Combinatorics and Physics

Chapter 2a
Dimers, Tilings, Non-crossing paths and determinant

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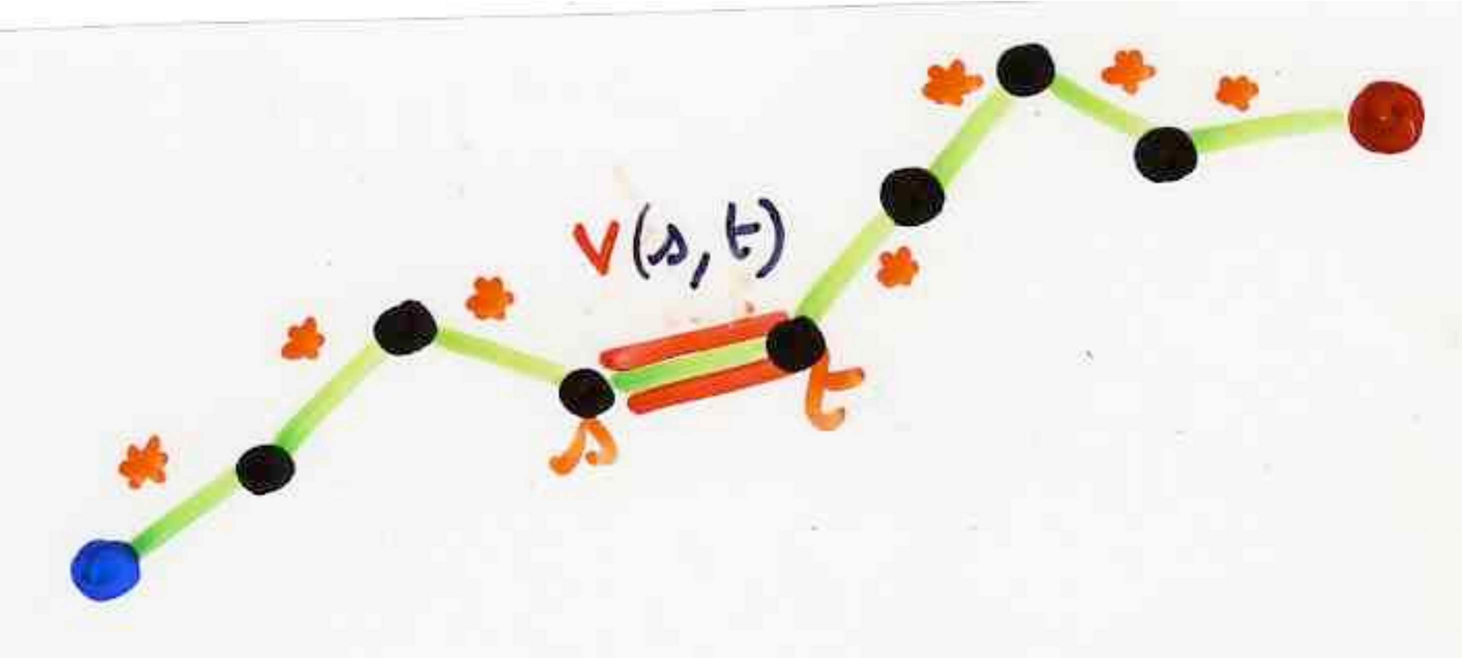
§1 The LGV Lemma


determinant

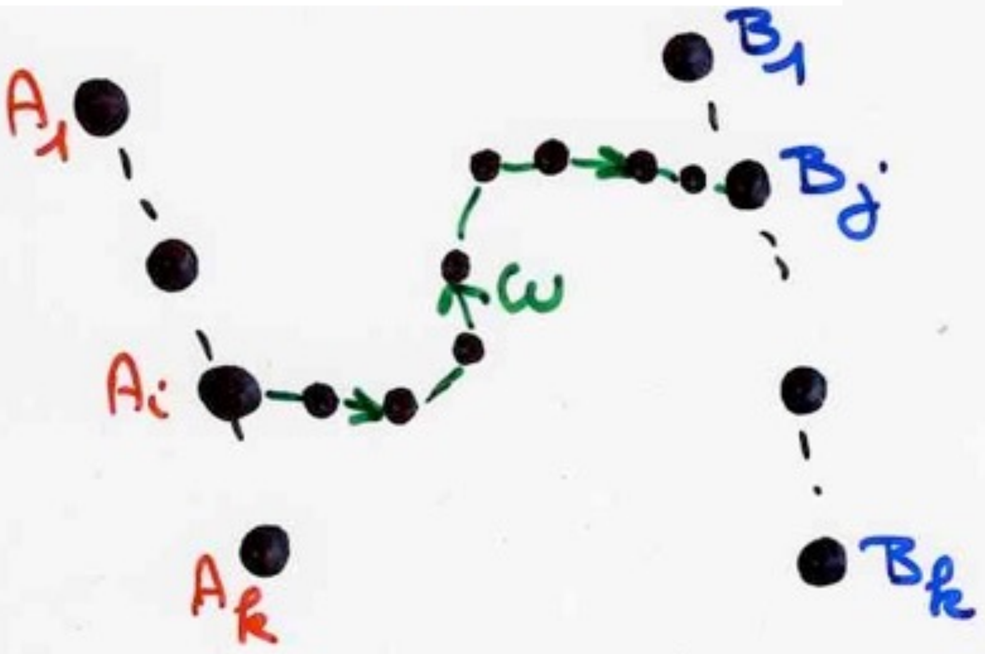
Path $\omega = (s_0, s_1, \dots, s_n)$ $s_i \in \Pi$

valuation $v : \Pi \times \Pi \to \mathbb{K}$ commutative ring

$$v(\omega) = v(s_0, s_1) \cdots\cdots v(s_{n-1}, s_n)$$



LGV methodology



$A_1, \dots, A_k$

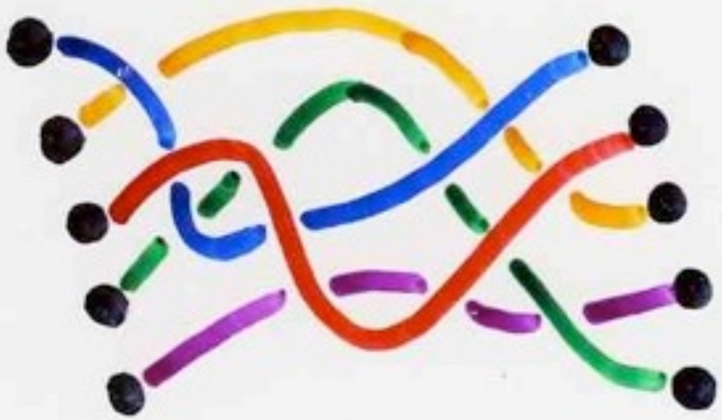
$B_1, \dots, B_k$

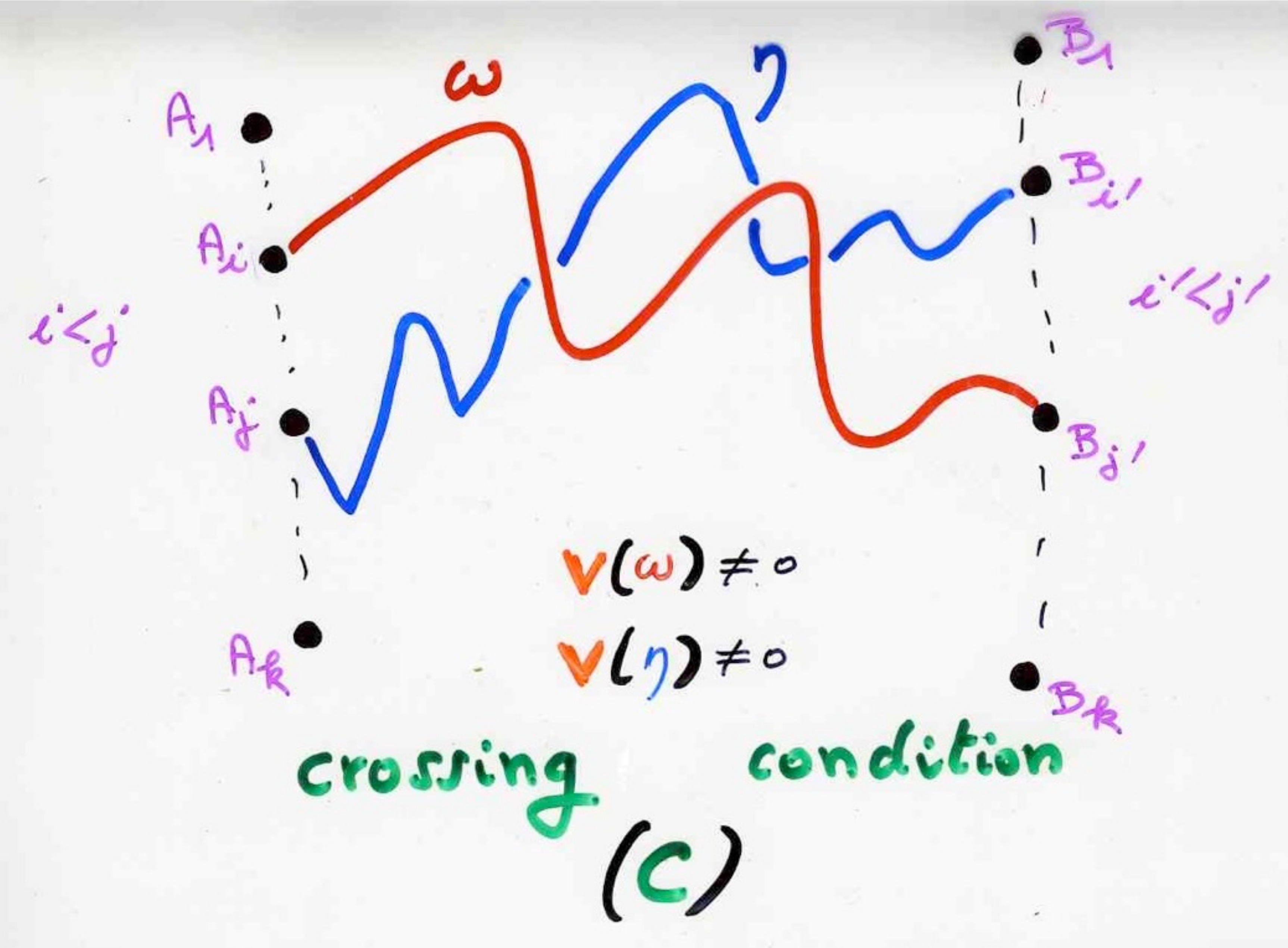
$$a_{ij} = \sum_{A_i \rightsquigarrow B_j} v(\omega)$$

suppose finite sum

$$\det(a_{ij}) = \sum_{(\sigma;\, \omega_1, \dots, \omega_k)} (-1)^{\mathrm{Inv}(\sigma)}\, v(\omega_1) \cdots v(\omega_k)$$

