Annals of Discrete Mathematics 23 (1984) 409-464

CHAIN AND ANTICHAIN FAMILIES
GRIDS AND YOUNG TABLEAUX

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Dedicated to Professor Corominas

Abstract. - Greene's Theorem relating chain and antichain families with maximal cardinalities in a poset (partially ordered set) was motivated by the Robinson-Schensted correspondence. This correspondence is a bijection between permutations and pairs (P, Q) of standard Young tableaux having the same shape.

This bijection originated in the representation theory of the symmetric group. The underlying combinatorics is very deep and also takes roots in the theory of symmetric functions and algebraic geometry.

The purpose of this talk is threefold. First we give a brief summary of the principal combinatorial properties of the Robinson-Schensted correspondence, especially those having an order-theoretical flavor. Second we shed some light on its relationship with poset theory and Greene's Theorem.

The third purpose of this talk is to solve the following open problem : give an interpretation of the value located in the (i,j) cell of the Young tableaux P and Q. This "local" characterization of the correspondence is completely symmetric in rows and columns and requires the concept of grids and extendable grids.

These last results can be extended to arbitrary posets. Complete proofs will be given elsewhere.

Résumé. - Le théorème de Greene sur les cardinaux maximaux des familles de chaînes et d'antichaînes extraites d'un ensemble (partiellement) ordonné a son origine dans la correspondance de Robinson-Schensted. Cette correspondance est une bijection entre les permutations et les paires (P, Q) de tableaux standards de Young de même forme.

Cette bijection provient en fait de la théorie des représentations du groupe symétrique. La combinatoire sous-jacente est fort riche et prend également racine dans la théorie des fonctions symétriques et en géométrie algébrique.

Le but de cet exposé est triple. D'abord nous donnons un bref aperçu des propriétés combinatoires les plus classiques de cette correspondance, notamment celles en relation avec la méthodologie des ensembles ordonnés. Le deuxième but de cet exposé est de faire sentir à un public motivé par les ensembles ordonnés l'intérêt de mieux connaître

cette merveilleuse correspondance.

Enfin une troisième partie est consacrée à la résolution d'un problème ouvert : donner une interprétation symétrique en lignes et colonnes de la valeur située dans la case (i,j) des tableaux de Young P et Q. Cette définition "locale" de la correspondance de Robinson-Schensted utilise les nouveaux concepts de grille et de grille prolongeable.

Ces dernières idées peuvent être étendues aux ensembles ordonnés quelconques. Les preuves complètes de ces nouveaux résultats seront données ailleurs.

§ 1 - Introduction.

In 1938, G. de B. Robinson introduced [77] a correspondence between the $n!$ permutations of the symmetric group $\mathfrak{S}_n$ and pairs of certain combinatorial objects called standard Young tableaux. This correspondence gives in fact a bijective (constructive) proof of the following identity

$$n! = \sum_{\lambda} f_\lambda^2 \quad , \tag{1}$$

where the summation extends over all irreducible representations (over the field of complex numbers) of the symmetric group $\mathfrak{S}_n$ and f_λ denotes the degree of the corresponding representation.

This identity is classical in the representation theory of finite groups, where the left hand-side is the order of the group.

In the case of the symmetric group $\mathfrak{S}_n$, each of the irreducible representations is in bijection with a very simple combinatorial object : a partition of the integer n. Such a partition is visualized as a Ferrers diagram (see figure 2 below) with n cells. In a series of papers (see the collected papers [118]), Young introduced his famous tableaux as a combinatorial tool in the study of the representations of the symmetric group. These so-called Young tableaux are labelings of the Ferrers diagrams with integers such that the numbers are increasing in rows and columns (see definition below).

The dimension f_λ is in fact the number of standard Young tableaux associated with a given partition λ. The right-hand side of (1) is interpretated as the number of pairs of standard Young tableaux having the same shape (i.e. Ferrers diagram).

The problem to find f_λ becomes a purely combinatorial problem, and

is in fact a poset problem : find the number of linear extensions of the poset defined by the cells of the Ferrers diagram.

Combinatorics of Young tableaux appears also in many other areas, as for example the theory of symmetric functions (Schur functions, ...), invariant theory or algebraic geometry (Schubert calculus, flag manifolds, ...). The interested reader will see the books [72], [80], the survey paper [7] and some recent work of Lascoux and Schützenberger. We shall restrict ourself to the combinatorial point of view.

The correspondence introduced by Robinson was rediscovered by Schensted [89] and defined more clearly in purely combinatorial way by a recursive algorithm : the "bumping process". Since then, this correspondence has been known as the Robinson-Schensted correspondence and appears also to be of purely combinatorial interest. Following Schützenberger, much combinatorial work has been done. A good survey of the state of the art in 1972 can be found in Knuth's book [52], section 5.1.4, and more recently in the book [13] edited by D. Foata and related to the Strasbourg "table ronde" in April 1976.

The Robinson-Schensted correspondence $\sigma \to (P(\sigma), Q(\sigma))$ has strong links with poset theory. With every permutation σ, one can associate a poset having dimension $\leqslant 2$, and conversely any such poset can be obtained in this way. Increasing subsequences correspond to chains and decreasing subsequences correspond to antichains. Some combinatorial properties of the correspondence are in fact properties of this poset. For example the well-known Schensted property gives the length of the longest increasing (resp. decreasing) subsequence as the number of elements of the first row (resp. column) of the Ferrers diagram common to $P(\sigma)$ and $Q(\sigma)$. A generalization has been given by Greene [35], interpretating the entire shape (Ferrers diagram) of $P(\sigma)$ and $Q(\sigma)$ in terms of a family of chains and antichains with maximum cardinalities.

This was the starting point for the deep and now classical Greene's Theorem [37] : the duality between family of chains and antichains coming from the Robinson-Schensted correspondence (posets with dimension $\leqslant 2$) is valid for any poset.

From a poset point of view, it may be fruitful to have a good understanding of the order-theoretical properties of the Robinson-Schensted correspondence (this can be already very difficult) and then to look for possible extensions to arbitrary posets. In particular some generalizations of the correspon-

dence have been proposed for arbitrary posets by Fomin [16] and Gansner [27].

Greene's proof of his theorem relies on some joint work with Kleitman [39] generalizing the well-known Dilworth's Theorem. This work is part of the so-called "extremal properties" and a survey of this field can be found in West [113]. Since the papers of Greene and Kleitman, extensive research has been done and several other proofs have been given (see below). In particular, Greene's Theorem can also be deduced from integer programming techniques, using the minimal cost flow algorithm of Ford and Fulkerson [23], [47]. For a survey of the use of linear programming ideas in poset theory, see Hoffman [45].

This paper is arranged in the following way. First in section 2, we recall Greene's Theorem (for arbitrary posets). Then we give different versions of the Robinson-Schensted correspondence : the original definition of Schensted in section 3, a "planarization" given by Viennot [109] in section 5, then the synthesis made by Schützenberger and Lascoux with the "jeu de taquin" and culminating in the plactic monoid [62], [97] in section 6. Some classical combinatorial properties of the correspondence are given in section 4. In section 7 we introduce the new concepts of grids and extendable grids. We are thus able to give an interpretation of the shape of the Young tableaux $P(\sigma)$ and $Q(\sigma)$ that is completely symmetric in rows and columns. Moreover, we give an explicit min-max formula for the value $P_{ij}(\sigma)$ (resp. $Q_{ij}(\sigma)$) located in the (i,j) cell of $P(\sigma)$ (resp. $Q(\sigma)$). In section 8 we extend these ideas to arbitrary posets. This extension relies on some theorems of Frank [23], proved by linear programming techniques, from which Greene's Theorem can be deduced. An open problem is to prove directly the results of section 8 without the use of linear programming.

In this paper, we do not pretend to give an exhaustive survey of this huge subject and apologize for omissions and unquoted papers.

Also we confess that we chose some of the most spectacular properties related to the high use of transparencies in the talk. Unfortunately, the simplicity of the combinatorial constructions, together with the magic of this very beautiful correspondence, cannot be written down in a paper as easily as it can be described in an oral communication with a friend or using superposition of pictures with transparencies. This is the rule in combinatorics. Nevertheless, we hope that the reader, not having attented the talk corresponding to this paper, will agree with Knuth's statement ([52], page 60, line 21) : "The unusual nature

of these coincidences might lead us to suspect that some sort of witchcraft is operating behind the scenes ! ".

§ 2 - Greene's Theorem.

Let P be a finite poset (partially ordered set). An antichain is a subset of pairwise incomparable elements of P. A chain is a subset that is totally ordered by the induced order of P. A chain k-family (resp. antichain k-family) is a subset of P that is the union of k chains (resp. k antichains). We denote by $c_k(P)$ (resp. $d_k(P)$) the maximum cardinality of chain families (resp. antichain families) of the poset P.

