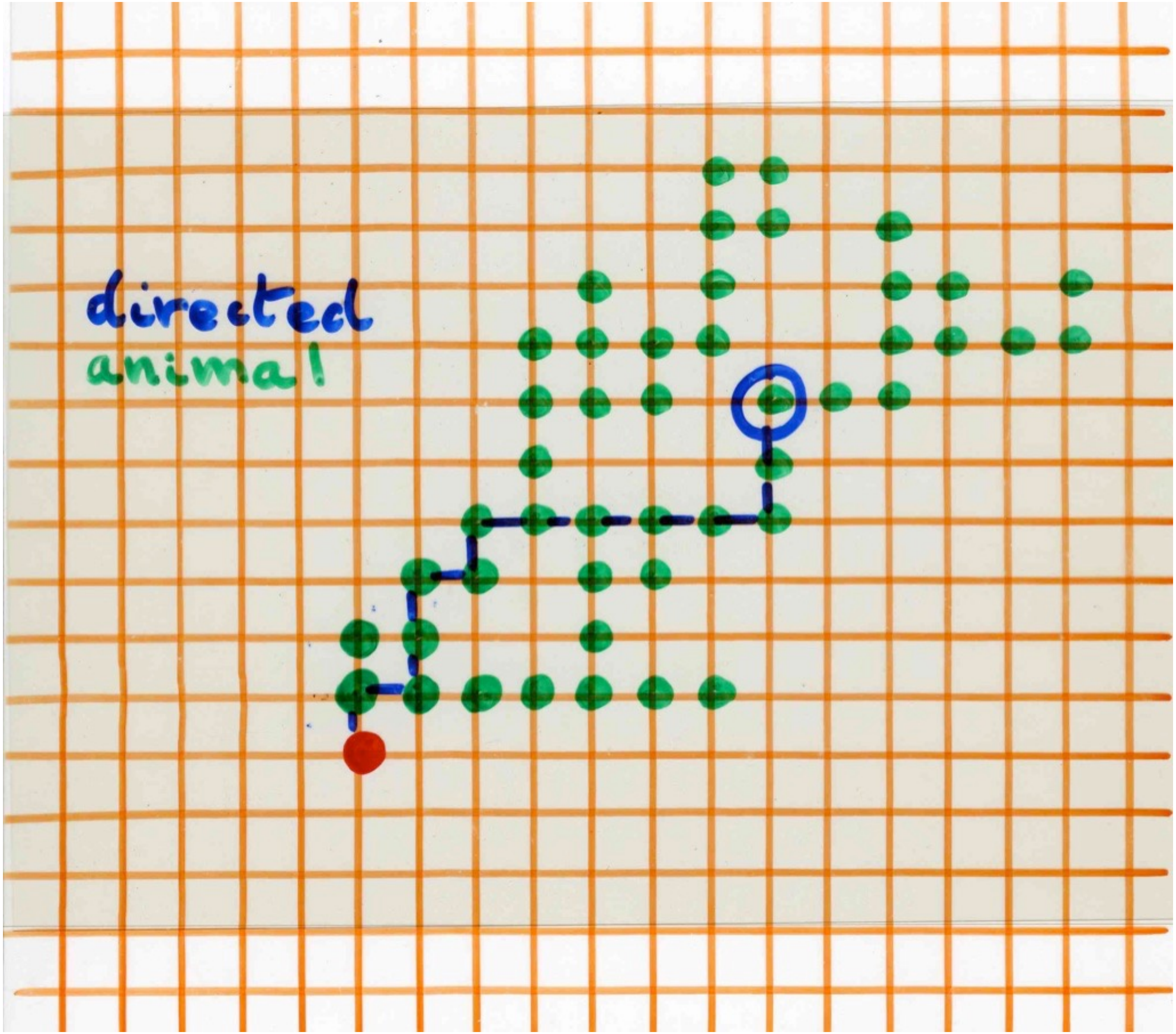
algebraicity

decomposable structures

example: directed animals


directed
animal


directed
animal

D.Dhar

equivalence with
hard gas model