$$\omega_i : A_i \rightsquigarrow B_{\sigma(i)}$$

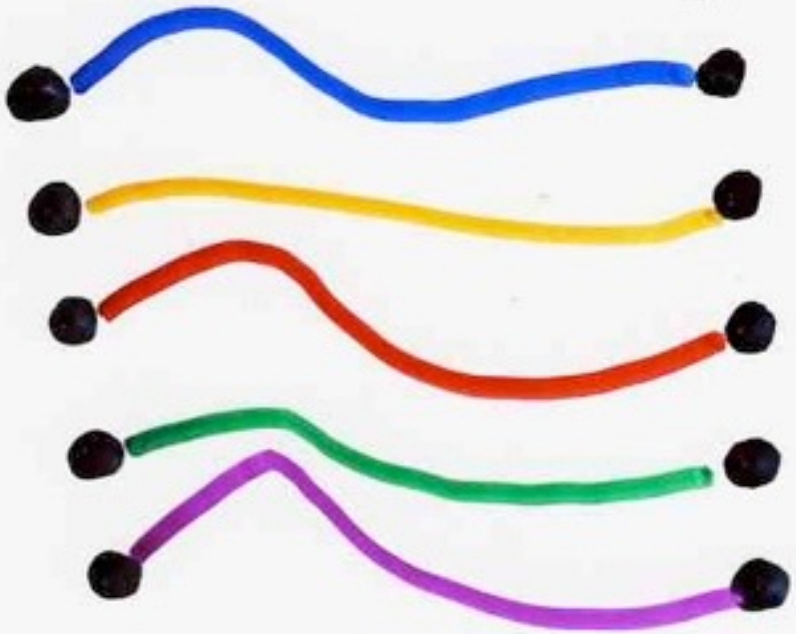


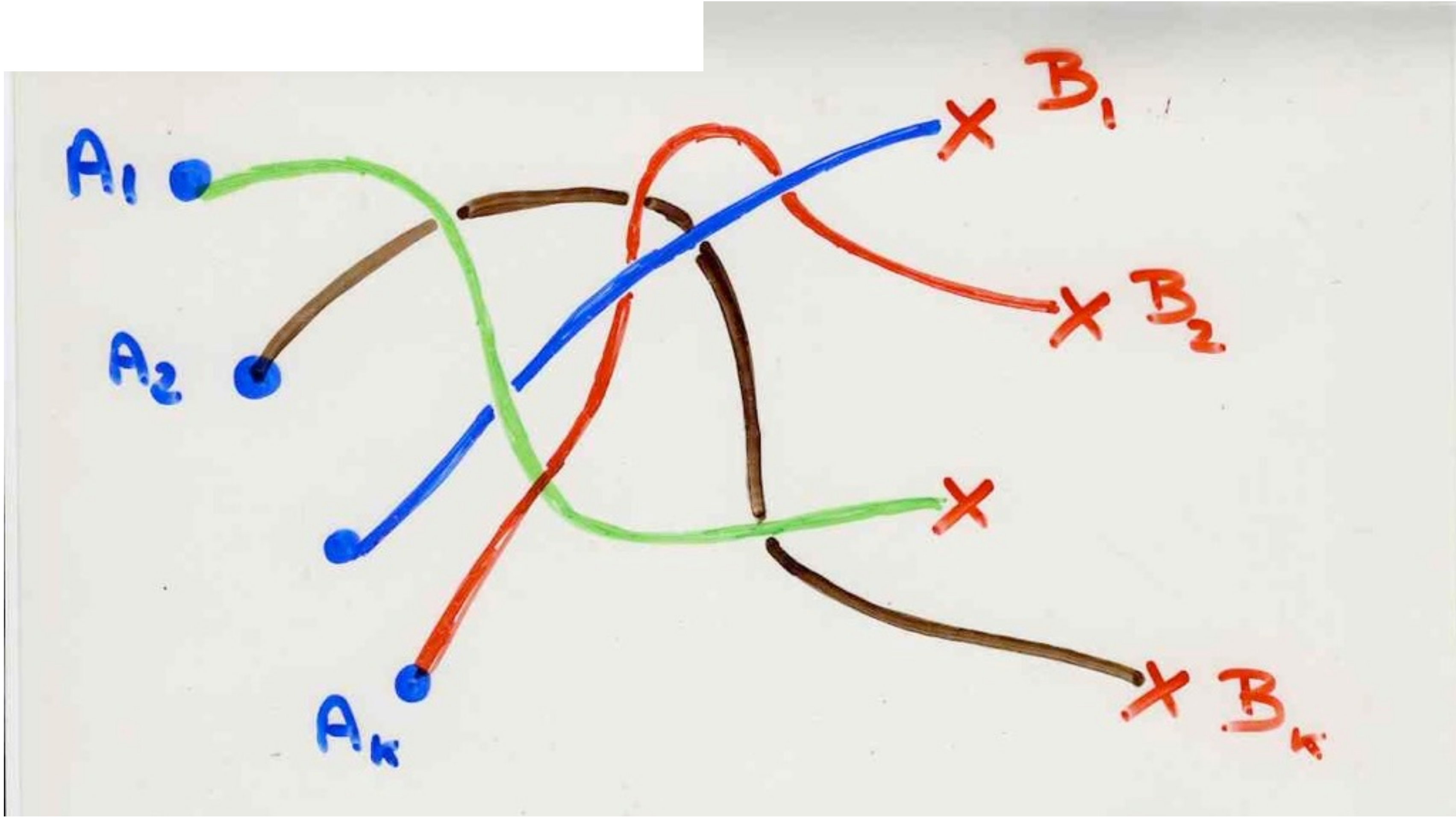


A_1
A_i
A_j
A_k
B_1
$B_{i'}$
$B_{j'}$
B_k
ω
η
$i<j$
$i'<j'$
$V(\omega)\neq 0$
$V(\eta)\neq 0$
crossing condition
(C)

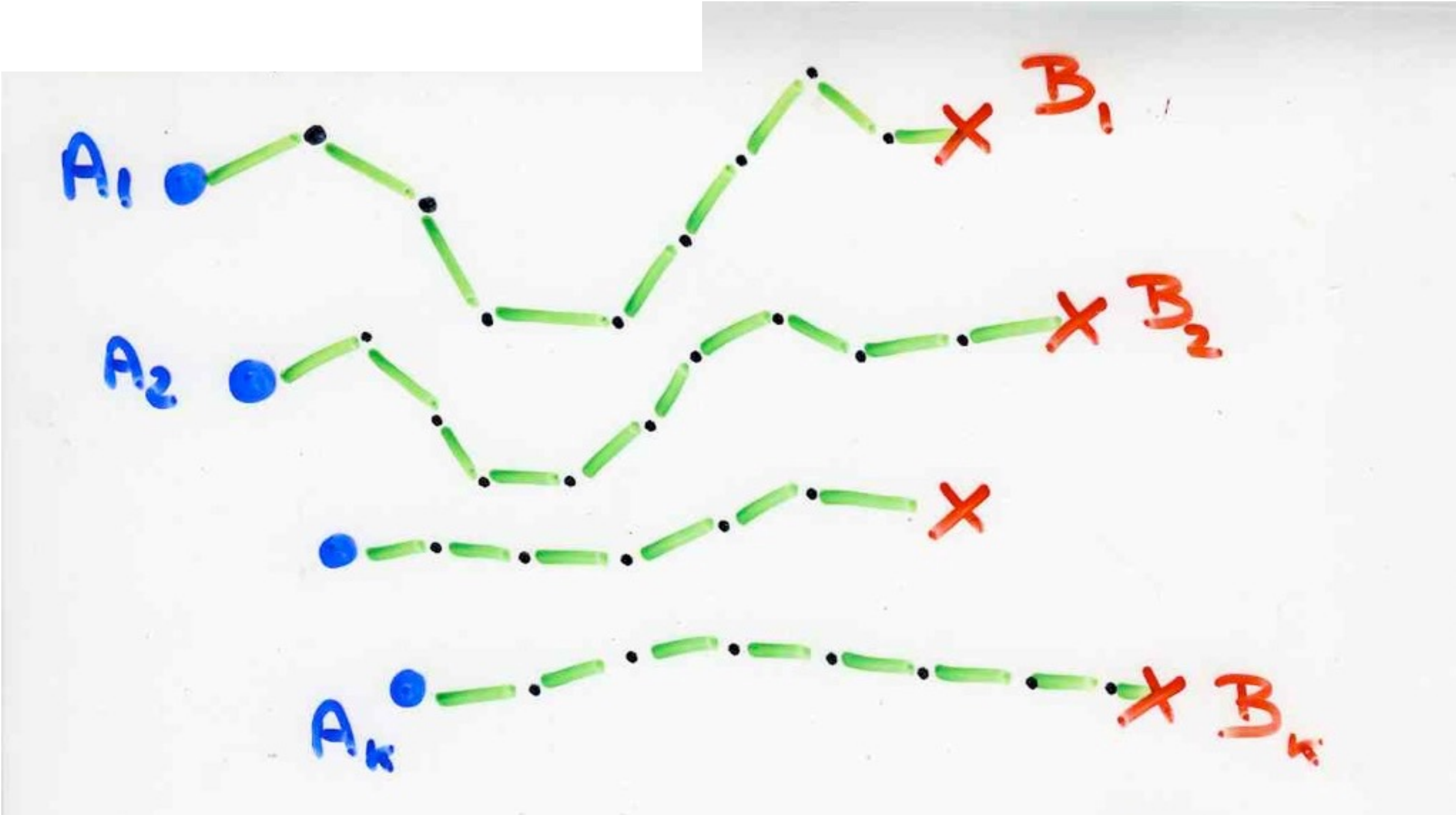
Prop. (C)

$$\det(a_{ij}) = \sum_{\substack{\Omega=(\omega_1,\dots,\omega_k) \\ \omega_i : A_i \rightsquigarrow B_i \\ \text{2 by 2 disjoints}}} v(\omega_1)\dots v(\omega_k)$$