Example : Let P be the poset defined by the diagram displayed in figure 1. The vertices are $1, 2, \ldots, 9$. For the reader unfamiliar with poset theory, we recall that the (partial) order relation $\preceq$ of P is defined as to be the transitive closure of the following relation : $x \mathcal{R} y$ iff and edge links x and y and x is below y.

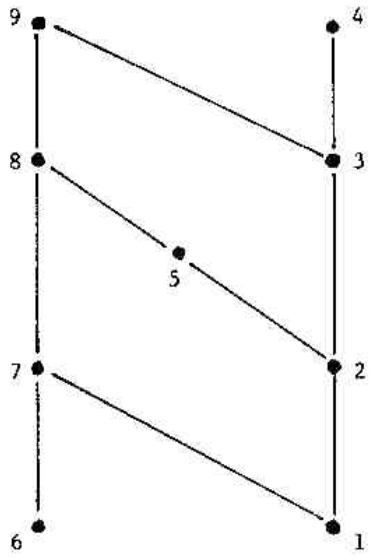


Figure 1 : A poset.

For $k = 1, 2, 3$ the following subsets are chain k-families with maximum cardinality

$$k = 1 , \qquad \gamma_1 = \{1, 2, 5, 8, 9\}$$

$$k = 2 , \qquad \gamma_2 = \{1, 2, 3, 4, 6, 7, 8, 9\}$$

$$k = 3 , \qquad \gamma_3 = P = \{1, \ldots, 9\} .$$

The subset γ_2 is a union of two chains, $\gamma_2 = \{1, 2, 3, 4\} \cup \{6, 7, 8, 9\}$, while γ_3 is a union of three chains $\gamma_3 = \{1, 2, 3, 4\} \cup \{5\} \cup \{6, 7, 8, 9\}$. Thus $c_1(P) = 5$, $c_2(P) = 8$, $c_3(P) = 9$.

Note that, in this particular example, there exists only one chain k-family with maximum cardinality, for $k = 1, 2, 3$.

For $k = 1, 2, 3, 4, 5$, the following subsets are antichain k-families with maximum cardinality.

$$k = 1 , \qquad \delta_1 = \{3, 5, 7\} ,$$

$$k = 2 , \qquad \delta_2 = \{1, 6\} \cup \{3, 5, 7\} ,$$

$$k = 3 , \qquad \delta_3 = \{1, 6\} \cup \{3, 5, 7\} \cup \{4, 8\} ,$$

$$k = 4 , \qquad \delta_4 = \{1, 6\} \cup \{2, 7\} \cup \{3, 8\} \cup \{4, 9\} ,$$

$$k = 5 , \qquad \delta_5 = P = \{1, 6\} \cup \{2, 7\} \cup \{3, 8\} \cup \{4, 9\} \cup \{5\} .$$

We have written each δ_k as a union of k antichains.
Thus $d_1(P) = 3$, $d_2(P) = 5$, $d_3(P) = 7$, $d_4(P) = 8$, $d_5(P) = 9$.
Here, for each k, $1 \leqslant k \leqslant 4$, there exists several distinct antichain k-families with maximum cardinality $d_k(P)$.

In the literature, chain k-families are often called k-cofamilies, antichain k-families are also called k-families, while antichain k-families with maximum cardinality $d_k(P)$ are called Sperner k-families. The notation $d_k(P)$ is in honor of Dilworth (Saks calls this the k^{th} Dilworth number). Usually $c_k(P)$ is denoted by $\hat{d}_k(P)$. We have changed this notation because throughout the paper, $c_k(P)$ is associated with the "shape" λ of the Young tableaux obtained by the Robinson-Schensted correspondence, while $d_k(P)$ is associated with the "conjugate shape", usually denoted by $\hat{\lambda}$ or λ^*.

A partition of an integer n is a non-increasing sequence of non zero

integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 0$ such that $n = \lambda_1 + \ldots + \lambda_p$. Usually a partition λ is visualized by a "Ferrers diagram" F (see figure 2, in "French notation"). The number of elements in the i^{th} row is λ_i. If λ_j^* denotes the number of elements in the j^{th} column, we obtain another partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_q)$, called the conjugate partition of λ. We have denoted by q the number of elements of the first row of P.

With these definitions we can state the well-known Greene's Theorem as follows

THEOREM 1. (Greene) Let P be a finite poset with n elements, q (resp. p) the length of the largest chain (resp. antichain). We define the numbers $\lambda_i(P) = c_i(P) - c_{i-1}(P)$ and $\mu_i(P) = d_i(P) - d_{i-1}(P)$ (with the convention $c_0(P) = d_0(P) = 0$). Then we have the following properties

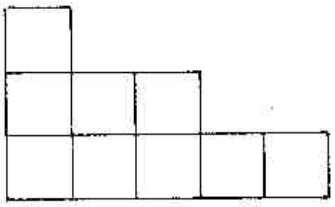
(i) $\lambda_1(P) \geq \ldots \geq \lambda_p(P)$,

(ii) $\mu_1(P) \geq \ldots \geq \mu_q(P)$,

(iii) the partitions defined by (i) and (ii) are conjugate.

The Ferrers diagram associated with the partition defined by (i) will be called the Greene diagram of P and denoted by $G(P)$.

Example : The Greene diagram associated with the poset defined by figure 1 is displayed in figure 2.



$\lambda = (5, 3, 1)$

$\lambda^* = (3, 2, 2, 1, 1)$

Figure 2. The Ferrers diagram of $\lambda = (5, 3, 1)$.

Note that relations (i) and (ii) of Greene's Theorem are not at all obvious. In particular, a chain k-family (resp. antichain k-family) with maximum cardinality is not necessarily obtained by adding a chain (resp. antichain) from a chain (k-1)-family (resp. antichain (k-1)-family) with maximum cardinality. The poset of figure 1 has a unique chain of length 5 and a unique chain 2-family of cardinality 8 which is a union of two chains of length 4.

Originally, Greene proves his remarkable theorem [37] by using some work done with Kleitman [39] that extends the classical Dilworth's Theorem [11], which states that the size $d_1(P)$ of the largest antichain in the poset P is the minimum number of chains which cover P. Given any chain partition C of P, let $m_k(C) = \sum_{c \in C} \min(k, c)$. Since each chain contributes at most k elements to a k-family, $d_k(P) \leq m_k(C)$. If equality holds then C is called a k-saturated partition. Greene and Kleitman proved that for any k there exists a chain partition C that is both k and $(k+1)$-saturated. This fact is known as the t-phenomenon, and implies condition (ii) of theorem 1.

Since then, many other proofs have been given : Saks considered antichain k-families in a product of two posets [86], [88] and gave a short proof of the existence of k-saturated partitions [87]. The tools of linear algebra can be used : Saks [86] and Gansner [27] proved that the numbers $c_k(P)$ can be considered as the invariants of a nilpotent matrix, that is the block sizes of its Jordan canonical form ; thus they proved condition (i) of theorem 1. Dual interpretations exist for the numbers $d_k(P)$. Another kind of proof comes from linear programming using the minimal cost flow algorithm of Ford and Fulkerson : Hoffman and Schwartz [47], Fomin [16], Frank [23], see also Hoffman's survey paper on the use of linear programming in poset theory [45].

§ 3 - The Robinson-Schensted algorithm.

In the case of a poset of dimension 2, there is a very convenient way of computing the Greene diagram : just apply the Robinson-Schensted algorithm.

We denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and by $\mathfrak{S}_n$ the symmetric group on $[n]$. For every permutation σ of $\mathfrak{S}_n$ we associate a poset $Pos(\sigma)$ to be the set of points $\hat{\sigma} = \{(i, \sigma(i)), i \in [n]\} \subset [n] \times [n]$ ordered by the induced order of the usual product order :

(2) $\qquad (x, y) \leqslant (x', y') \quad \text{iff} \quad x \leqslant x' \quad \text{and} \quad y \leqslant y' .$

Every poset of dimension $\leqslant 2$ is a subposet of the product of two chains, and can in fact be identified with a poset $\hat{\sigma}$ for a certain permutation σ.

Let λ be a partition of n and F_λ the associated Ferrers diagram. A Young tableau of shape λ is a labeling of the cells of F_λ with integers such that these numbers are strictly increasing in the columns (down-up in the "French notation") and are weakly increasing in the rows (left to right). If the entries of the Young tableau are distinct integers and are the integers $1, 2, \ldots, n$, then we have a standard Young tableau on $[n]$ (see figure 3). In other words, a standard Young tableau is an embedding in the chain $[n]$ of the poset associated with F_λ (where the order is the induced order of the product order).