relation with
crystal growth model
stochastic lattice gas

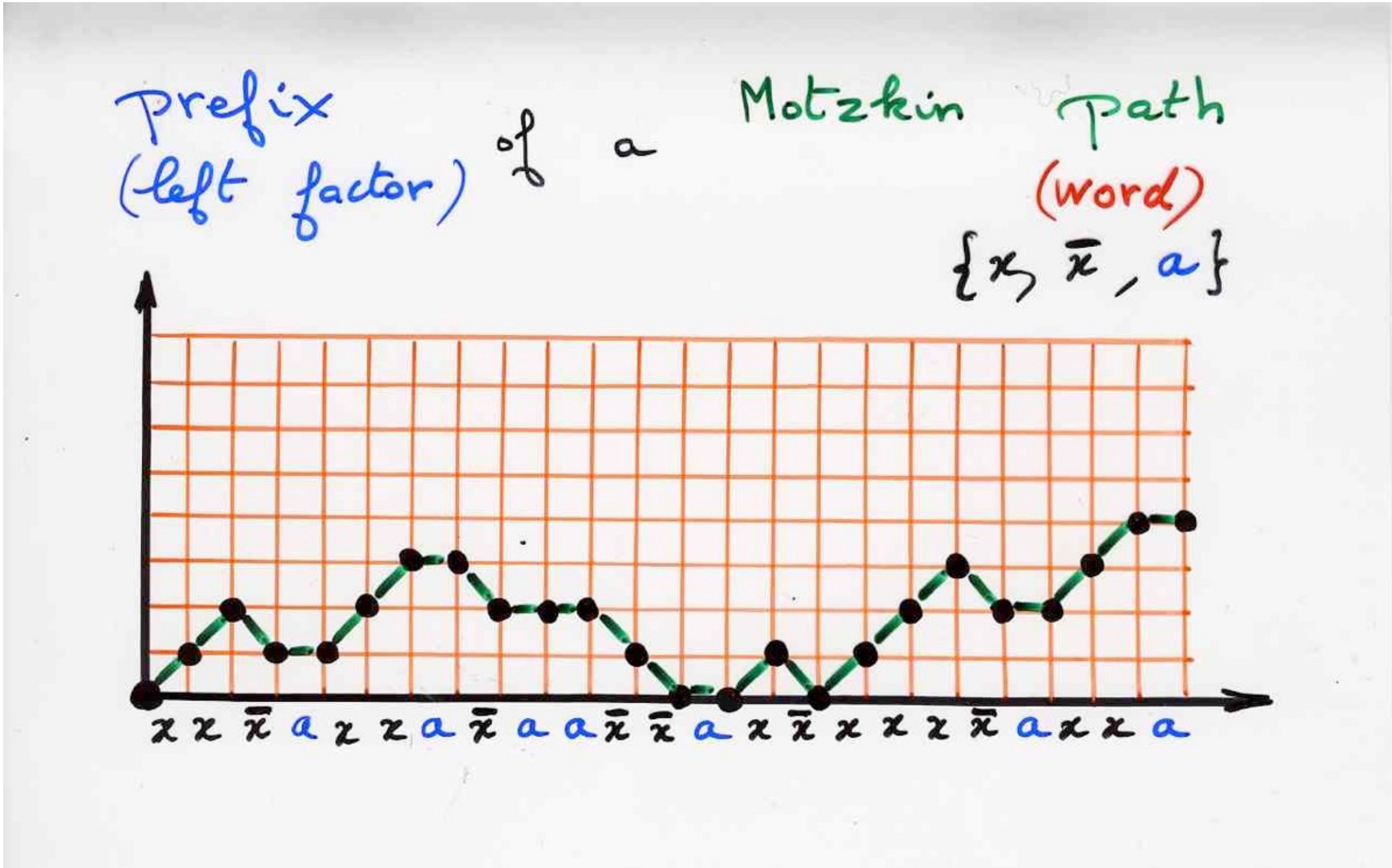
y generating function for
the number of directed animals with n points