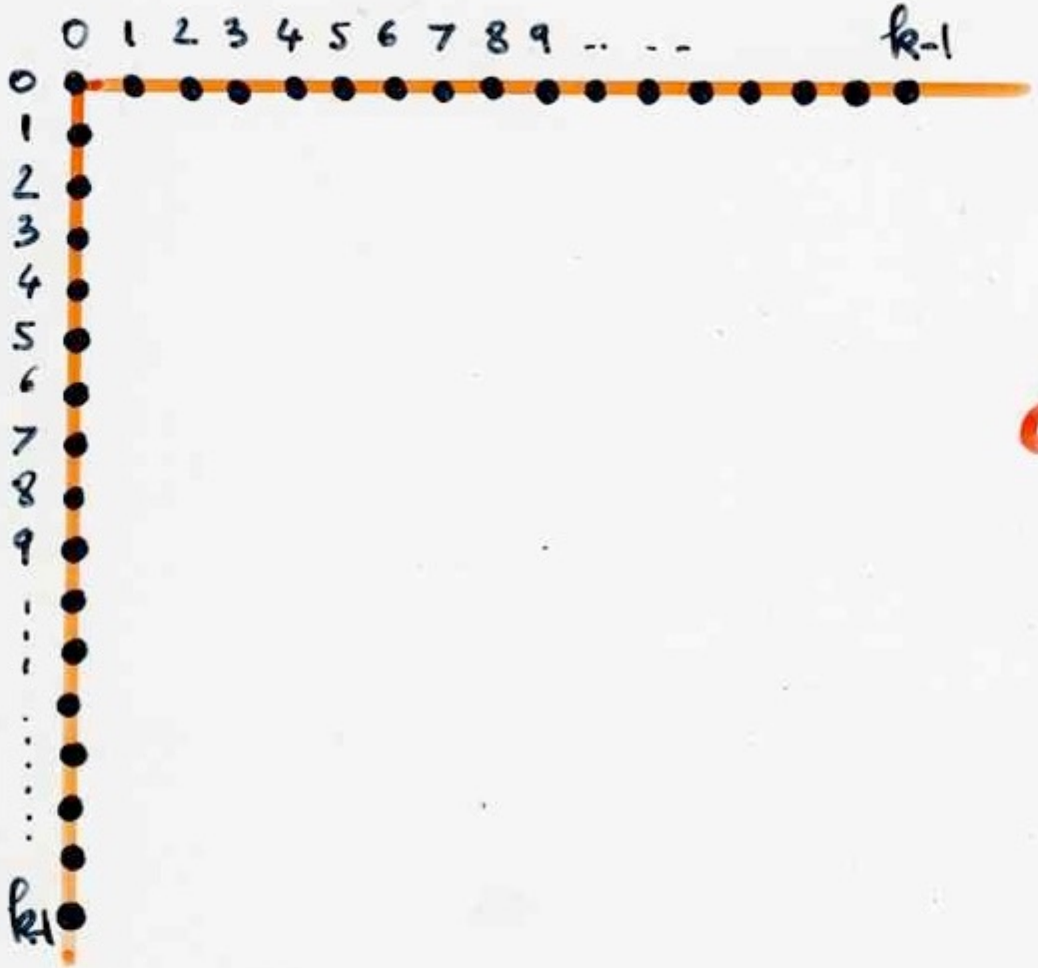


A1
A2
Ak
B1
B2
Bk

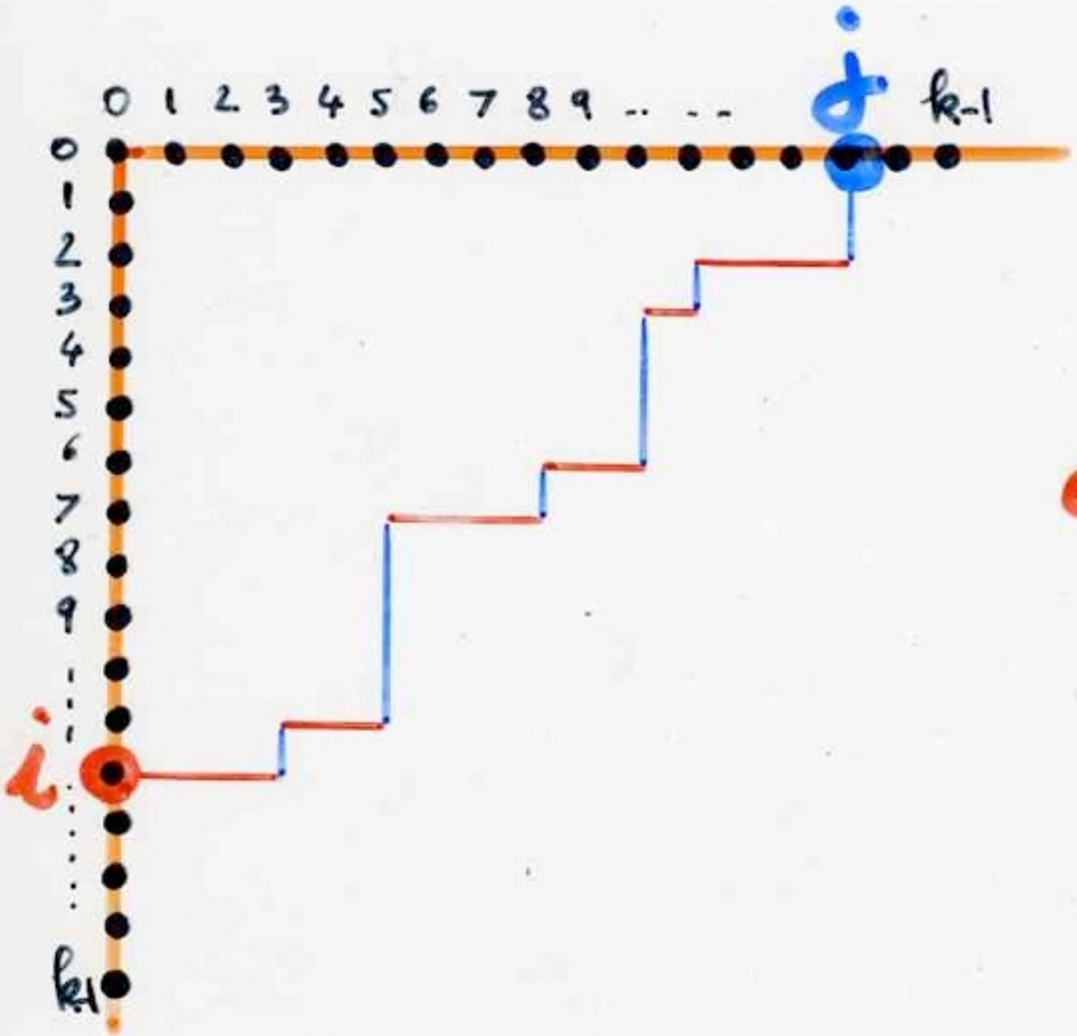


A_1
B_1
A_2
B_2
A_k
B_k

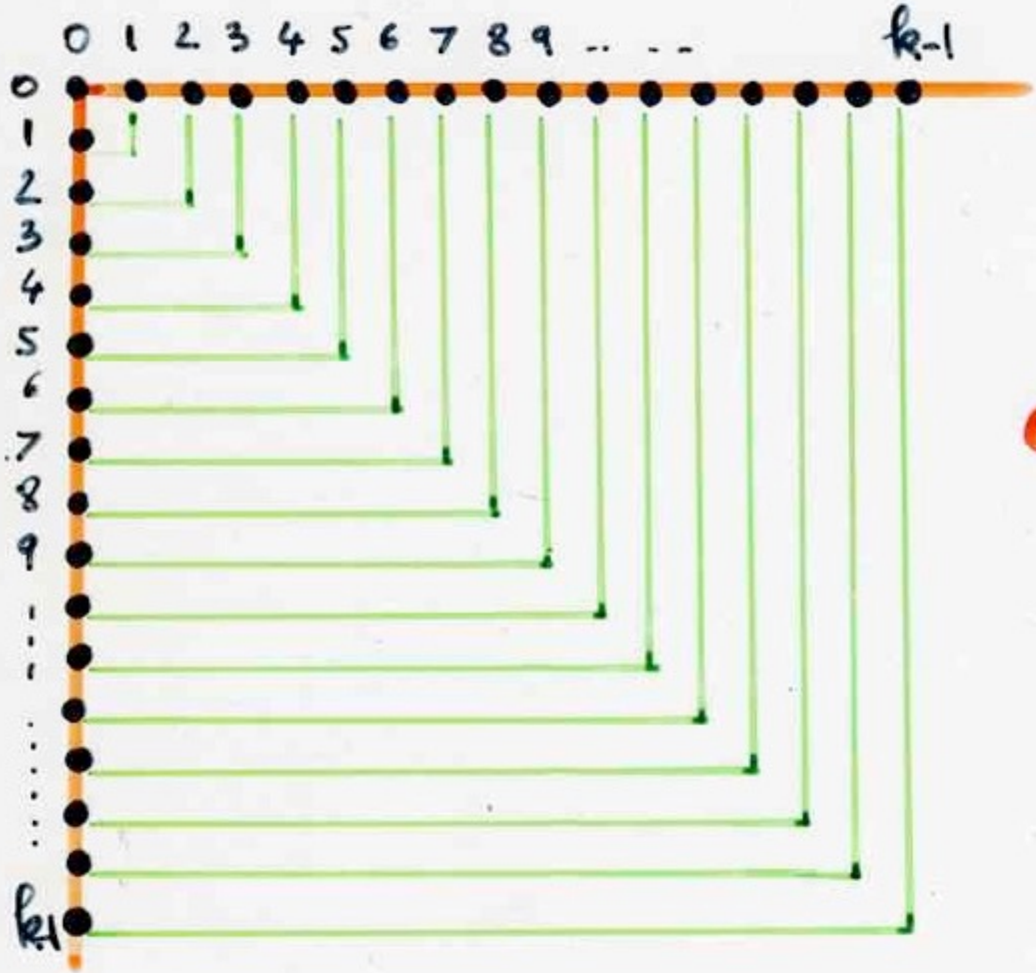
a simple example



$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k \times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots & \\ 1 & 3 & 6 & 10 & \cdots & & \\ 1 & 4 & 10 & \cdots & & & \\ 1 & 5 & \cdots & & \binom{i+j}{i} & & \\ 1 & \cdots & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} =$$



$$\det\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & 5 & & \\ 1 & 3 & 6 & 10 & & & \\ 1 & 4 & 10 & & & & \\ 1 & 5 & & & \binom{i+j}{i} & & \\ 1 & & & & & & \\ \vdots & & & & & & \end{bmatrix}_{k\times k} = 1$$

proof of LGV Lemma

Proof: Involution ϕ

$$E = \left\{ (\sigma ; (\omega_1, \dots, \omega_k)) ; \begin{array}{l} \sigma \in S_n \\ \omega_i : A_i \rightsquigarrow B_{\sigma(i)} \end{array} \right\}$$

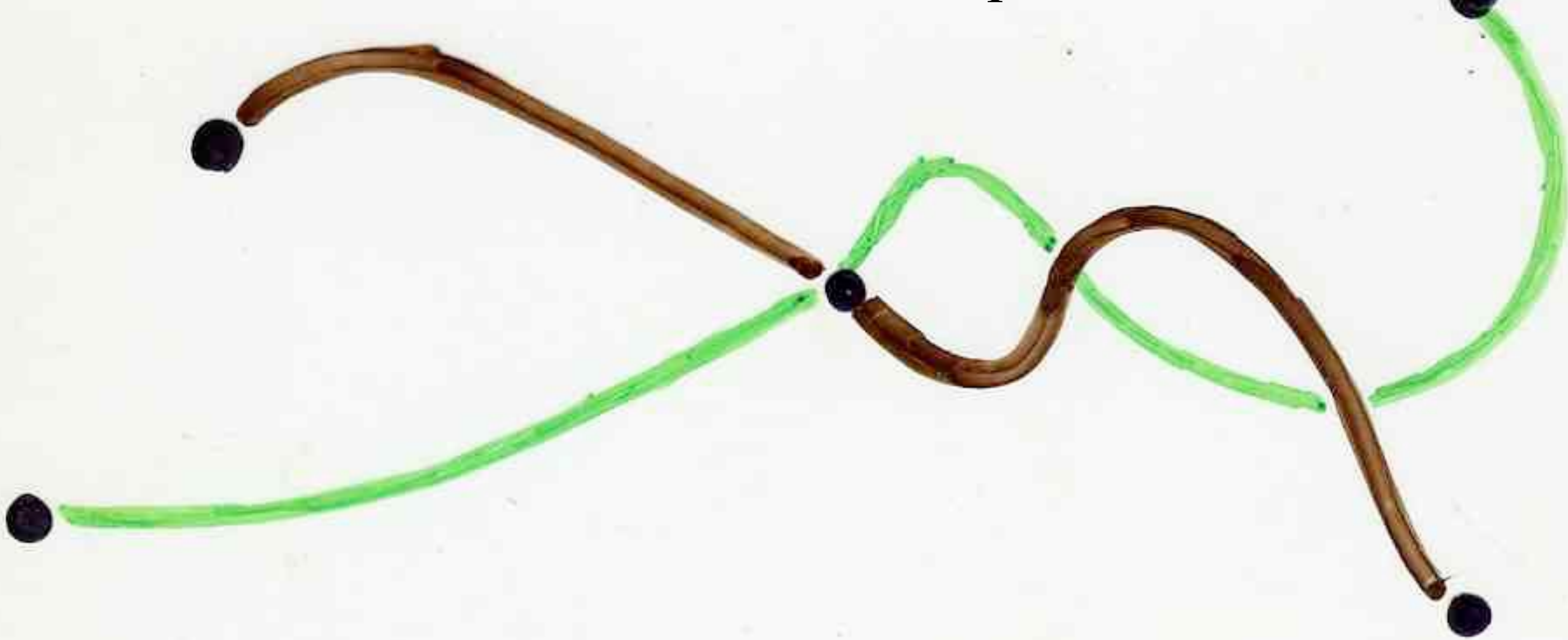
$NC \subseteq E$ non-crossing configurations