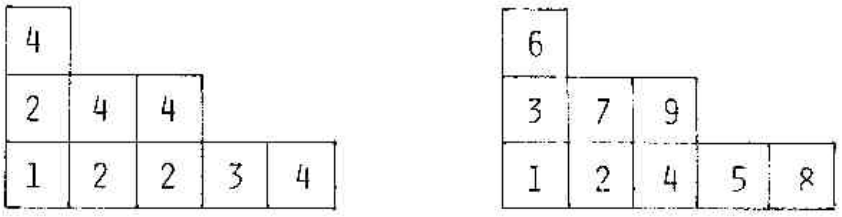


Figure 3. A Young tableau, and a standard Young tableau on $[9]$ with shape $\lambda = (5, 3, 1)$.

We define now recursively a process that inserts an integer x in a Young tableau T. We suppose that the entries of T are all distinct and distinct from x. The result is a Young tableau denoted by $I(T, x)$.

If x is greater than every element of the first row of T, then $I(T, x)$ is the Young tableau obtained by adding x at the end of this first row.

If not, then let z be the smallest value of the first row of T greater than x. Let T_1 be the Young tableau obtained by deleting the first row of T. Then $I(T, x)$ is the tableau obtained from T by replacing z by x

and by inserting (recursively) the value z in the tableau T_1.

The arrangement is a Young tableau at each stage because the replacement of z by x satisfies the required inequalities in its row and column. Note that the shape of (T,x) is obtained by adding a cell on the border of the shape of T. An example of this "bumping process" is given in figure 4.

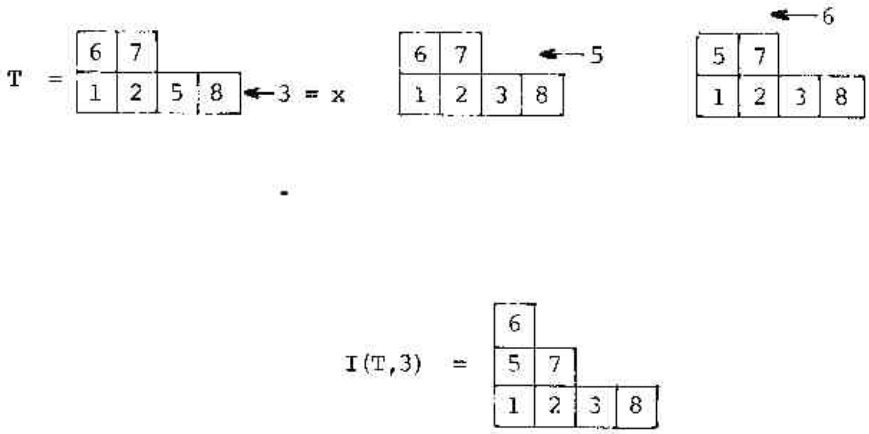


Figure 4. The bumping process.

Now if σ is a permutation of $\mathfrak{S}_n$, we define two standard Young tableaux $P(\sigma)$ and $Q(\sigma)$ by the following.

The tableau $P(\sigma)$ is obtained by successive insertions :
$P_0 = \emptyset$, $P_1 = (\sigma(1)), \ldots, P_n(\sigma) = P(\sigma)$ where, for $i = 1, \ldots, n-1$,
$P_{i+1} = I(P_i, \sigma(i+1))$.

The tableau $Q(\sigma)$ is obtained by constructing a sequence of tableaux $Q_0 = \emptyset, \ldots, Q_n = Q(\sigma)$ such that, for $i=1, \ldots, n-1$, Q_{i+1} is obtained from Q_i by adding a new cell labeled $i+1$ on the border of the tableau Q_i, in the same position as the unique cell that is in the shape of P_{i+1} but not in the shape of P_i.

In other words, the tableau $Q(\sigma)$ is a coding of the successive shapes occuring in the sequence of insertions $P_1, \ldots, P_n$.

Example : $\sigma = 6\ 1\ 7\ 2\ 5\ 8\ 3\ 9\ 4$.

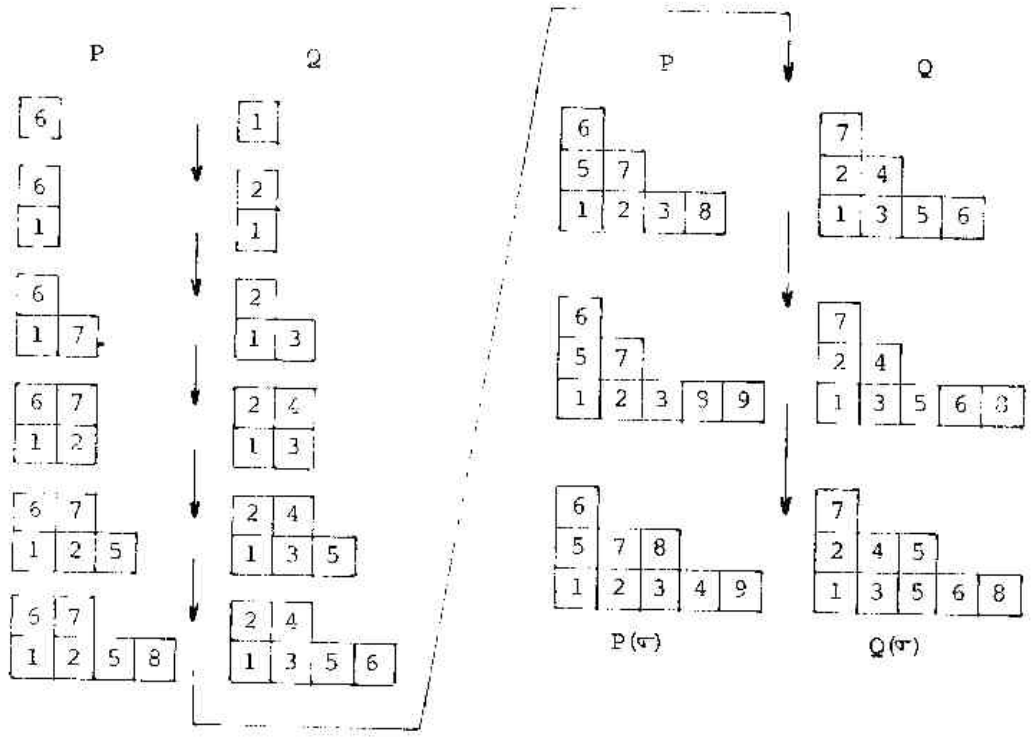


Figure 5. The Robinson-Schensted algorithm.

It is easy to see that this process can be reversed and the correspondence $\sigma \to (P(\sigma), Q(\sigma))$ is a bijection. This is the Robinson-Schensted correspondence. The tableau $P(\sigma)$ is the P-symbol, while $Q(\sigma)$ is the Q-symbol.

THEOREM 2 - (Robinson-Schensted) The correspondence $\sigma \to (P(\sigma), Q(\sigma))$ defined above is a bijection between the $n!$ permutations of $\mathfrak{S}_n$ and the

pairs of standard Young tableaux on $[n]$ having the same shape.

This correspondence gives a bijective proof of identity (1).

The number f_λ of standard Young tableaux of shape λ can be computed explicitly. Different formulae have been found and have received much investigations.

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_p)$ be a partition of n and for i, $1 \leq i \leq p$ let $\lambda'_i = \lambda_i - p - i$.

Using complicated algebraic methods (group characters, symmetric polynomials), Young [118] and Frobenius [25] independently gave the formula

$$f_\lambda = \frac{n! \prod_{i>j} (\lambda'_i - \lambda'_j)}{\lambda'_1! \ldots \lambda'_p!} \quad . \tag{3a}$$

A combinatorial proof of (3a), using difference methods, was found by Mac Mahon, in terms of a "n-candidate ballot problem" [74], or equivalently of "lattice permutations" [73], page 133.

Another way to prove (3a) is to derive this formula from the determinantal expression

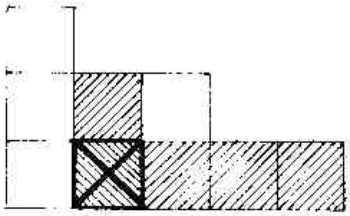
$$f_\lambda = n! \det\left(\frac{1}{(\lambda_i - i+j)!}\right), \tag{3b}$$

(with the convention $\frac{1}{(\lambda_i - i+j)} = 0$ when $\lambda_i - i+j < 0$). See for example Zelevinsky [120], page 92. A very simple "bijective" proof has been given by Gessel and Viennot [30] as part of a more general work interpretating many determinants of combinatorics as the number of certain non-crossing configurations of paths (and a bijection between Young tableaux and such configurations).