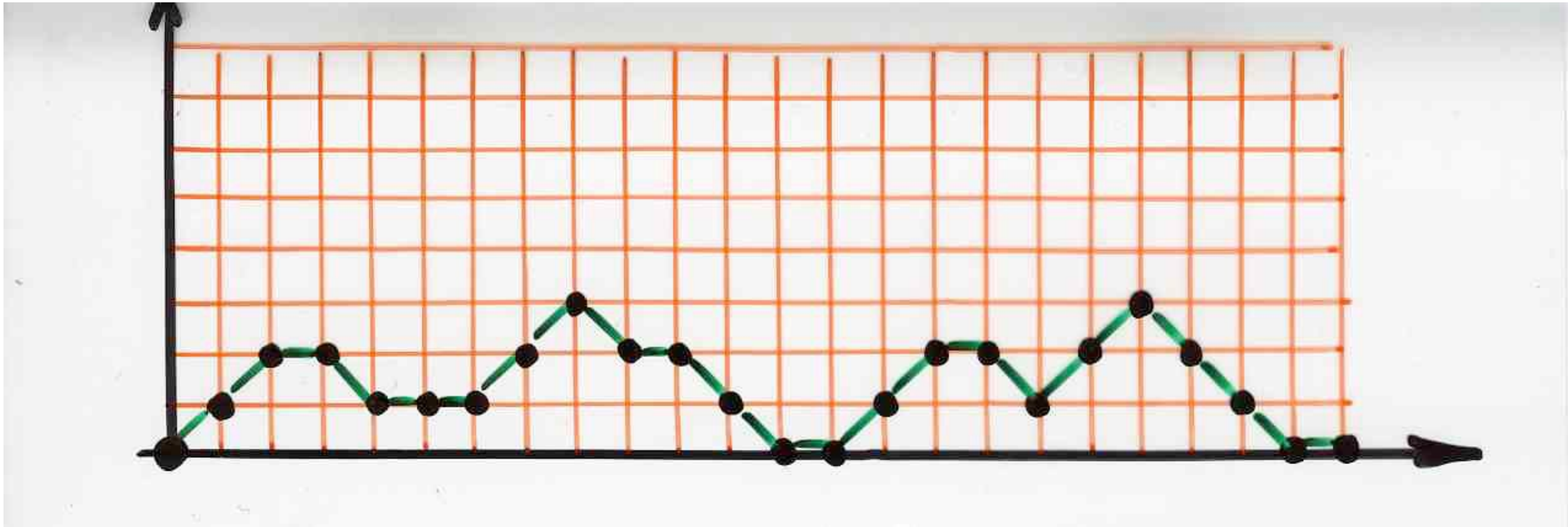
$$y = z + yz$$

$$z = t + tz + tz^2$$

see the proof at the chapter
about « heaps of pieces »