$$\phi : (E - NC) \to (E - NC)$$

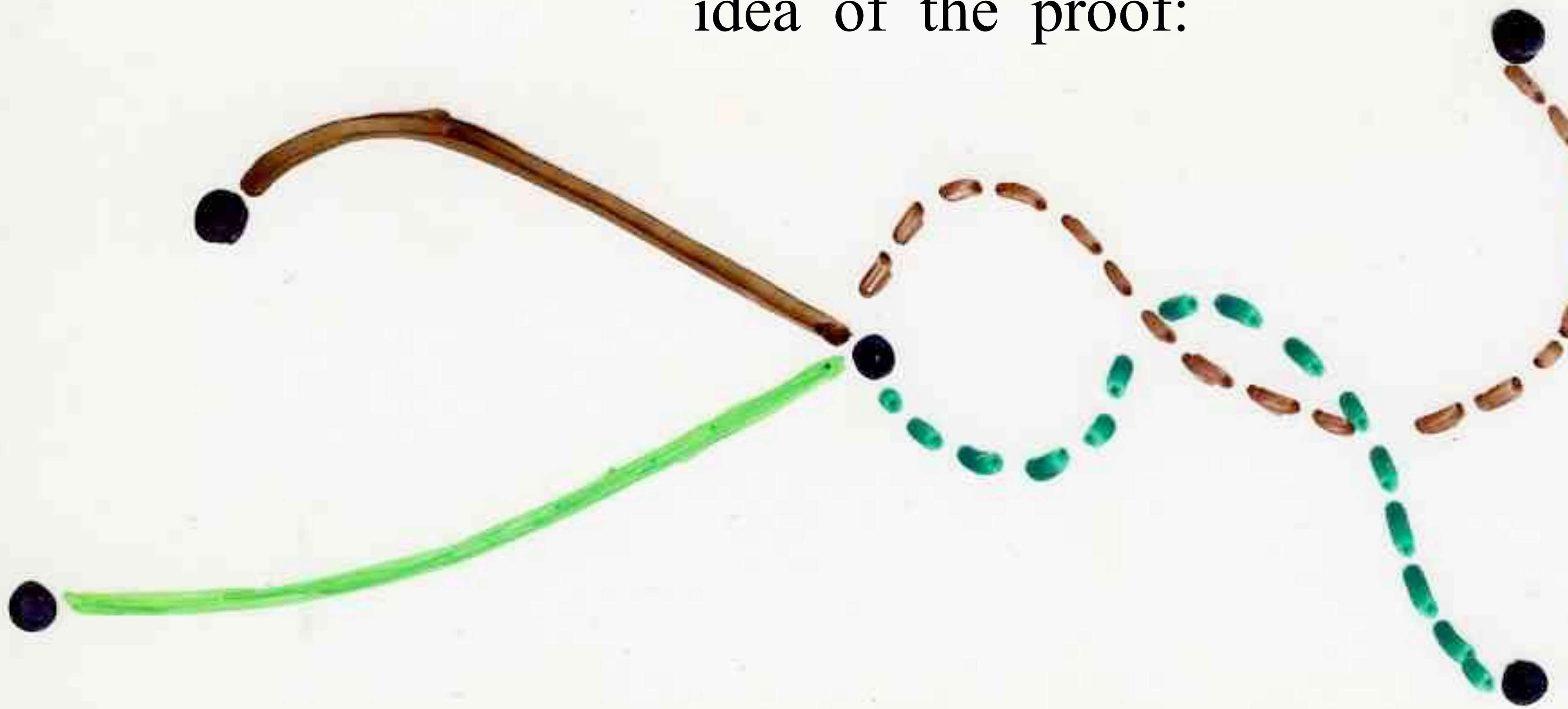
$$\phi(\sigma ; (\omega_1, \dots, \omega_k)) = (\sigma' ; (\omega'_1, \dots, \omega'_k))$$

$$\begin{cases} (-1)^{\mathrm{Inv}(\sigma)} = -(-1)^{\mathrm{Inv}(\sigma')} \\ v(\omega_1) \dots v(\omega_k) = v(\omega'_1) \dots v(\omega'_k) \end{cases}$$

idea of the proof:

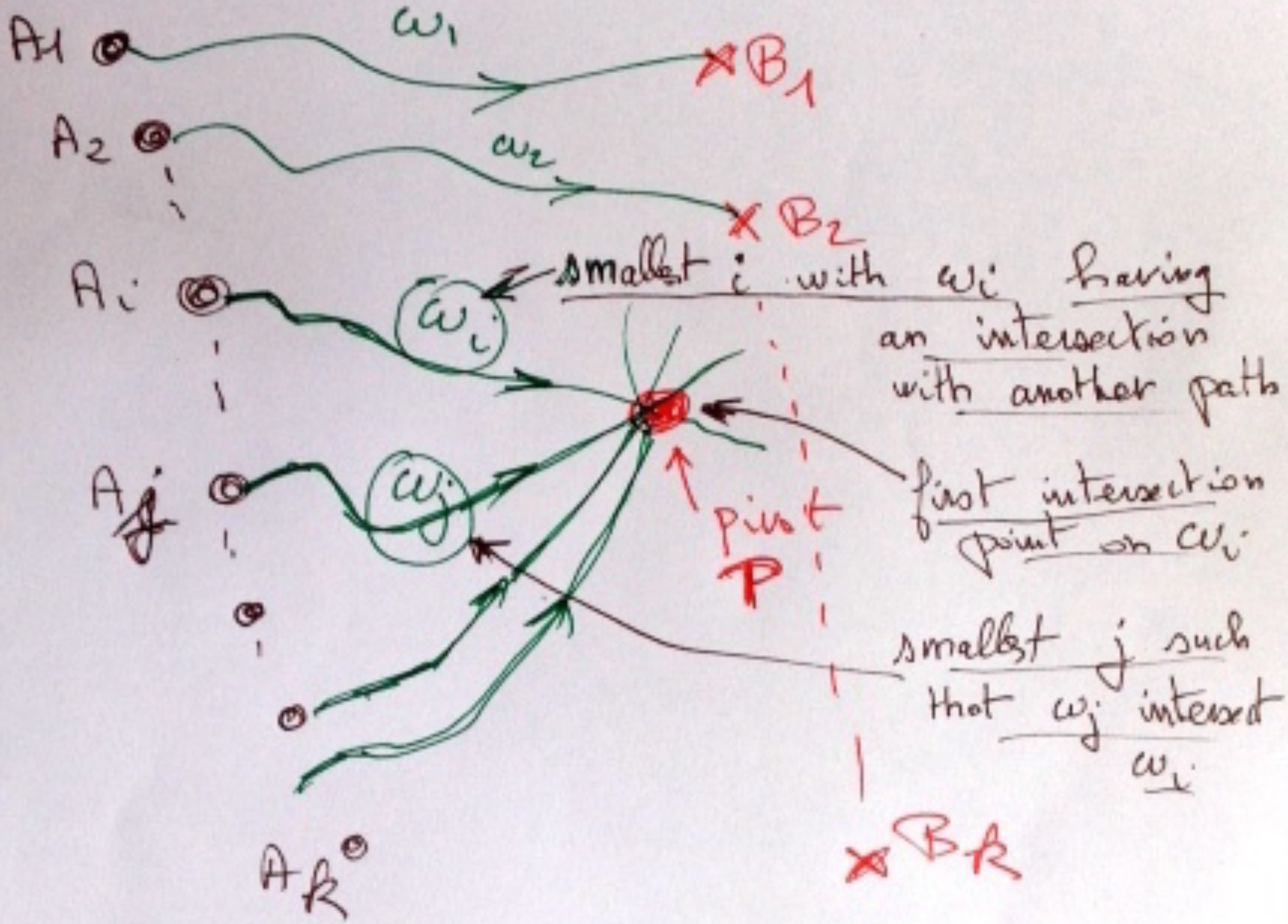


idea of the proof:



choice of the pivot P

choice of the two intersecting paths



Thm Lindström 1973 (matroids)

Gessel, Viennot 1985 (C)

$$\det(a_{ij}) = \sum_{(\omega_1,\dots,\omega_k)} v(\omega_1)\cdots v(\omega_k)$$

Chaundy 1932

Karlin-McGregor 1959

$\omega_i : A_i \rightsquigarrow B_i$

2 by 2 disjoints

B. Lindström, *On the vector representation of induced matroids*, Bull. London Maths. Soc. 5 (1973) 85-90.

I. Gessel and X.G. Viennot, *Binomial determinants, paths and hook length formula*, Advances in Maths., 58 (1985) 300-321.

I. Gessel and X.G. Viennot, *Determinants, paths and plane partitions, preprint (1989)*

combinatorics

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statistical physics: (wetting, melting)

Fisher, Vicious walkers, Botzmann lecture (1984)

combinatorial chemistry:

John, Sachs (1985)

Gronau, Just, Schade, Scheffler, Wojciechowski (1988)

probabilities, birth and death process,

Karlin , McGregor (1959)

quantum mechanics: Slater determinant

Slater(1929) (1968), De Gennes (1968)

Why « LGV Lemma » ?

from Christian Krattenthaler:

« Watermelon configurations with wall interaction: exact and asymptotic results »

J. Physics Conf. Series 42 (2006), 179--212,

4Lindström used the term "pairwise node disjoint paths". The term "non-intersecting," which is most often used nowadays in combinatorial literature, was coined by Gessel and Viennot [24].

5By a curious coincidence, Lindström's result (the motivation of which was matroid theory!) was rediscovered in the 1980s at about the same time in three different communities, not knowing from each other at that time: in statistical physics by Fisher [17, Sec. 5.3] in order to apply it to the analysis of vicious walkers as a model of wetting and melting, in combinatorial chemistry by John and Sachs [30] and Gronau, Just, Schade, Scheffler and Wojciechowski [28] in order to compute Pauling's bond order in benzenoid hydrocarbon molecules, and in enumerative combinatorics by Gessel and Viennot [24, 25] in order to count tableaux and plane partitions. Since only Gessel and Viennot rediscovered it in its most general form, I propose to call this theorem the "Lindstrom–Gessel–Viennot theorem." It must however be mentioned that in fact the same idea appeared even earlier in work by Karlin and McGregor [32, 33] in a probabilistic framework, as well as that the so-called "Slater determinant" in quantum mechanics (cf. [48] and [49, Ch. 11]) may qualify as an "ancestor" of the Lindstro¨m–Gessel–Viennot determinant.

6There exist however also several interesting applications of the general form of the Lindstro¨m–Gessel–Viennot theorem in the literature, see [10, 16, 51].