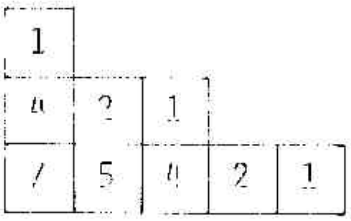
The formula (3a) can be rearranged into a more simple and elegant form. Let x be a cell of the Ferrers diagram F_λ associated with λ. We define the <u>hook length</u> of x as the number of cells of F_λ located above or at the right of x, including x itself (see figure 6). Using the fact that, for any i, $1 \leq i \leq p$, the product of the hook length of the i^{th} row is $\lambda'_i! / \prod_{j>i} (\lambda'_i - \lambda'_j)$, we obtain from (3a) the very well known and classical formula of Frame, Robinson and Thrall [22]

(3c) (hook length formula) $f_\lambda = \dfrac{n!}{\prod_x h_x}$,

where the product is taken over all hook lengths h_x of F_λ .



Hook length $h_{(1,2)} = 5$ of the (1,2) cell



Hook lengths of the Ferrers diagram (5,3,1)

Figure 6. Hook lengths.

A probabilistic proof of (3b) has been given by Greene, Nijenhuis and Wilf [41]. Bijective proofs of (3c) have been given by Remmel [76] and Zeilberger [119]. In fact, Remmel combined the non-crossing configurations of paths mentioned above and interpretating (3b), with a bijective derivation of (3a) and (3c) from (3b), using the "involution principle" introduced by Garsia and Milne [29] in their bijective proof of the celebrated Rogers-Ramanujan identity.

Nevertheless, no simple proof of (3c) is known and no direct combinatorial correspondence explains the role of the hook lengths. As for the Rogers-Ramanujan identities, it is still an open problem to know whether there exists a "simple" and "natural" bijective derivation of (3c).

Perhaps, the best combinatorial interpretation of the hook length is due to Grassl and Hillman [31]. The hooks appear naturally in a combinatorial correspondence involving the so-called plane partitions. This subject is very close to our topic, but will not be touched in this paper. From Grassl and Hillman's correspondence, formula (3) can be derived using an asymptotic argument.

Kreweras [53] gave an extension of formula (3b) to "skew Young

tableaux" (see definition in § 6 below). Canfield and Williamson [117] gave an "operator" flavour for hook lengths. The numbers f_λ are also related to the (weak) Bruhat order (see the end of this paper).

The problem of computing f_λ is a particular case of a classical problem in poset theory. If P is a poset with n elements, a linear extension of P (or order-preserving map) is a bijection $\varphi : P \to [n]$ such that $\varphi(x) \leqslant \varphi(y)$ if $x \lessdot y$ (in P). This is also called topological ordering (or topological sorting). The integers $\varphi(x)$ can be considered as labels (the labels are $1, 2, \ldots, n$ and each number appears once and only once). Sometimes, such labeling with labels increasing along the chains of P, are also called "natural labeling". Such example is given in figure 1.

The Ferrers diagram F_λ can be considered as a lower ideal of $\mathbb{N} \times \mathbb{N}$ (ordered by the restriction of the product order (2)). The number f_λ is the number of linear extensions of this poset F_λ. The problem for general posets is connected with many other considerations (see for example Stanley [99]) and is widely open.

Two other kinds of posets are known, giving rise to an analogous formula for the number of linear extensions as a ratio of $n!$ by a product of numbers playing the role of "hook lengths". One kind is the shifted Ferrers diagrams, that is the induced posets obtained by deleting the cells above the diagonal from Ferrers diagrams (Thrall [108]). Another family of posets is the family of trees (where the order means "to be a descendant of"). It is easy to prove the analog of formula (3) where the "hook length" is taken as the size of a subtree (see Knuth [52], exercise 20, section 5.1.4.). It is a major problem to find an other family of posets having a "hook length" formula (see Sagan [82]).

The correspondence of theorem 2 can easily be generalized to sequences with repetition of letters, that is words. The tableau P is a Young tableau while the tableau Q is a standard Young tableau. Knuth gave in [51] an ultimate generalization where both indices i and values $\sigma(i)$ can be repeated.

§ 4 - Some classical properties of the Robinson-Schensted correspondence.

The dihedral group D_4 (symmetries of the square) acts on the set $\mathfrak{S}_n$ of permutations ; in other words, permutation matrices can be flipped or

rotated to obtain other permutation matrices. To relate this to the Robinson-Schensted correspondence, it suffices to view a permutation σ as the obvious embedding $\tilde{\sigma} = \{(i, \sigma(i))\}$ in the square $[n] \times [n]$. The action can be generated by the two symmetries : $\sigma \to \sigma^{-1}$ (inverse) and $\sigma \to \sigma^*$ with $\sigma^* = \sigma(n) \ldots \sigma(1)$ (mirror image). To study the action, let us examine the effect of the operations $\sigma \to \sigma^{-1}$ and $\sigma \to \sigma^*$ on the P-symbol and Q-symbol.

Even though the construction of the P and Q-symbol are completely different, we have the surprisingly simple (and not trivial) property (Schützenberger [91]) :

PROPOSITION 3 - $\quad P(\sigma^{-1}) = Q(\sigma)\ , \quad Q(\sigma^{-1}) = P(\sigma)\ .$

For the effect of the mirror image we need to define two operations on standard Young tableaux. The transpose of a standard Young tableau Y is the tableau obtained by reversing rows and columns and is denoted by Y^T. The second operation, called dual, was introduced by Schützenberger [91] and is more elaborate.

Let Y be a standard Young tableau on $[n]$. First we define the trace $Tr(Y)$ to be the longest sequence $\{x_1 < \ldots < x_k\}$ defined by the following : x_1 is the value of the (1.1) cell (that is the smallest entry of Y) ; for $1 \leqslant p < k$, if x_p is the value of the cell (i, j), then x_{p+1} is the smallest of the two entries located in the cells $(i, j-1)$ and $(i-1, j)$, (if one of these cells is not in the shape of Y, let x_p be the value in the cell that is). By "longest sequence", we mean that the value x_k is located in a cell (i, j) such that the cells $(i, j+1)$ and $(i+1, j)$ are not in the shape of Y. Such a cell of Y will be called a corner cell of the Ferrers diagram of Y.

We define the tableau $d(Y)$ as the tableau obtained from Y by replacing each value x_p $(1 \leqslant p < k)$ of $Tr(P)$ by the value x_{p+1}, deleting the cell with value x_k and keeping invariant all other values. Obviously, $d(P)$ is a Young tableau with distinct entries $2, \ldots, n$, and shape contained in the shape of Y.

LEMMA 4 - (Schützenberger) Let σ be a sequence of distinct integers, and τ be the sequence obtained by deleting the minimum element. Then we have

$$P(\tau) = d(P(\sigma))\ .$$

Let Y be a standard Young tableau. We define the tableau Y^J by the following labeling. The cell containing the greatest value x_k of $Tr(Y)$ is relabeled 1. For ℓ, $1 \leqslant \ell < n$, the cell which is in $d^{\ell}(Y)$, but not in $d^{\ell+1}(Y)$ (that is the cell containing the greatest value of $Tr(d^{\ell}(Y))$), is relabeled $\ell+1$. The tableau Y^J is a labeling of the Ferrers diagram underlying Y with the integers $1, 2, \ldots, n$ such that rows and columns are strictly decreasing. Such a tableau is called a reverse standard Young tableau and is named the dual of Y. An example is shown in figure 7.

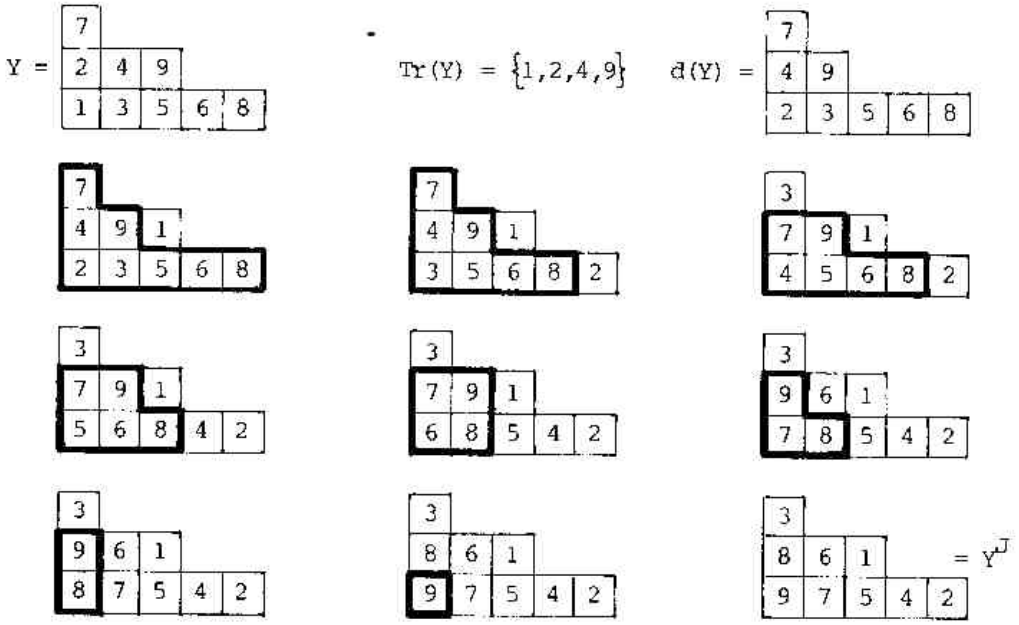


Figure 7. The dual of a standard Young tableau, obtained by "vidage-remplissage".