exercise: algebraic equations for Motzkin paths and prefix of Motzkin paths





Motzkin path

$y = t_p$

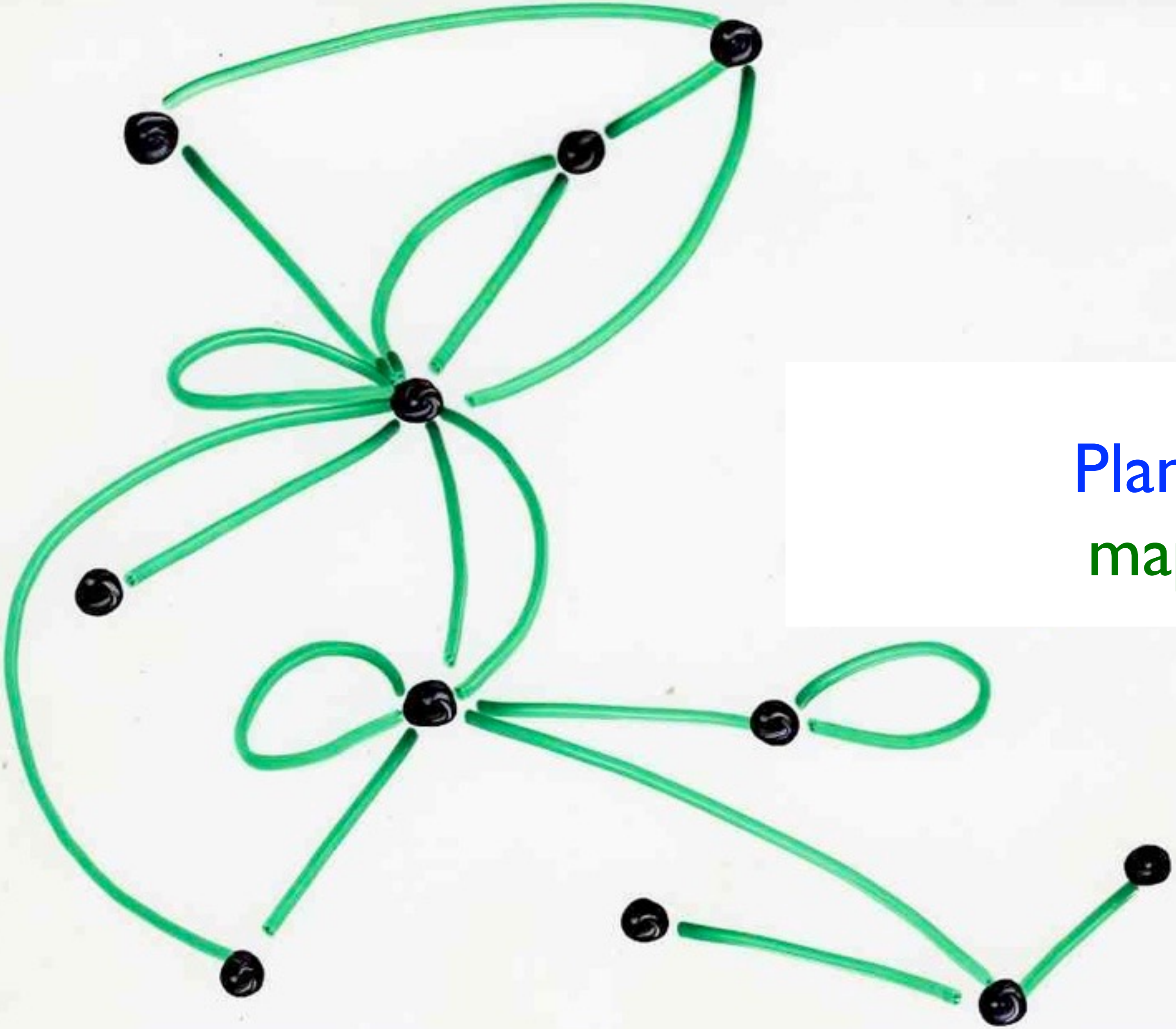
$z = t_m$

number of directed animals n points $=$ number of prefix of Motzkin paths length $(n-1)$