§2 Binomial determinants

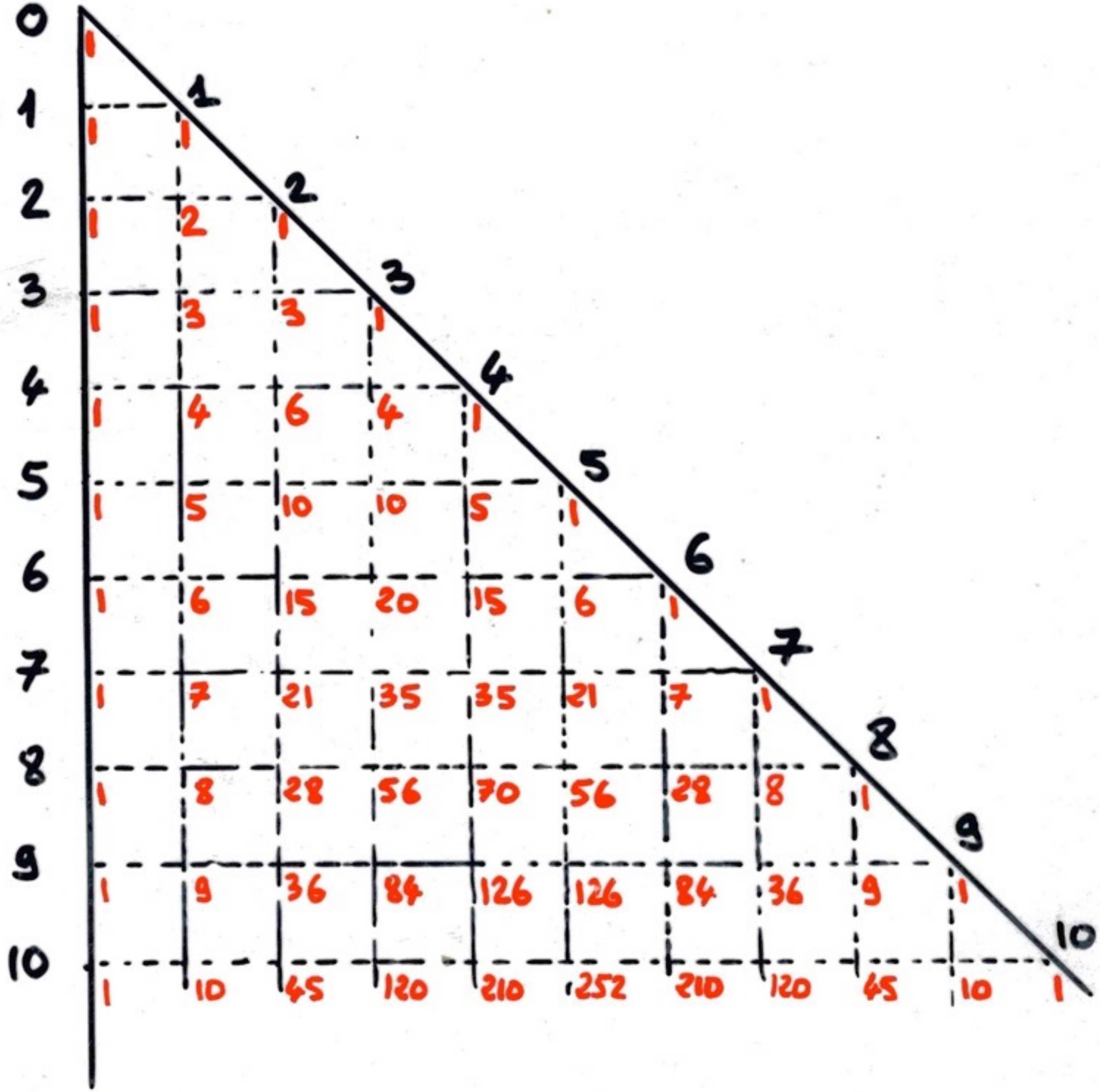
$0 \leq a_1 < \cdots < a_k$

$0 \leq b_1 < \cdots < b_k$

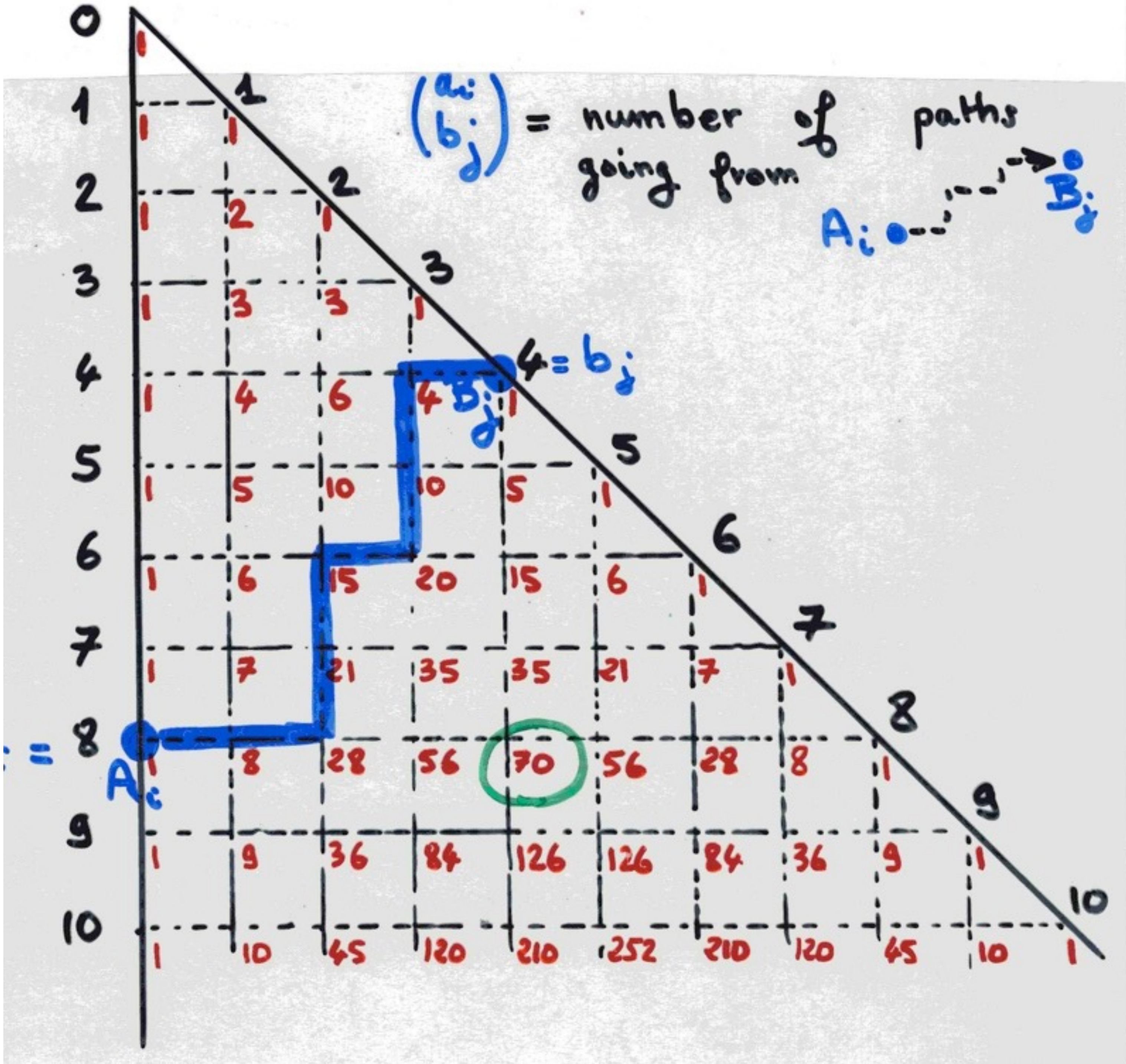
$$\binom{a_1, \ldots, a_k}{b_1, \ldots, b_k} = \det\left(\binom{a_i}{b_j}\right)_{1 \leq i, j \leq k}$$

I. Gessel, X.G. Viennot
(Adv. in Maths
58 (1985) 300-321)

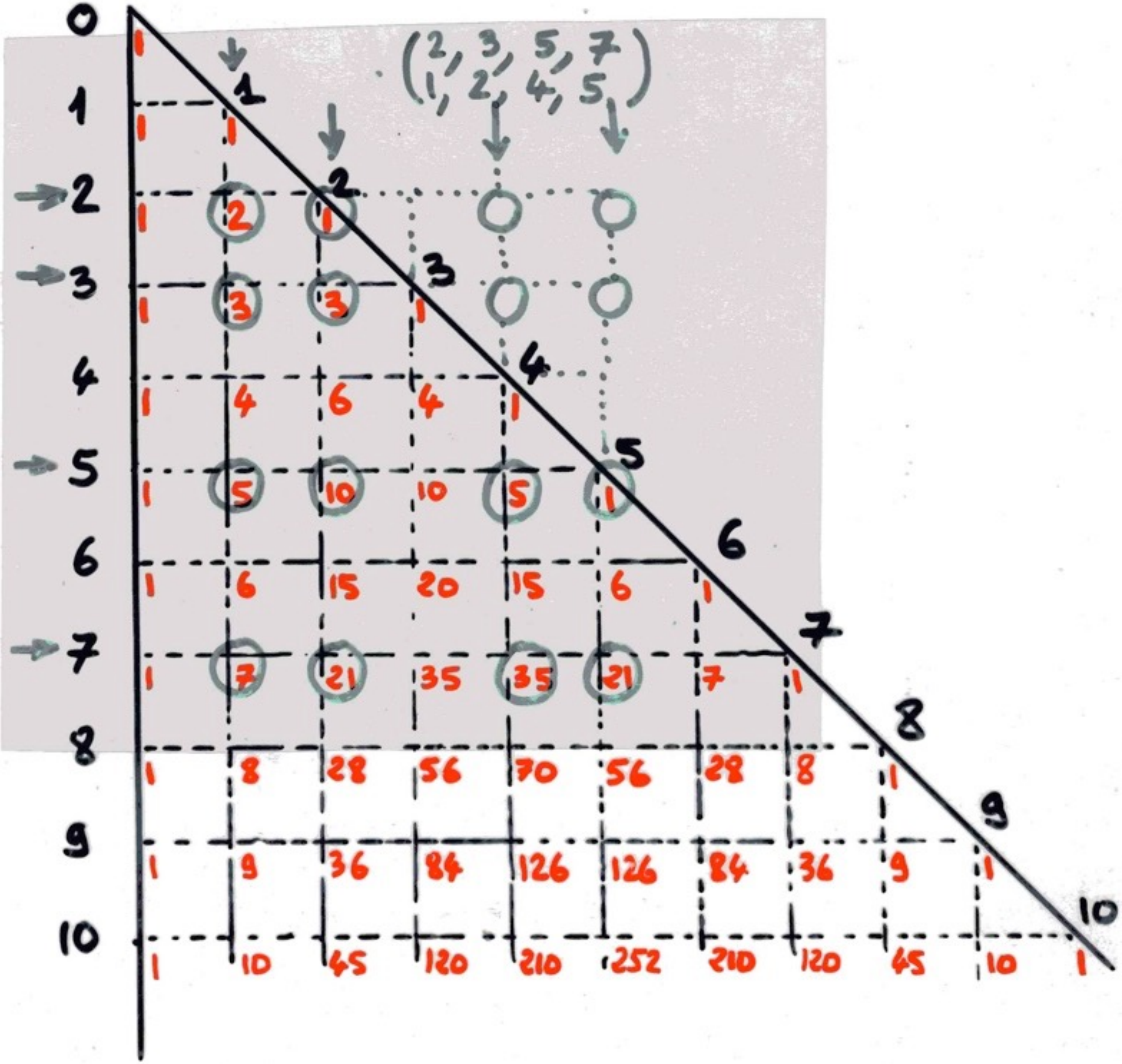
Binomial Determinant

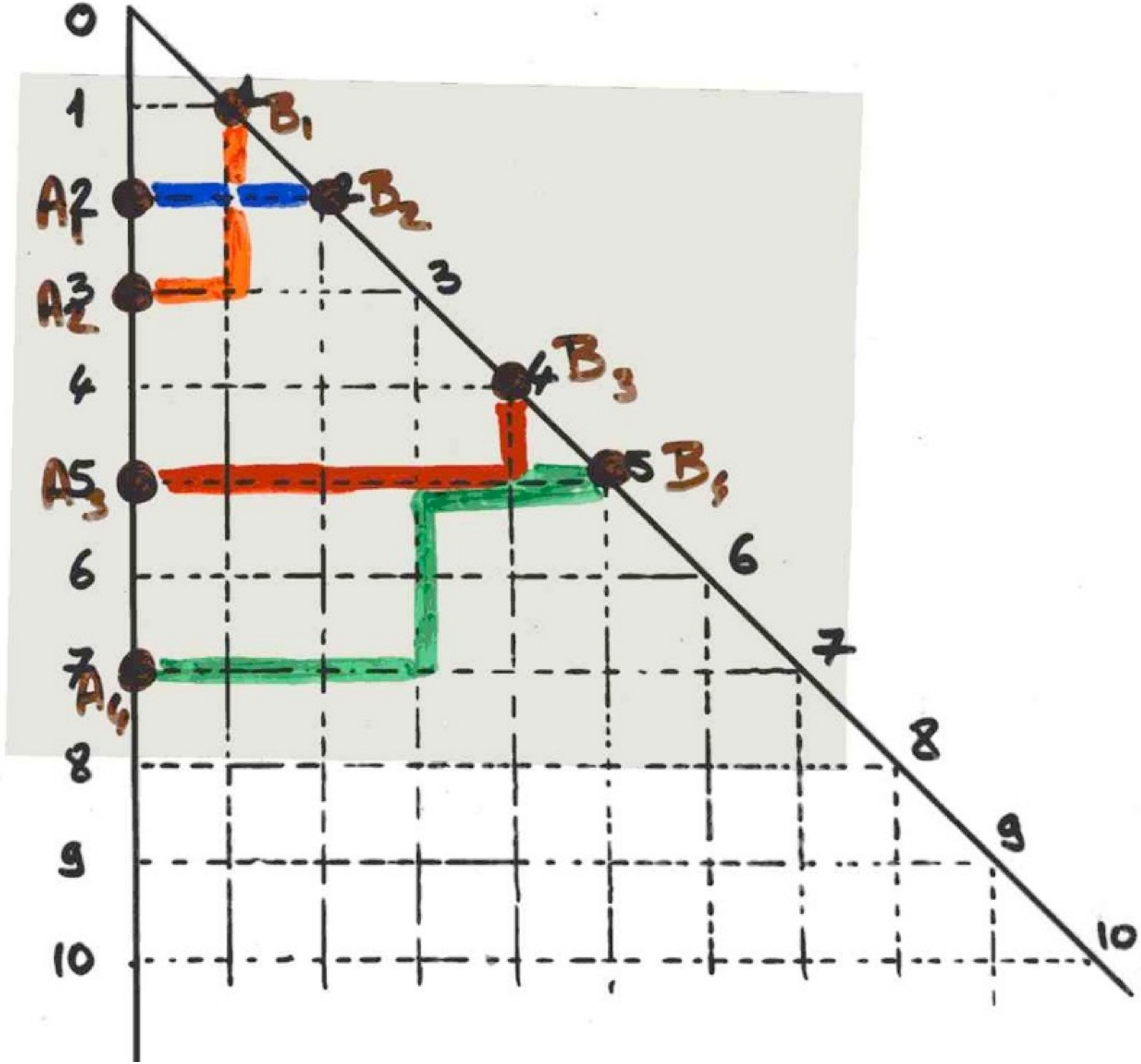


0
1
1
1 1
2
1 2 1
3
1 3 3 1
4
1 4 6 4 1
5
1 5 10 10 5 1
6
1 6 15 20 15 6 1
7
1 7 21 35 35 21 7 1
8
1 8 28 56 70 56 28 8 1
9
1 9 36 84 126 126 84 36 9 1
10
1 10 45 120 210 252 210 120 45 10 1