PROPOSITION 5 - (Schensted, Schützenberger). For any permutation σ of $\mathfrak{S}_n$, $P(\sigma^*) = P^T(\sigma)$, $Q^c(\sigma^*) = (Q^J)^T(\sigma)$, where Q^c denote the reverse tableau obtained from Q by replacing each value i by $n+1-i$.

The reader is urged to test this property with the permutation of figure 5 using the construction of figure 7.

By reversing the order, one can define the dual of a reverse standard Young tableau. The above proposition implies the relation

(4) for any standard Young tableau $(Y^J)^J = Y$.

In [94], Schützenberger has extended the construction $Y \to Y^J$ for arbitrary posets. Let P be a poset and P_L a linear embedding in $[n]$. As for standard Young tableaux, we can define $Tr(P_L)$, an increasing sequence of labels called the trace of P_L. Using the operator analogous to $d(Y)$, it is possible to define a dual "reverse natural" labelling of P_L. A deep result is that relation (4) is still valid.

Knuth's transpositions.

We study the class of permutations having the same P-symbol. The number of such permutations is obviously f_λ where λ is the shape of P. We may ask how to generate all these permutations from one of them, using elementary transformations. This is done by the so-called Knuth transpositions (Knuth [51]).

Let $\sigma \in \mathfrak{S}_n$ and $x = \sigma(i)$, $y = \sigma(i+1)$ be two consecutive elements of σ. If the value $z = \sigma(i-1)$ or $z = \sigma(i-1)$ is between x and y (that is $x < z < y$ or $y < z < x$), then the transposition of x and y is called a Knuth transposition.

The P-symbol is invariant under Knuth's transpositions. Conversely, any permutation τ such that $P(\sigma) = P(\tau)$ can be obtained from σ by a sequence of Knuth's transpositions.

A permutation is called row canonical (resp. column canonical) if the sequence $\sigma(1)\ldots\sigma(n)$ can be obtained by reading the entries of the P-symbol in the following way : read each row from left to right with rows ordered top-down (resp. read each column top-down with the columns ordered from left to right). In each class having the same P-symbol there is one and only one canonical permutation.

Example - The permutation $\sigma = 6\ 3\ 5\ 8\ 1\ 2\ 4\ 7\ 9$ is row canonical. Possible Knuth's transpositions are the transpositions of consecutive values $(6, 3)$

and (8,1).

Of course, we can dually define Knuth's transpositions on the values instead of the positions and keep the Q-symbol invariant.

Greene's interpretation of the shape.

We come back to the main motivation of this talk.

PROPOSITION 6 - For any permutation σ the Greene diagram associated with the poset $Pos(\sigma)$ (defined in § 2) is the same as the Ferrers diagram of $P(\sigma)$ and $Q(\sigma)$ obtained by the Robinson-Schensted correspondence.

In the other words, the number of elements of the first k rows (resp. columns) is the maximum cardinality of subsequences of σ which are union of k increasing (resp. decreasing) subsequences.

The particular case $k = 1$ is known as Schensted's Theorem [89] and was the motivation of Schensted's original construction.

Greene proved proposition 6 in [35] by showing its invariance under Knuth's transpositions, and noting that the property is trivial for row canonical permutations. We shall see below a direct geometric construction of such subsequences (that is, unions of k increasing (decreasing) subsequences with maximum cardinality), and also a characterization of the shape of the P-symbol completely symmetric with respect to rows and columns.

A nice corollary of Schensted's theorem is a proposition of Erdös and Szekeres [12] : any permutation containing more than n^2 elements has monotonic subsequence (increasing or decreasing) of length greater than n (this proposition is also a corollary of Dilworth's Theorem, as applied to posets of dimension 2). Note that there exist permutations with n^2 elements having no monotonic subsequences of length greater than n. These permutations are exactly those having a P-symbol with a square $n \times n$ shape.

Many investigations have been made of the number of permutations having a maximum-length increasing subsequence. No exact formula is known (see [79] for example). This problem is equivalent to enumerating standard Young tableaux having k rows, and is related to problems in algebra (see Regev [75]). It has been proved by Hammersley [43] that the mean of the length of maximal increasing subsequence is asymptotically $c\, n^{\frac{1}{2}}$. Logan and Shepp [71] proved

$c \geqslant 2$, while Kerov and Vershik [50] proved $c \leqslant 2$. Thus $c = 2$, as was suggested by Baer and Brock [122]. Also Fredman [24] gave an optimal algorithm in nlogn-nloglogn comparisons to compute this maximal length.

"Line-of-route" and up-down sequence

An important notion in the enumerative theory of permutations is the up-down sequence of a permutation $\sigma \in \mathfrak{S}_n$. It is defined as a word of length n-1 with two letters + , - as follows

(5) $UD(\sigma) = w_1 \ldots w_{n-1}$ with $w_i = +$ iff $\sigma(i) < \sigma(i-1)$ (rise)
$= -$ if $\sigma(i) > \sigma(i+1)$ (fall)

The dual notion of a rise (resp. fall) is that of advance (resp. retreat) : the value $x = \sigma(i)$ is an advance (resp. retreat) if $x+1 = \sigma(j)$ with $i < j$ (resp. $i > j$). The up-down sequence of σ^{-1} yields the advances and retreats of σ .

Shützenberger showed that the value x is an advance of the permutation σ iff the value x-1 is located to the South-East of the value x in the Young tableau $P(\sigma)$ (that is x is in the cell (i, j) and x+1 in the cell (i', j') with $i' \geqslant i$, $j' \leqslant j$).

Note that if x+1 is not South-East of x , then x+1 is at the North-West (that is $i' \leqslant i$, $j' \geqslant j$). This succession of South-East or North-West steps has been called the line of route by Foulkes.

Dually, by taking the inverse σ^{-1} of σ as mentioned above, the line of route of $Q(\sigma)$ gives the up-down sequence of the permutation (see example on figure 8).

Example - For $\sigma = 6\,1\,7\,2\,5\,8\,3\,9\,4$, $UD(\sigma) = - + - + - - + -$ and
$UD(\sigma^{-1}) = + + + - - + + +$.

The lines of route of the P-symbol and Q-symbol are the following

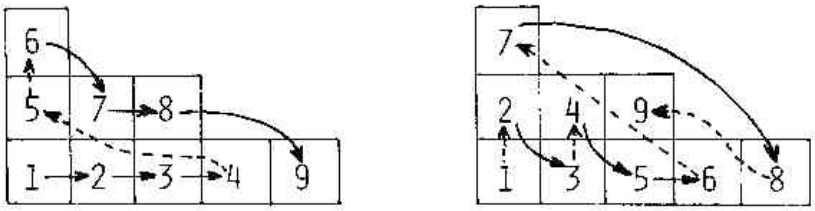


Figure 8 - Lines of route of the P-symbol and Q-symbol.

Foulkes gave beautiful applications of this interpretation to enumerative problems of permutations [19], [20], [21]. Other applications (using also the duality $Y \to Y^J$) can be found in Foata, Schützenberger [15].

k-matchings - A major question is to give a direct interpretation of the values in the P and Q-symbol. Greene gave an interpretation of the set of values which are above the k^{th} row in the P-symbol. For that purpose, he introduced the new concept of k-matchings.

A k-matching of a permutation σ is an array $(a_{ij})_{1 \leq i \leq p,\ 1 \leq j \leq k}$ of integers such that each row is a decreasing subsequence of σ and the elements of each column are distinct. The set $a_{11}, \ldots, a_{p1}$ is called the source of the k-matching. First, the number of elements in the rows above and including the k^{th} row of $P(\sigma)$ is equal to the maximum size (among k-matchings, k fixed) of the source of a k-matching of the permutation [38].