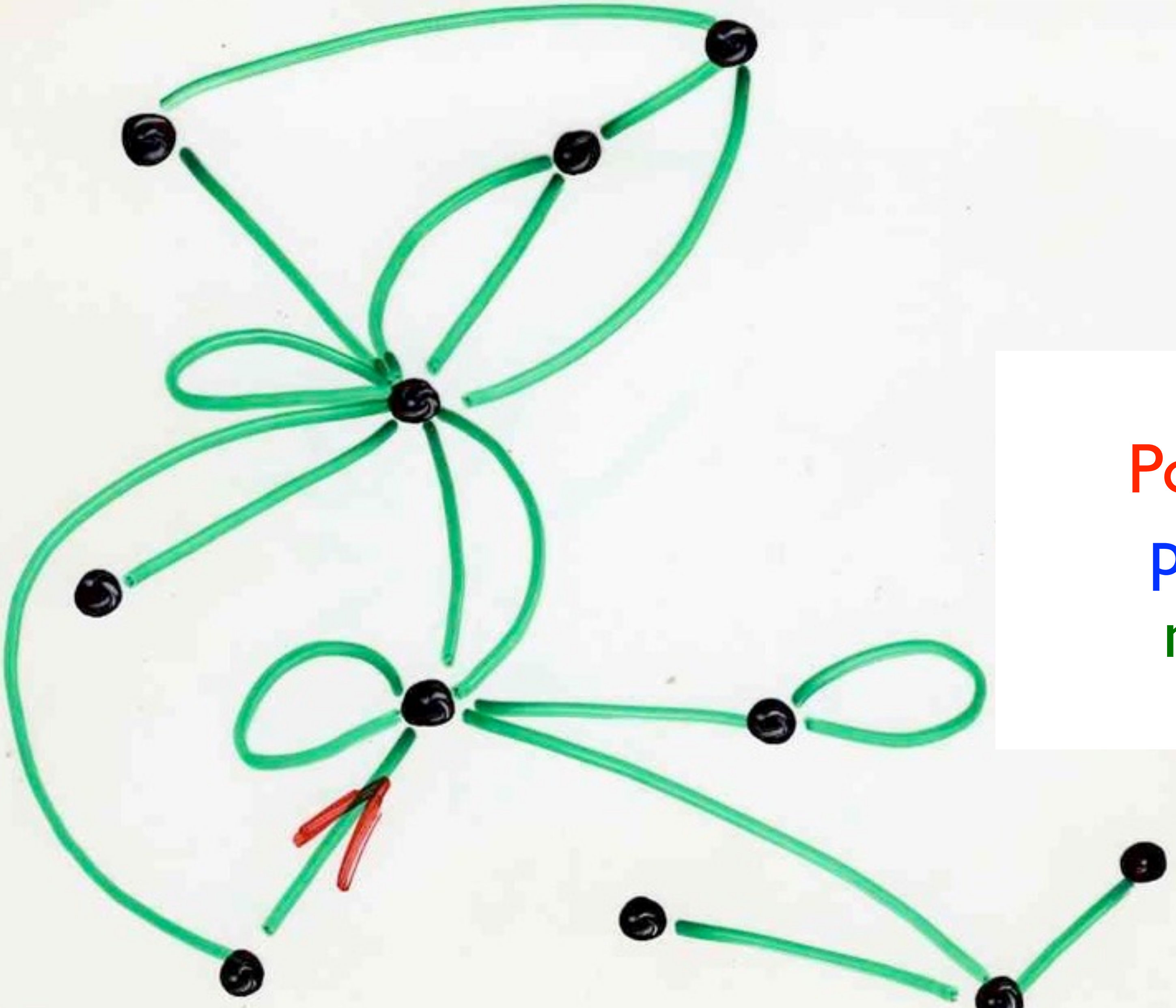
algebraicity

decomposable structures

example: planar maps



Planar
maps



Pointed
planar
maps

$$y = A - tA^3$$
$$A = 1 + 3tA^2$$

Tutte (1968)

$$y = A - tA^3$$
$$A = 1 + 3tA^2$$

Tutte (1968)

Tutte (1968)

$$\frac{2 . 3^m}{(n+2)} C_m$$

Catalan

m edges

$$y = A - tA^3$$
$$A = 1 + 3tA^2$$

Tutte (1968)

Tutte (1968)

$$\frac{2 . 3^m}{(n+2)} C_m$$

Catalan

m arêtes

Cori, Vauquelin (1970, ...)

Arquès (1980, ...)

Schaeffer (1997, ...)

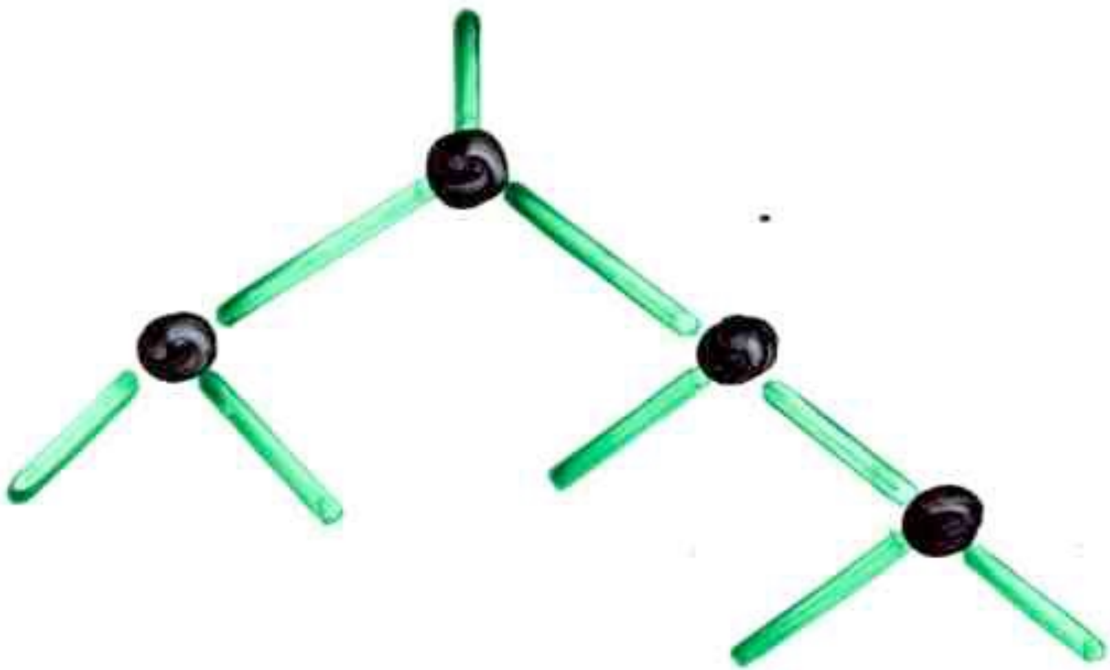
Bouttier, Di Francesco, Guitter (2002, ...)