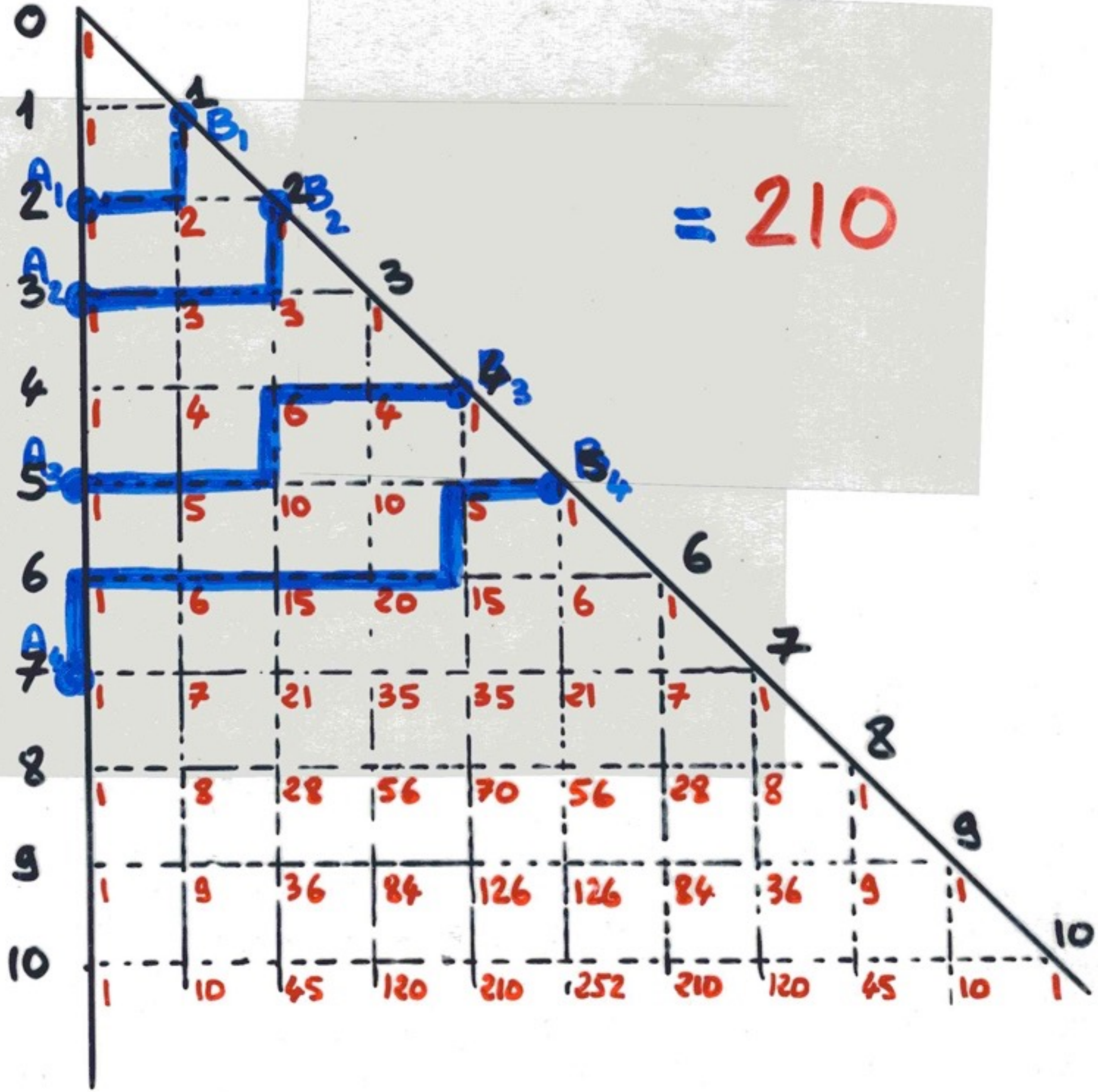


$\binom{a_i}{b_j}$ = number of paths going from A_i to B_j
$4 = b_j$
B_j
A_i


(2, 3, 5, 7
1, 2, 4, 5)
0 1 2 3 4 5 6 7 8 9 10
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



0
1
2
3
4
5
6
7
8
9
10
B_1
B_2
B_3
B_4
A_1
A_2
A_3
A_4



0
1
2
3
4
5
6
7
8
9
10
A_1
A_2
A_3
A_4
B_1
B_2
B_3
B_4
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
= 210

Chern classes calculus

tensor product fiber bundles

Schur functions

Lascoux 1978

Cor 1. $\begin{pmatrix} a_1, \dots, a_k \\ b_1, \dots, b_k \end{pmatrix} \geq 0$

Cor 2. Nb of nonzero minors of $A_n = \left[\binom{i}{j}\right]_{0 \leq i,j \leq n}$ is C_{n+2}

Catalan nb

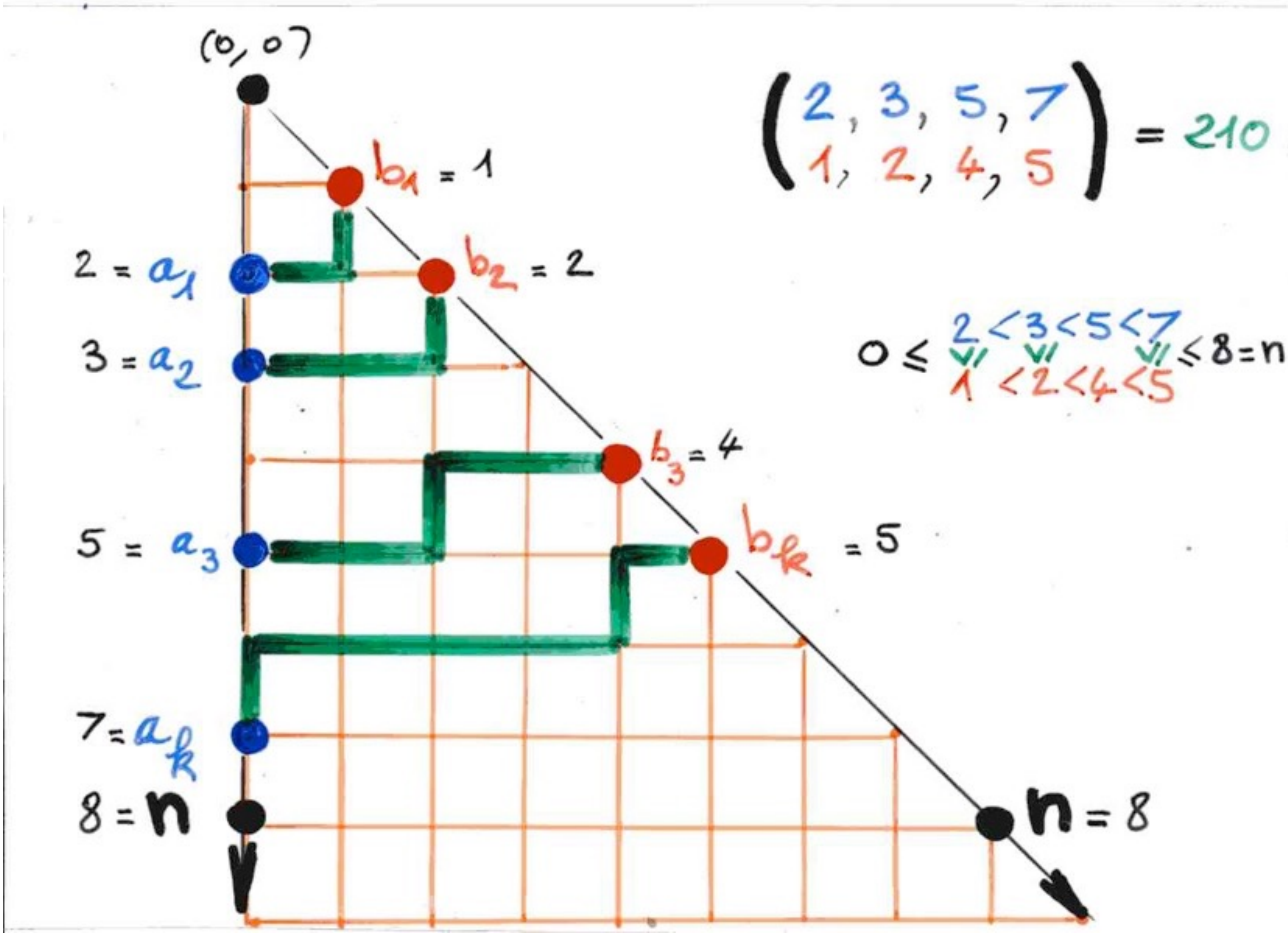
$$0 \leq \begin{matrix} a_1 < \dots < a_k \\ b_1 < \dots < b_k \end{matrix} \leq n$$

Cor 2. Nb of nonzero minors

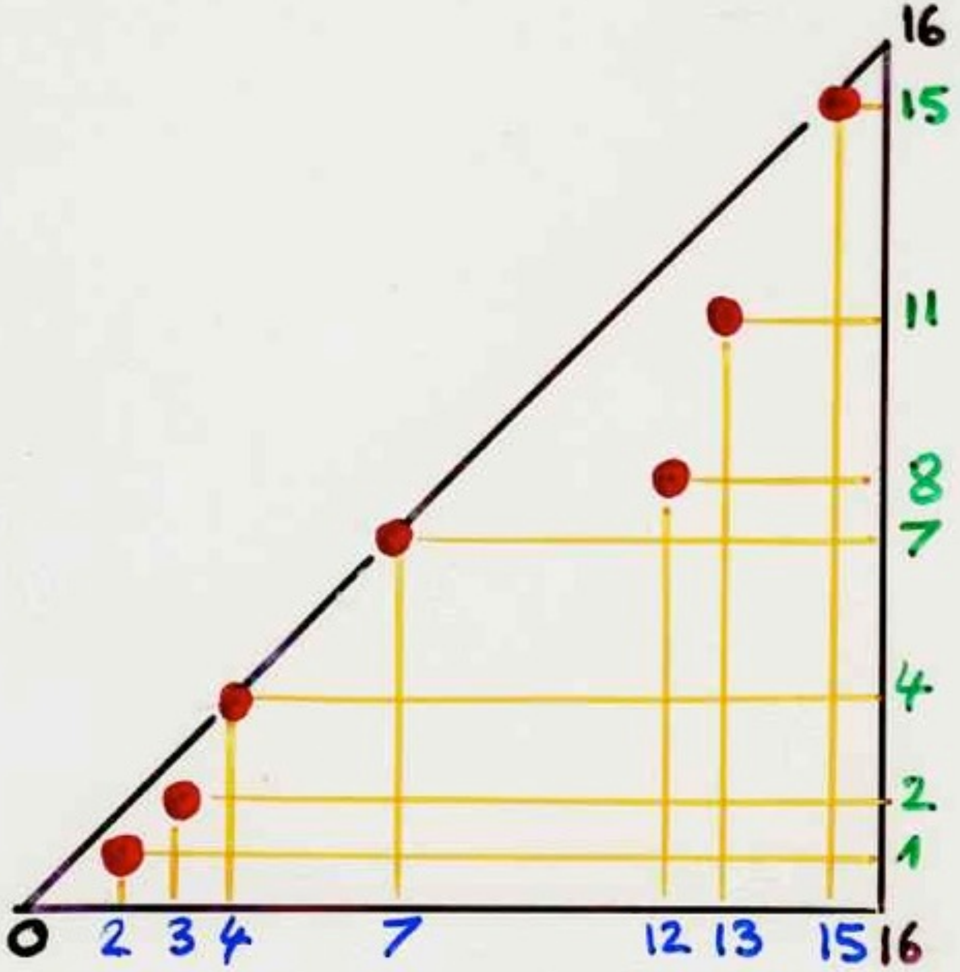
of $A_n = \left[\binom{i}{j}\right]_{0\le i,j\le n}$ is C_{n+2}