For the second result, we order subsets of $[n]$ by lexicographic order, that is $A < B$ iff the smallest element that occurs in only one of A and B appears in A. Greene [38] showed that the set of elements located in the rows above and including the k^{th} row of $P(\sigma)$ is the lexicographically minimum source among the sources of maximum-sized k-matchings of the permutation.

Gansner [27] gave proofs of these two facts using tools from linear algebra, and gave an analog of the first property for columns, by introducing the concept of k-scatters of a permutation.

We shall discuss below the extension of this to arbitrary posets.

§ 5 - Planarization .

In order to give a simple explanation of proposition 3 (and other properties such as Greene's interpretations), Viennot introduced [109] a geometric version of the Robinson-Schensted correspondence, that is a pictorial description with "shadows" and "borders of shadows" in the plane.

We consider the poset $\Gamma = \mathbb{Z} \times \mathbb{Z}$ ordered by the usual product order (2). We imagine some light coming from the South-West.

The shadow of a point (i,j) is the set of points (x,y) of $\mathbb{R} \times \mathbb{R}$ with $x \geqslant i$, $y \geqslant j$. The shadow of a finite subset F of Γ is the union of the shadows of each point of F . The border of this shadow (in the topological sense) is a "zig-zag line" supported by the points that are "in full light", that is the points which are not in the shadow of another point of F .

A finite subset F of Γ is said to be a quasi-permutation iff no two distinct points are on the same vertical or horizontal line. This terminology is not classical. Several other names are used, in particular independent set. When Γ is considered as a poset (direct product) such sets are called semiantichains. We introduce here the term "quasi-permutation" because it will correspond below to a subsequence of a permutation viewed as a sequence, in which the information about the positions held by the elements is retained.

Let F be a quasi-permutation of Γ . We define a sequence $L_1, \ldots, L_k$ of broken lines by the following process : L_1 is the border of the shadow of F , and for i, $1 \leqslant i < k$, L_{i+1} is the border of the shadow of the quasi-permutation F_{i+1} obtained by deleting from F the points that are on the lines $L_1, \ldots, L_i$ (see figure 9). The lines $L_1, \ldots, L_k$ will be called the outstanding of F .

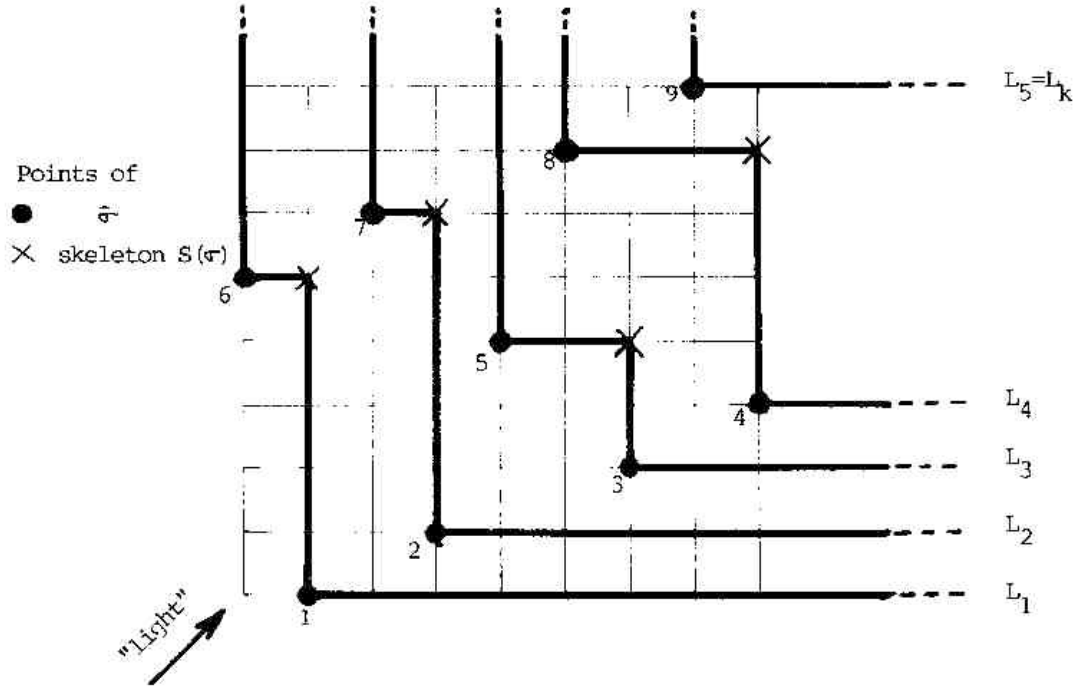


Figure 9 - The outstanding lines and the skeleton of the permutation $\sigma = 6\,1\,7\,2\,5\,8\,3\,9\,4$.

In this construction, some remarkable points appear. Following the outstanding lines of F , one changes direction for each point of F . There is also a change of direction for other points that are not in F . The set of these points is a quasi-permutation of $\mathbb{Z} \times \mathbb{Z}$ called the skeleton of F and denoted $S(F)$. They are marked with crosses in figure 9.

Note that $|S(F)| + k = |F|$.

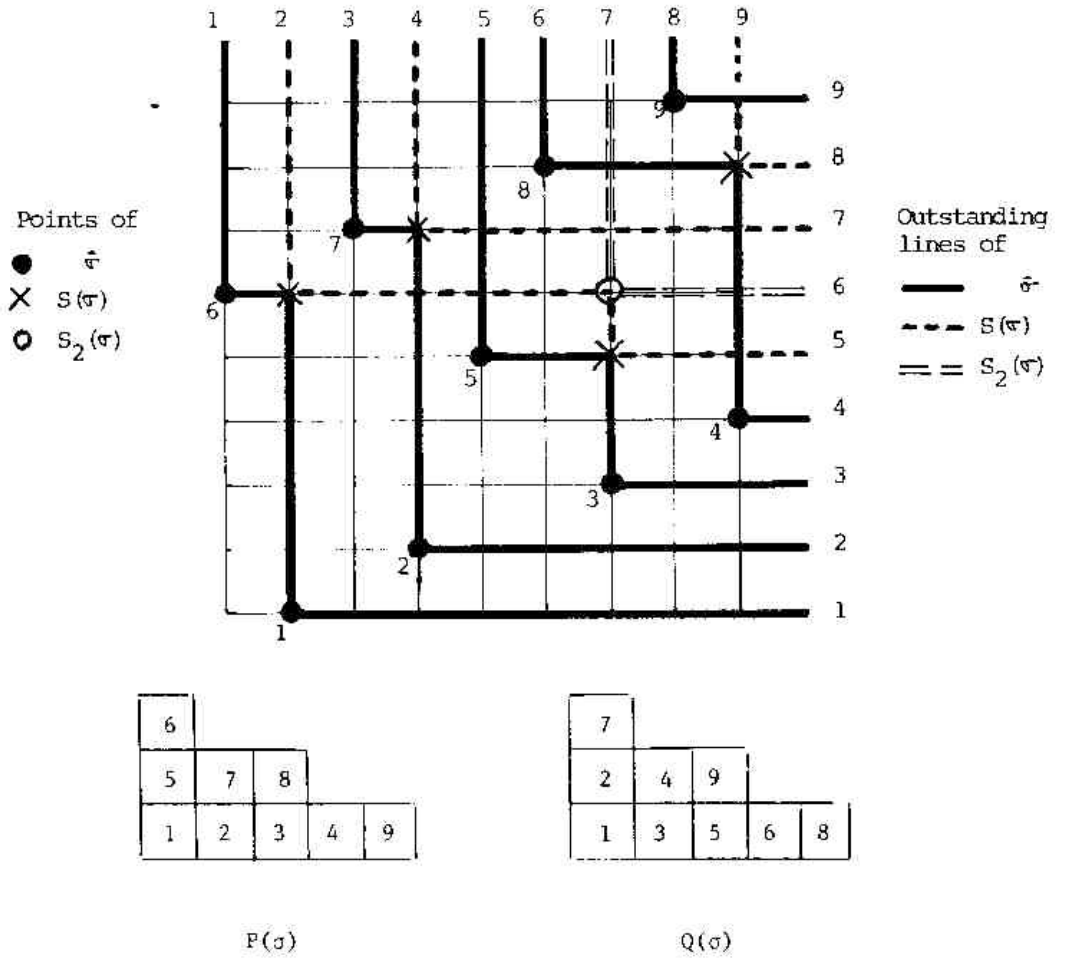
Let σ be a permutation of $\mathfrak{S}_n$. We consider the quasi-permutation $\hat{\sigma} = \{(i, \sigma(i)),\ i \in [n]\}$ and apply the construction $F \to S(F)$ as many times as we need to obtain the empty quasi-permutation. We thus construct a sequence $S_0(\sigma) = \hat{\sigma}, \ldots, S_{p-1}(\hat{\sigma})$ where for i, $1 \leqslant i < p$, $S_{i+1}(\hat{\sigma}) = S(S_i(\hat{\sigma}))$, and $S_p(\hat{\sigma})) = \emptyset$ (see figure 10).