bijection

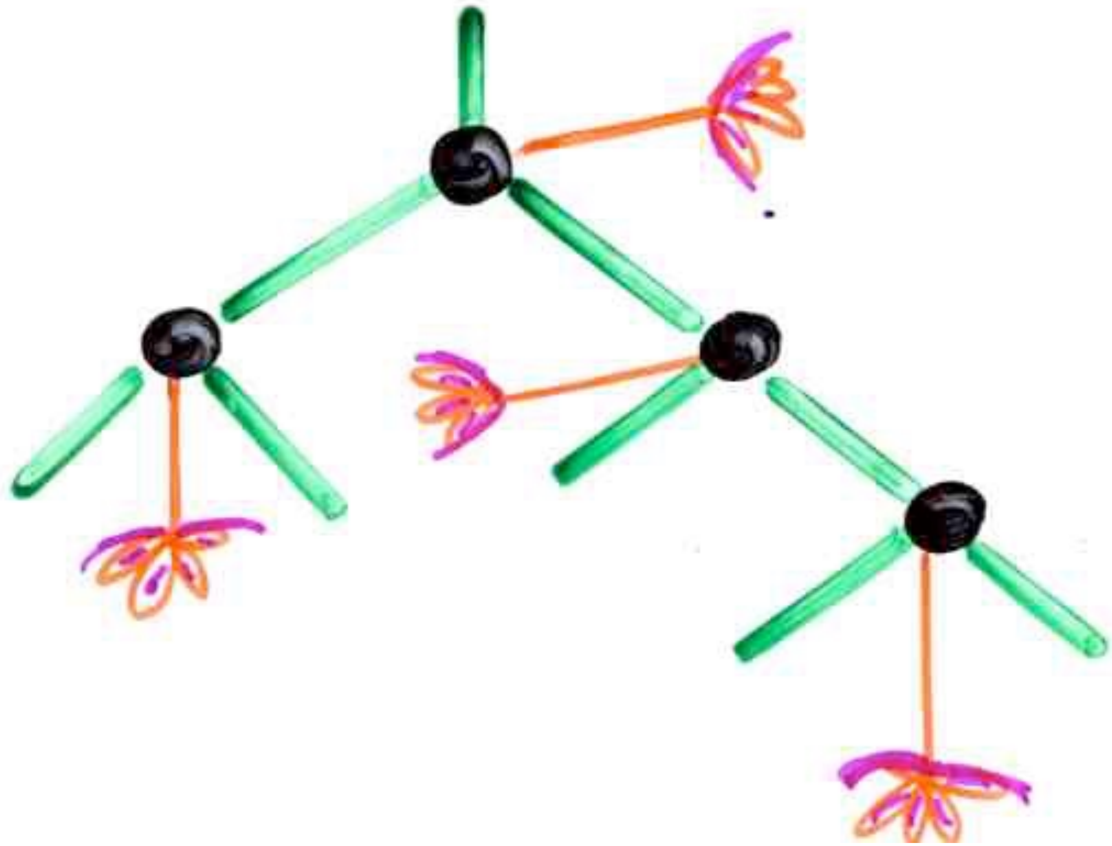
- planar maps (n edges)
- balanced blossoming trees (n nodes)

Schaeffer (1997)