Catalan nb

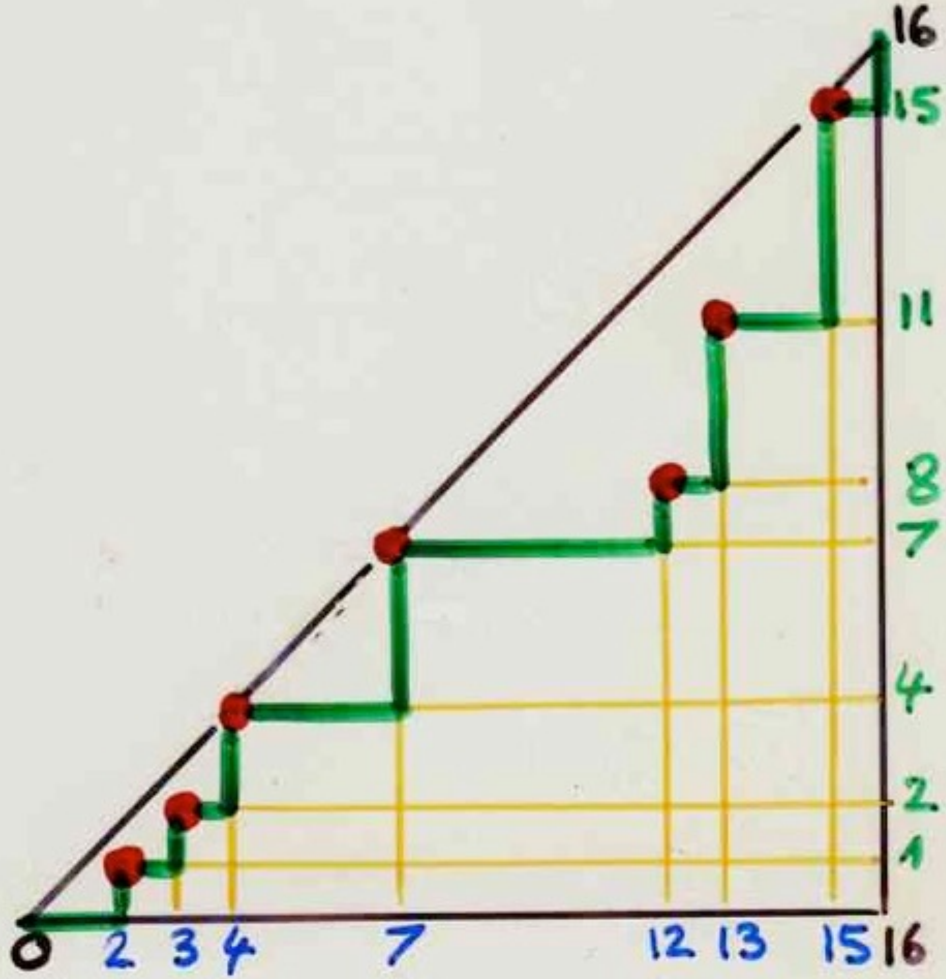
$$0 \le \begin{array}{c} a_1 < \cdots < a_k \\ \rotatebox{90}{$\geq$} \qquad\quad \rotatebox{90}{$\geq$} \\ b_1 < \cdots < b_k \end{array} \le n$$



(0, 0)
$b_1 = 1$
$2 = a_1$
$b_2 = 2$
$3 = a_2$
$b_3 = 4$
$5 = a_3$
$b_k = 5$
$7 = a_k$
$8 = n$
$n = 8$
$\begin{pmatrix} 2, & 3, & 5, & 7 \\ 1, & 2, & 4, & 5 \end{pmatrix} = 210$
$0 \le 2 < 3 < 5 < 7 \le 8 = n$
$1 < 2 < 4 < 5$



$$\begin{array}{ccccccccccccccccc} & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\ & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\ 1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n \end{array}$$



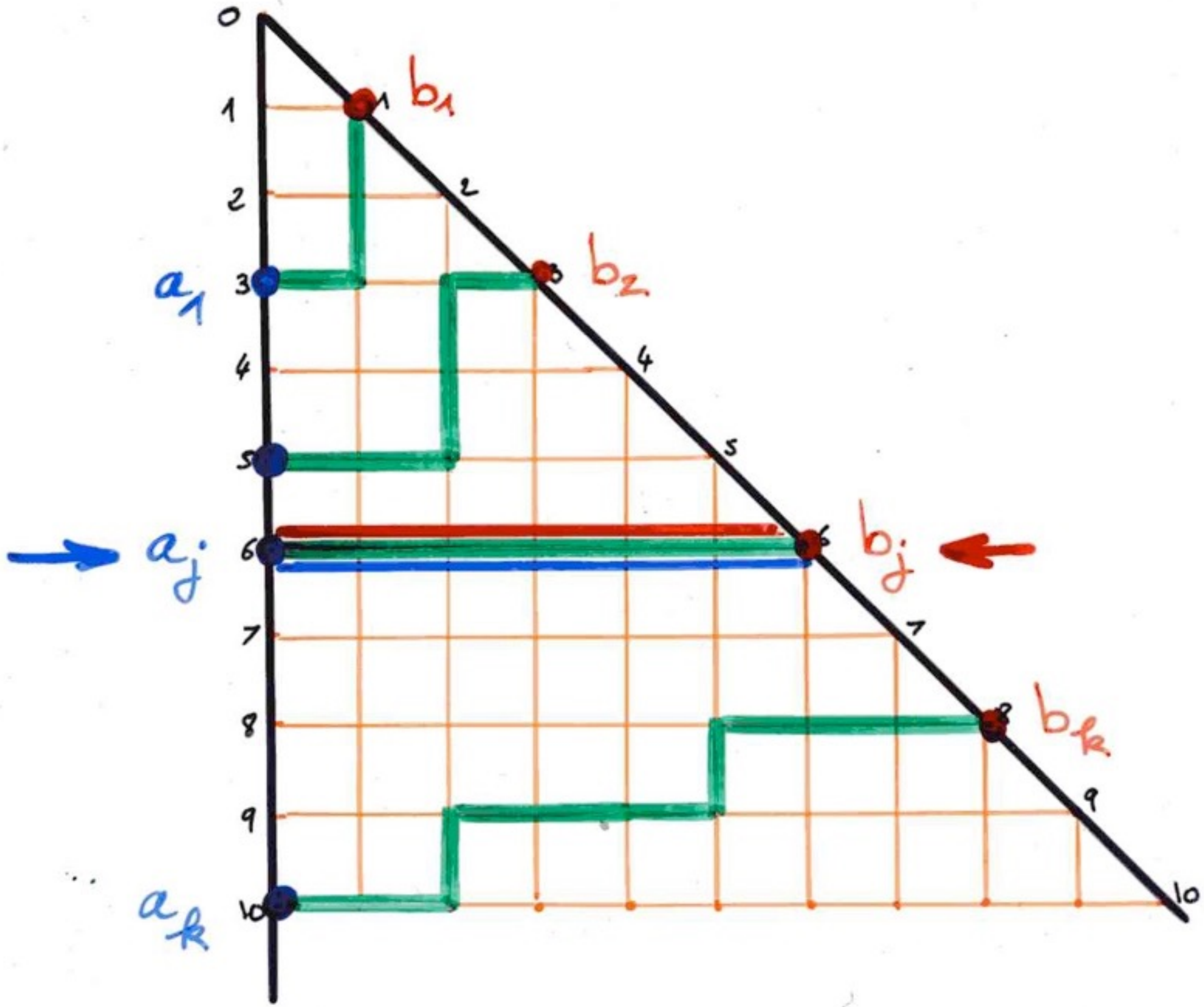
$$
\begin{array}{ccccccccccccccccc}
 & & 2 & < & 3 & < & 4 & < & 7 & < & 12 & < & 13 & < & 15 & & \\
 & & \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & & \vee\!| & & \\
1 & \leq & 1 & < & 2 & < & 4 & < & 7 & < & 8 & < & 11 & < & 15 & \leq & n
\end{array}
$$

Cor 3. If $a_j = b_j$

$$\begin{pmatrix} a_1, & \dots\dots, & a_j, & \dots\dots, & a_k \\ & & \parallel & & \\ b_1, & \dots\dots, & b_j, & \dots\dots, & b_k \end{pmatrix}$$

Cor 3. If $a_j = b_j$

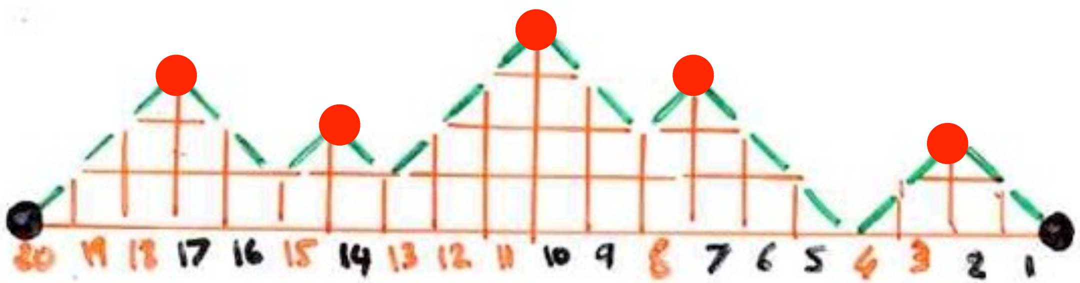
$$\begin{pmatrix} a_1, \dots, a_{j-1} \\ b_1, \dots, b_{j-1} \end{pmatrix} \begin{pmatrix} a_{j+1}, \dots, a_k \\ b_{j+1}, \dots, b_k \end{pmatrix}$$