An outstanding line of a quasi-permutation F is formed with finite segments (joining a point of F with a point of $S(F)$) and with two infinite half-lines, one is vertical and the other is horizontal. This vertical (resp.

horizontal) half-line has coordinate $v(L)$ (resp. $h(L)$) with respect with the x's axis (resp. y's axis).

We are now ready to state the "geometric" version of the Robinson-Schensted correspondence :

PROPOSITION 7 - (Viennot) Let σ be a permutation of $\mathfrak{S}_n$. The i^{th} row of the P-symbol $P(\sigma)$ (resp. Q-symbol $Q(\sigma)$) is the sequence $h(L_1), \ldots, h(L_{k_i})$ (resp. $v(L_1), \ldots, v(L_{k_i})$) of coordinates of the half-lines composing the outstanding lines of $S_{i-1}(\tilde{\sigma})$.



6				
5	7	8		
1	2	3	4	9

$P(\sigma)$

7				
2	4	9		
1	3	5	6	8

$Q(\sigma)$

Figure 10 - Planarization of the Robinson-Schensted correspondence.

The permutation σ can be reconstructed when knowing, for every $i \geq 1$, the coordinates of the half-lines appearing in the outstanding lines of the i^{th} skeleton $S_i(\hat{\sigma})$. This comes from the following geometric property.

For a quasi-permutation F, let I and J be the projections of F on the two coordinate axis. The direct product $I \times J \subseteq \mathbb{Z} \times \mathbb{Z}$ is called the support of F and is denoted by $Supp(F)$. A fundamental lemma is that the map $F \to (Supp(F), S(F))$ is an injection from the set of quasi-permutations of $[n] \times [n]$ into the set of pairs (A, B) where A is a subset of $[n] \times [n]$ of the form $I \times J$, and B is a quasi-permutation included in A. At first sight, it does not seem obvious to give a construction of F when $(Supp(F), S(F))$ is known. Nevertheless, there exists an easy geometric construction (see Viennot [109]).

F is given by $F = T(\tilde{S}(F) \cup C)$, where C is a certain subset of $\mathbb{Z} \times \mathbb{Z}$ (with $2k$ points if $S(F)$ has k outstanding lines) and T is the notation for an analogous definition of the skeleton of a quasi-permutation, but with the "light" coming from the North-East.

Thus, by repetitive application of this relation, the permutation σ can be reconstructed when knowing the (decreasing) sequence of supports of the different skeletons $S_0(\hat{\sigma}), \ldots, S_{p-1}(\hat{\sigma})$. This sequence is completely defined by the coordinates of the half-lines appearing in the outstanding lines of $S_0(\hat{\sigma}), \ldots, S_{p-1}(\hat{\sigma})$.

The reader is urged to take a rectangular piece of paper, hide the points of $\hat{\sigma}$ for the permutation displayed on figure 10, then move this piece of paper from left to right, and look carefully what happens each time a new column of the $[n] \times [n]$ grid appears. When passing from column i to $i+1$, the change of the outstanding lines crossing the left-hand side of the piece of paper is nothing but a geometric coding of the "bumping process" inserting $i+1$ in the P-symbol of the word $\sigma(1) \ldots \sigma(i)$. A new vertical half-line appears in column $i+1$. The corresponding outstanding line belongs to a certain skeleton $S_\ell(\sigma)$. The index ℓ is a coding of the "Q-part" of the Robinson-algorithm. This is essentially the proof of proposition 7.

It would be possible to move the piece of paper from the bottom to the top and thus described a dual version of the bumping process. In fact this duality is nothing but the transformation $\sigma \to \sigma^{-1}$, which corresponds to exchanging rows

and columns in the picture. Vertical coordinates of the half-lines of the outstanding lines are exchanged with the horizontal coordinates. Thus proposition 3 is made crystal clear : the duality $\sigma \to \sigma^{-1}$ corresponds to exchanging the P- and Q-symbols.

The symmetry $\sigma \to \sigma^{*}$ (mirror image) corresponds to making the "light" come from the South-East. In fact, there are four different possible directions for the "light" and thus four different constructions of sets of outstanding lines and skeletons. Unfornately, there do not seem to be simple geometric relations between these four constructions. In particular, proposition 5 does not seem to have a simple explanation in this geometric model.

Nevertheless, Greene's Theorem (proposition 6 and k-matching interpretation) can be proved in a constructive geometric way, without changing the permutation to a canonical one with Knuth's transpositions.

The reader will find in [109] a geometric construction of a maximal sized antichain k-family, using an operator, somewhat "inverse" to the operator $F \to S(F)$, sending every subset of $S(F)$ to a subset of F according to some "light" coming from the North-East.

A k-matching with lexicographically minimum maximum-size source can be obtained in the following way. We start with the set of y-coordinates of the points of $S_{k-1}(\sigma)$. Below each of these points, there is a unique point of $S_{k-2}(\sigma)$. The second column of the k-matching will be formed with the y-coordinate of these points, (which are distinct values). Going down from skeleton to skeleton till reaching points of $\hat{\sigma}$ gives values that can be arranged in a rectangular array having the k-matching property.

Foata's matrix characterization and kernel of the inversion pair graph.

When using transparencies, the different skeletons $S_i(\sigma)$, $i \geq 0$, are colored black, red, blue, green, ... and everything is clear. Here, we label the points of $S_i(\sigma)$ by the value $i+1$. All other points of $[n] \times [n]$ are labeled 0. We have a $n \times n$ matrix with integer entries. A question is to find a characterization of this matrix. This has been done by Foata in [14].

We need to define the "hook" joining two points. Let (x, y) and (x', y') be two points of $\mathbb{Z} \times \mathbb{Z}$ such that the second is at the South-East of the first one. The hook joining these two points is the set of points (i, j) with $j = y$, $x < i \leq x'$ or $i = x'$, $y \geq j \geq Y'$ (see figure 11).

A matrix with integer entries has the form defined above (coding of the different skeletons of a permutation σ of $\mathfrak{S}_n$) iff it satisfies the three following conditions :

(i) there exists one and only one 1 in each row and column

(ii) for every entry $i+1 \geq 2$, there exists an entry i both on the left and below the entry $i+1$,

(iii) for every hook joining two points labeled $i \geq 1$, there exists an entry $i+1$ located on the hook between them.

Such matrices have been introduced by Foata [14] under the name of Viennot's matrices.

Example -

$$\begin{matrix} 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&0&2 \\ 0&0&1&2&0&0&0&0&0 \\ 1&2&0&0&0&3&0&0 \\ 0&0&0&0&1&0&2&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \end{matrix}$$

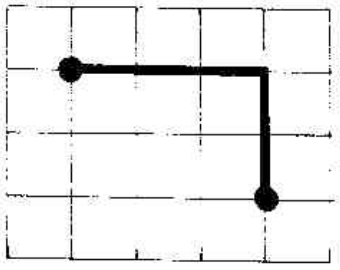


Figure 11 - A hook joining two points.

The above characterization can also be written in a nice way in terms of graph theory.

Let σ be a permutation of $\mathfrak{S}_n$ and $\hat{\sigma} \subseteq [n] \times [n]$ its planar representation. We denote by $GI(\sigma)$ the graph whose vertices are all the points of $[n] \times [n]$ such that there exists a point of $\hat{\sigma}$ on the left and below them (that is the union of all the corners of the hooks joining points of $\hat{\sigma}$). We put an edge between the vertices (x, y) and (x', y') iff $x = x'$ and $y \geq y'$ or $y = y'$ and $x \geq x'$. The number of vertices of this graph is the number of pairs (i, j) such that $i < j$ and $\sigma(i) \geq \sigma(j)$, i.e. the so-called inversions number of the permutation σ. Such diagrams have been introduced by Rothe [81] in order to give a nice (geometric) proof of the fact that the inversion

number is invariant under the transform $\sigma \to \sigma^{-1}$.

In their recent work relating combinatorics (Young tableaux, plactic monoid, Schur functions, ...) and algebraic geometry (flag manifolds, ...) [61], [65], [66], Lascoux and Schützenberger showed that these "inversion pairs" graphs play the same role for Schubert functions as Ferrers diagrams for Schur functions. They called these graphs Riguet diagrams.