random
planar
maps

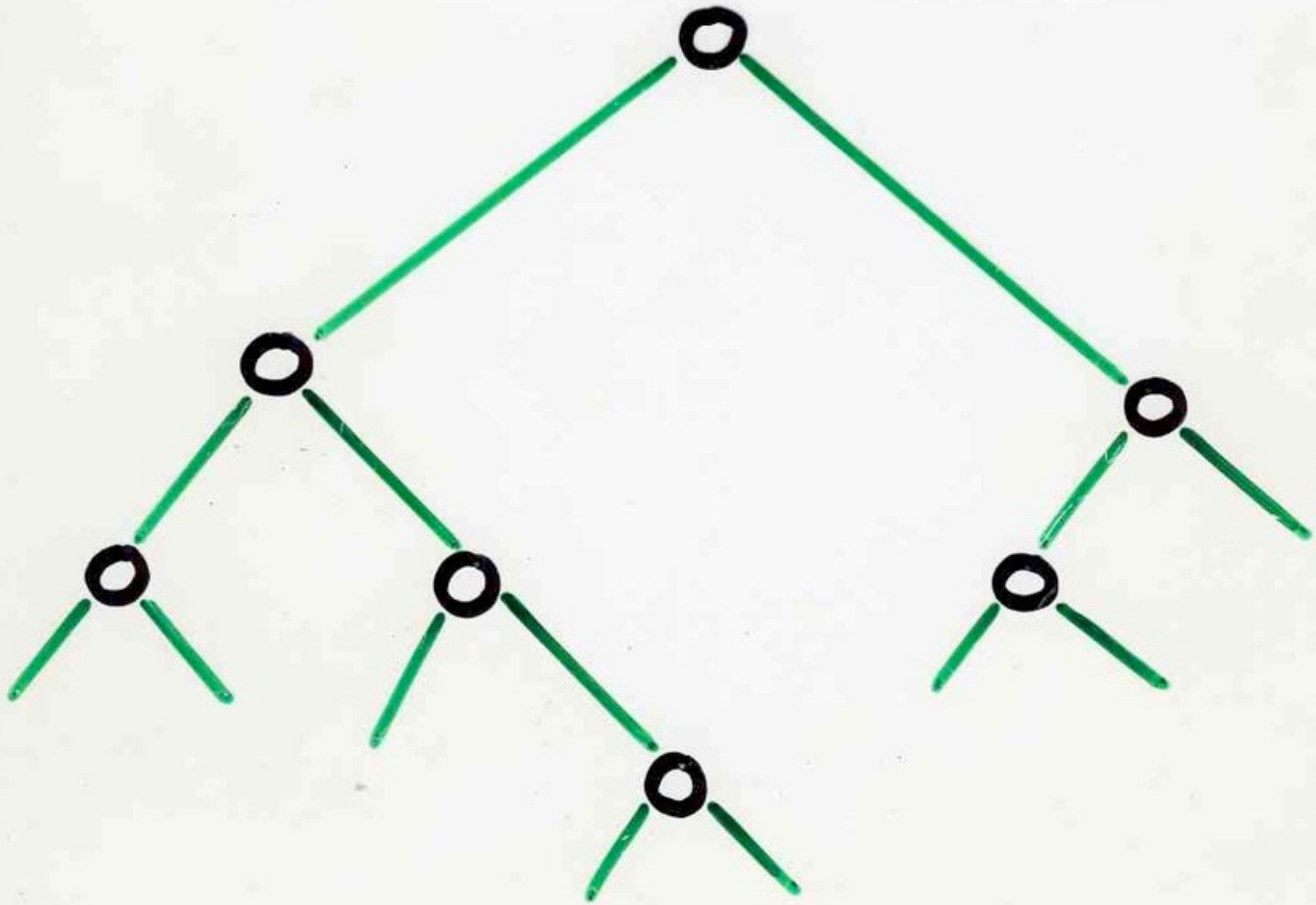


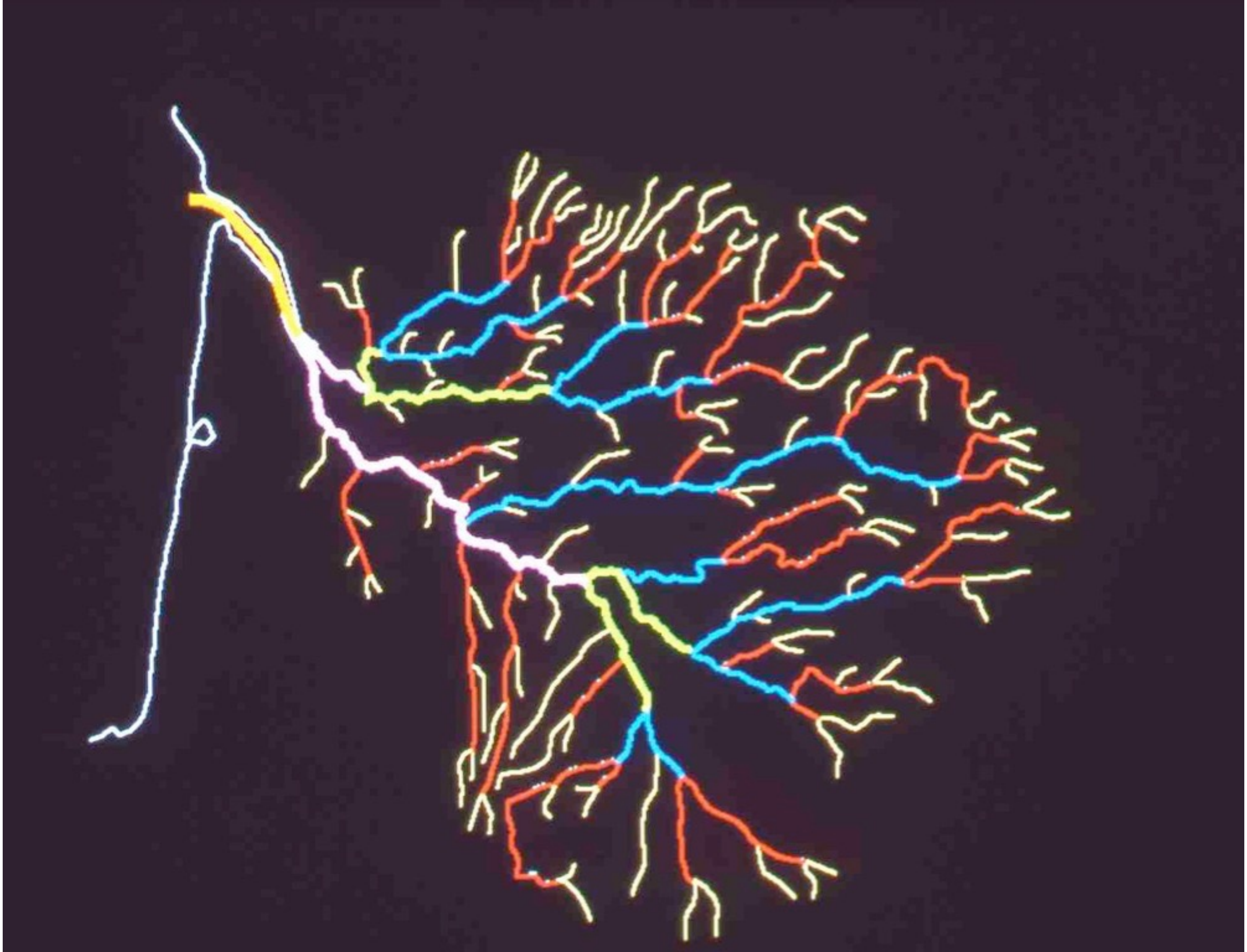
many works in physics (quantum gravity)
and probabilities using Schaeffer's bijection



Substitution in generating function

example:
Strahler number of binary trees





$$S_{n,k} = \left\{ \begin{array}{l} \text{number of binary trees } B \\ \text{with } n \text{ internal vertices} \\ \text{and } \quad St(B) = k \end{array} \right.$$

$$S_k(t) = \sum_{n \geq 0} S_{n,k}\, t^n$$

$$S_{k+1}(t) = t\, S_k^2(t) + 2t\, S_{k+1}(t) \left[\sum_{1 \leq i \leq k} S_i(t) \right]$$

$$S_1 = 1$$

$$S_2 = \frac{t}{1-2t}$$

$$S_3 = \frac{t^3}{1-6t+10t^2-4t^3}$$

$$S_4 = \frac{t^7}{1-14t+78t^2-220t^3+330t^4-252t^5+84t^6-8t^7}$$

Pascal triangle

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

1									
1	1								
1	2	1							
1	3	3	1						
1	4	6	4	1					
1	5	10	10	5	1				
1	6	15	20	15	6	1			
1	7	21	35	35	21	7	1		
1	8	28	56	70	56	28	8	1	
1	9	36	84	126	126	84	36	9	1
1	10	45	120	210	252	210	120	45	10
1	11	55	165	330	462	462	330	165	55
1	12	66	220	495	792	924	792	495	220
1	13	78	286	715	1287	1716	1716	1287	715
1	14	91	364	1001	2002	3003	3432	3003	2002
1	15	105	455	1365	3003	5005	6435	6435	5005

$$S_k(t) = \frac{t^{(2^{k-1}-1)}}{R_{2^k-1}(t)} = S_{\leq k}(t) - S_{\leq (k-1)}(t)$$

$$S_{\leq k}(t) = \frac{R_{2^k-2}(t)}{R_{2^k-1}(t)}$$

please replace R by F

Fibonacci polynomials

Strahler number (B)

$$\left\lfloor \log_2 (1 + \text{height}(\omega)) \right\rfloor$$

$$LH(\omega) = k \iff 2^{k-1} - 2 < H(\omega) \leq 2^{k} - 2$$