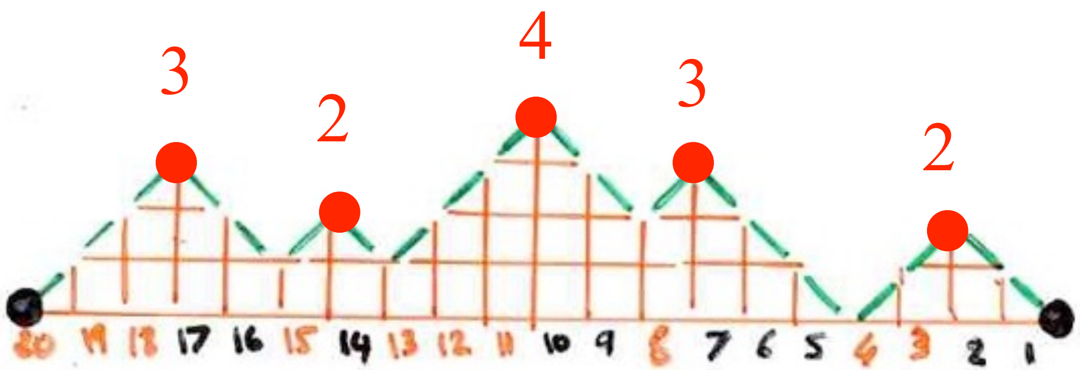


0
1
2
3
4
5
6
7
8
9
10
a_1
a_j
a_k
b_1
b_2
b_j
b_k

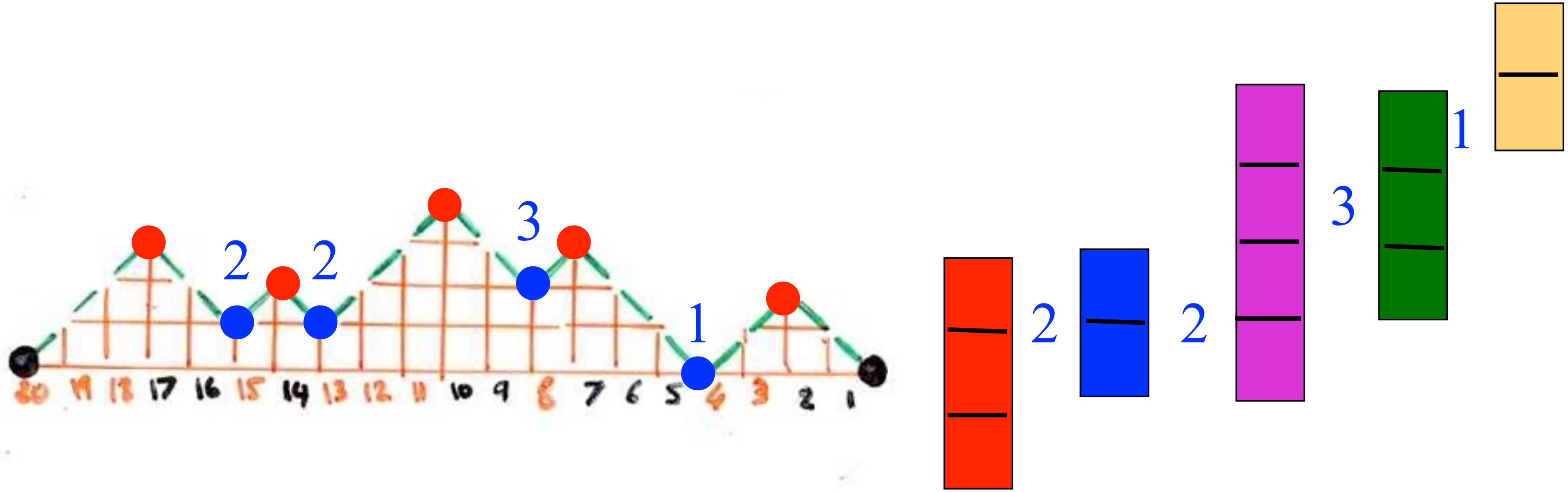
another example:
Naranaya numbers

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

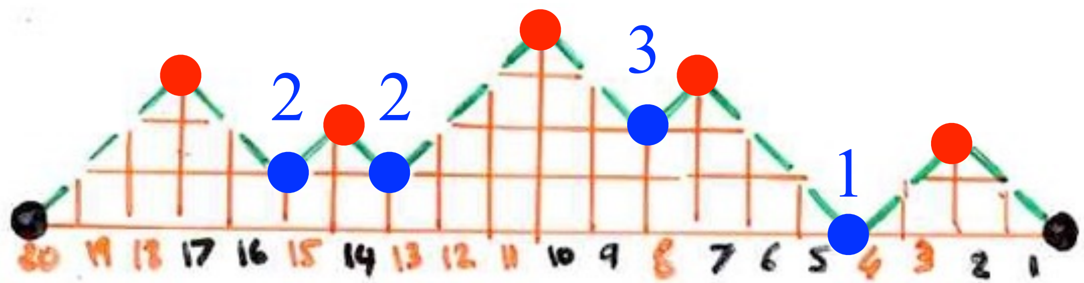


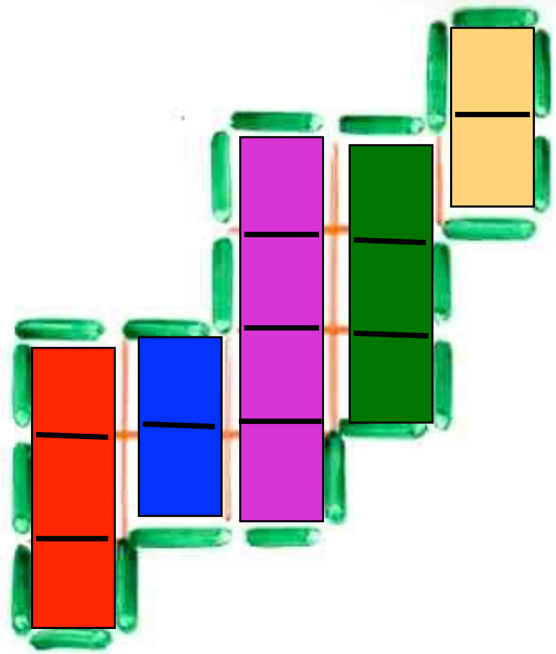


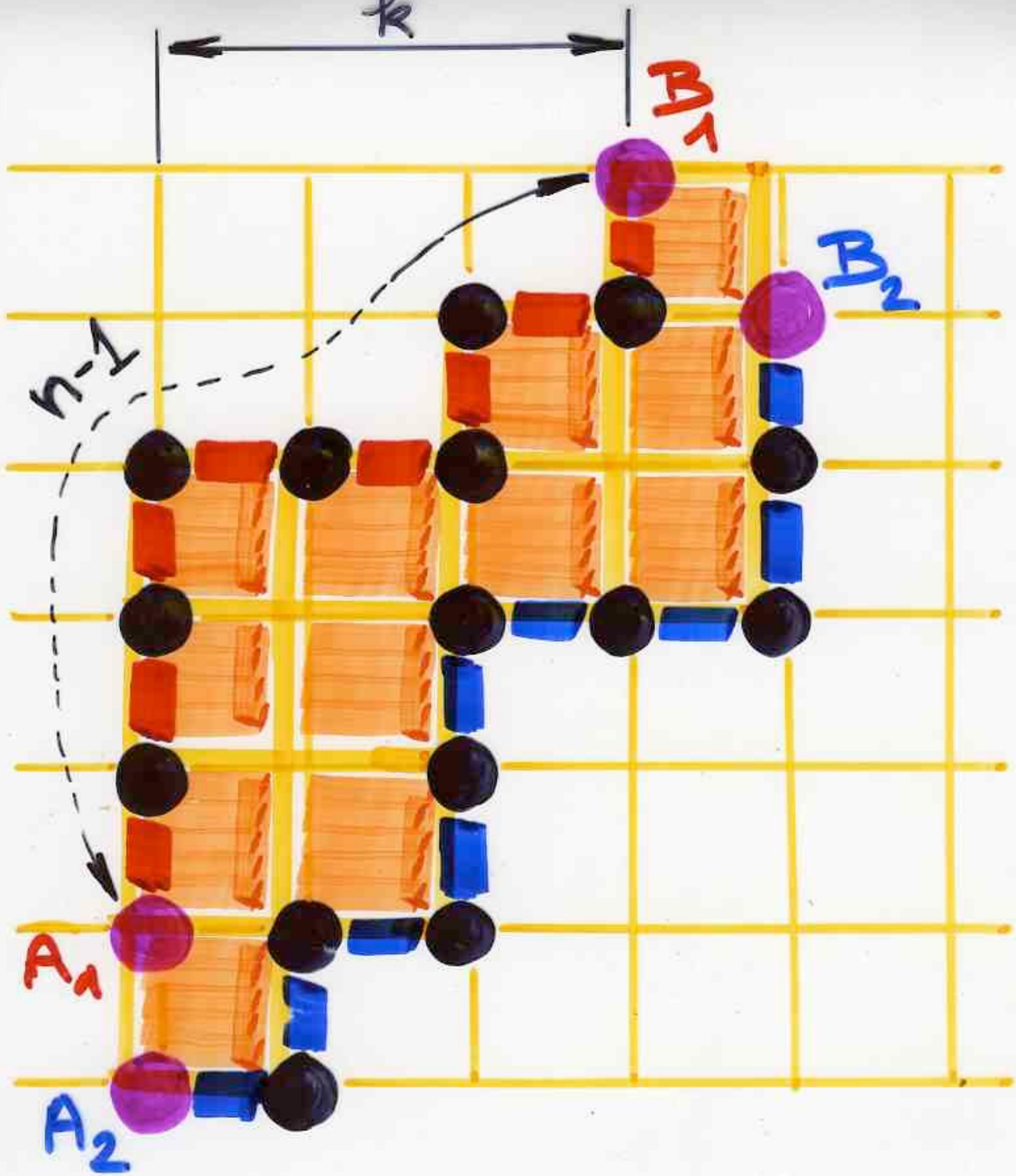
3
2
4
3
2
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1



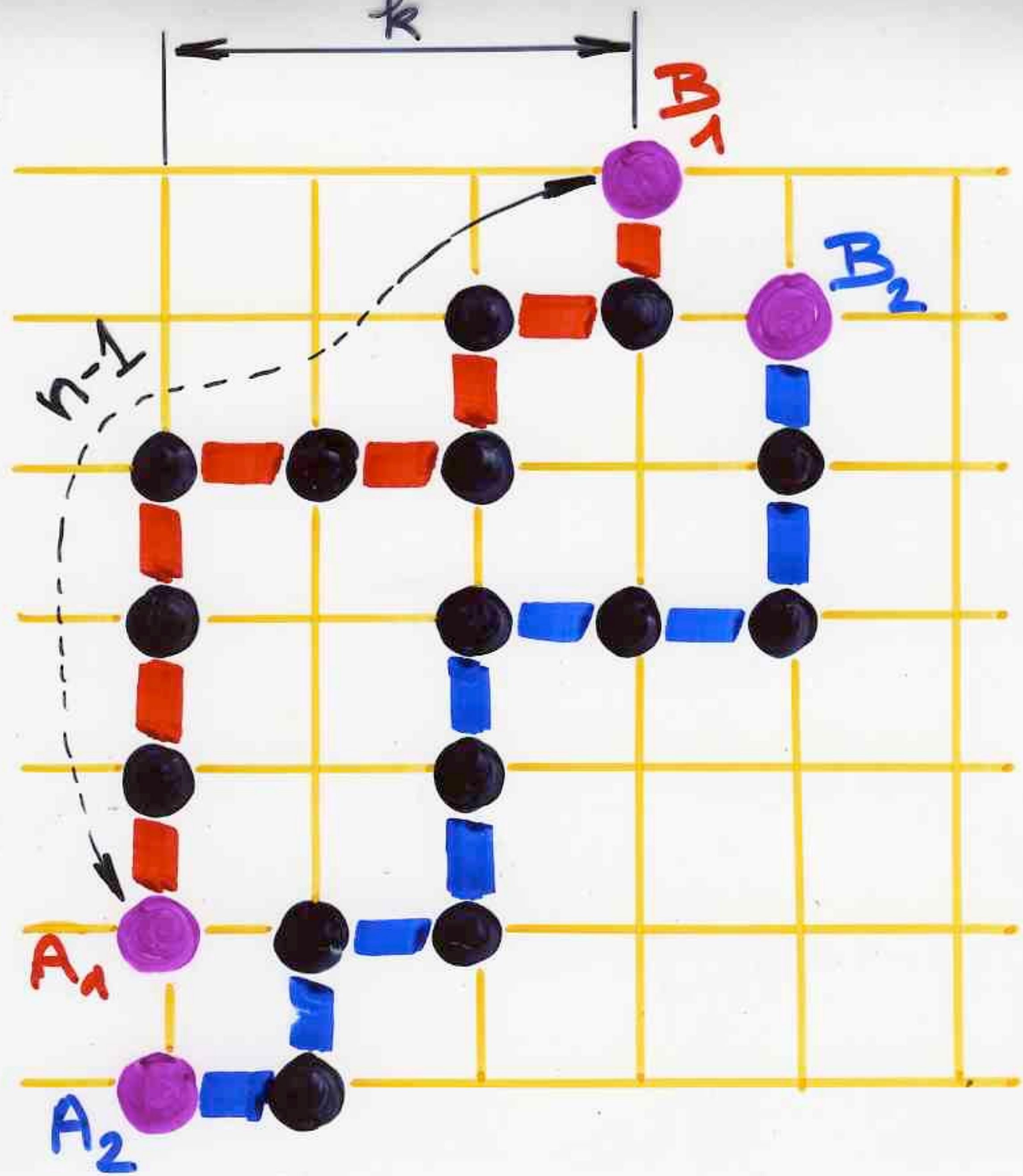
2
2
3
1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1
2
2
3
1


2
2
3
1
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1





k
B_1
B_2
n-1
A_1
A_2



$$\begin{vmatrix} \binom{n-1}{k} & \binom{n-1}{k+1} \\ \binom{n}{k} & \binom{n}{k+1} \end{vmatrix} = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$