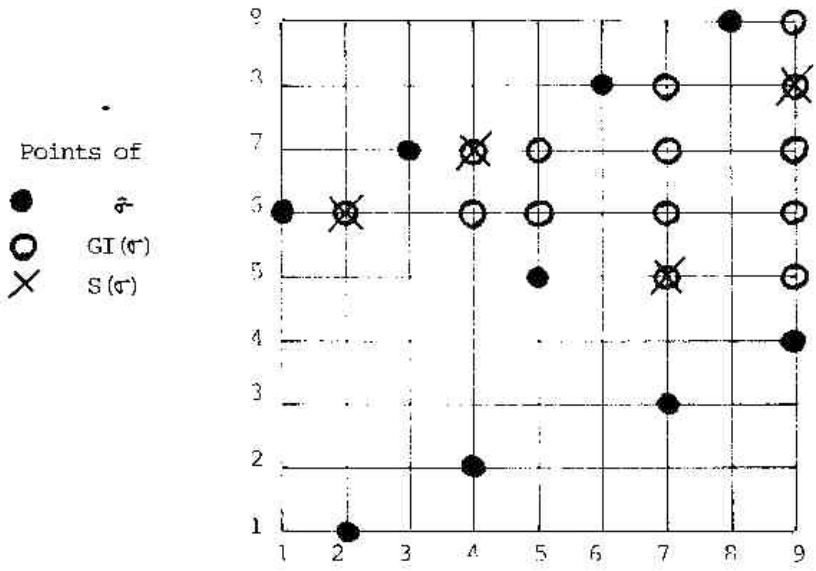


Figure 12 - The inversion pairs graph (on Riguet diagram) of a permutation.

A classical notion in the theory of directed graphs is the kernel of a graph, that is a set of points such that every vertex of the graph is the source of an edge ending in the kernel, and every edge having its source in the kernel has its end not in the kernel (see Berge [5]). Such a set is unique if it exists. It is also the set of "winning positions" of a "Nim game" played on the graph.

From [109] the skeleton $S(\sigma)$ of the permutation σ is exactly the

kernel of its inversion pairs graph. Writing this property for all skeletons $S_i(\sigma)$ leads to Foata's characterization.

From Fredman [24] and Viennot [109] the construction of the longest subsequence of a permutation σ needs the construction of the skeleton. Thus, the construction of the skeleton can be done by an optimal algorithm (for the worst case in "on-line" reading) in nlogn-nloglogn comparisons. For more understanding about these theoretical computer science concepts, see for example Knuth [52] or Aho, Hopcroft, Ullman [2].

§ 6 - Plactic monoid.

We have seen that the Robinson-Schensted correspondence is related to the representation theory of the symmetric group. The combinatorics of Young tableaux is also intimately connected with the theory of symmetric functions.

A basis of symmetric functions has been introduced by Jacobi [48] as a quotient of antisymmetric functions (expressed as determinants). Following Frobenius [25], Schur [90] discovered their relevance to the representation theory of the symmetric groups and the general linear groups. These functions have been called Schur functions (or S-functions) by Littlewood and Richardson [70]. Also they are related to some work initiated by Pieri in 1873, followed by Schubert, Giambelli and others, and which is nowadays named a "cohomology ring of Grassmann varieties".

We shall not touch this deep and huge subject. We shall only give the well-known combinatorial definition of the Schur functions (see for example Littlewood [69]).

Let λ be a partition and Y be a Young tableau of shape λ (with entries strictly increasing in columns and weakly increasing in rows). We denote by $m(Y)$ the monomial $x_1^{i_1} \dots x_p^{i_p}$, where i_ℓ is the number of entries equal to ℓ in the tableau Y. The Schur function $S_\lambda(x_1, \dots, x_n)$ is the sum of all monomials $m(Y)$ extended over all Young tableaux with shape λ and entries taken among $1, 2, \dots, n$.

Example - $S_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 + x_1^2 x_3 + x_2^2 x_1 + x_3^2 x_2 + 2x_1 x_2 x_3$.

corresponding to the eight Young tableaux displayed in the following figure.

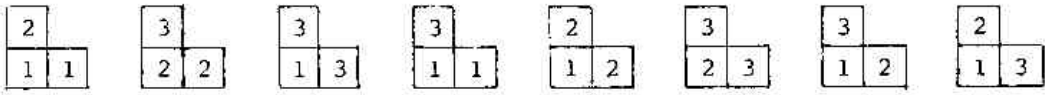


Figure 13 - A Schur function.

Lascoux and Schützenberger introduced the plactic monoid as a non-commutative calculus, unifying and extending previous properties of the Robinson-Schensted correspondence and of the theory of symmetric functions [62]. For example, the rather mysterious, so-called "Littlewood-Richardson rule" (giving the product of two Schur functions as sum of Schur functions) is made clear.

The combinatorial part of this theory is based on the "jeu de taquin", [62],[97]. This rule for moving labels in diagrams generalizes the operation $Y \to d(Y)$ made in the construction of the dual Y^J of a tableau Y.

We shall only give here a few hints of this beautiful theory. The interested reader will see the papers of Lascoux and Schützenberger [58],[59],[60],[62], [63],[95],[97] and of Thomas [103],[104],[105],[106],[107], the extension to the nilplactic monoid [66] and the survey paper of Cartier at the "Séminaire Bourbaki [7]. For the classical theory of symmetric functions, see the book of Macdonald [72] or Stanley [98], Foulkes [18].

Jeu de taquin.

Let λ and μ be two partitions such that the Ferrers diagram F_μ is contained in F_λ. The diagram $F_\lambda \setminus F_\mu$, the difference of the two Ferrers diagrams, is an interval of $\pi = \mathbb{Z} \times \mathbb{Z}$ (ordered by the product order (2)).

A skew Young tableau of shape $\lambda \setminus \mu$ is a labeling of $F_\lambda \setminus F_\mu$ with integers such that they are weakly increasing in the rows (from left to right) and strictly increasing in the columns (down-up, in the "French notation").

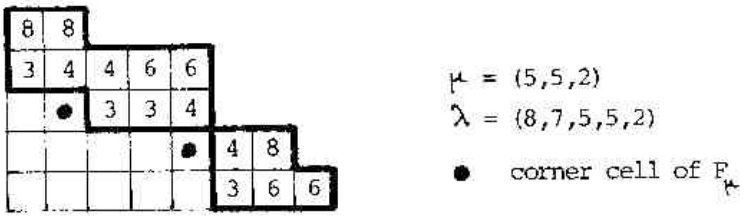


Figure 14 - A skew Young tableau.

The "jeu de taquin" is described as follows. First we choose one of the corner cells (defined in § 4) of the Ferrers diagram F_μ. We take the minimum of the two values that are located at the North and East, and move this value into the corner cell. This creates a "hole" in the skew Young tableau. In a similar way to the construction of the dual in § 4, we repeat the process until the hole has moved to the North-East outside of F_λ. In case the two values at the North or the East of a hole are equal, we move in the hole the value located at the North (thus preserving the condition to have a skew Young tableau). Also note that if the hole is at the "border" of F_λ, at most one of the North and East cells is non-empty. We move also this single value in the hole. The process terminates when both the North and East cell of the hole are empty (that is in fact when the hole is a corner cell of F_λ).

The terminology (and the theory !) "jeu de taquin" is due to Schützenberger. The "jeu de taquin" is a game played with a $n \times n$ board. Inside the board, there are n^2-1 elementary cells. On each cell is printed a letter. At each step of the game, one can move one of the nearest cells to the unique empty position. At each step this empty position takes the place of the cell which has moved. The rule is to obtain some words (or letters in alphabetic order).

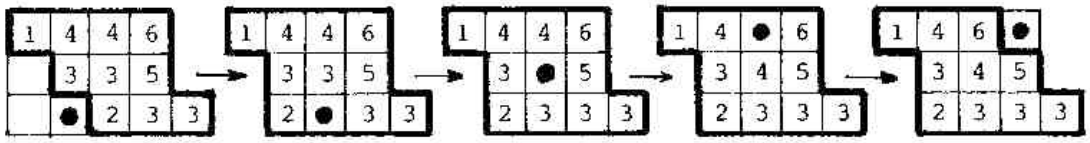


Figure 15 - The "jeu de taquin".

We obtain another skew Young tableau with shape $\lambda'\backslash\mu'$, where $F_{\lambda'}$ (resp. $F_{\mu'}$) is obtained from F_λ, (resp. F_μ) by removing a corner cell.

Now, if we repeat the process going from a tableau Y with shape $\lambda\backslash\mu$ to a tableau with shape $\lambda'\backslash\mu'$ until the diagram F_μ is empty, we obtain a Young tableau, called the "redressé" of the skew Young tableau Y under the successive choices of the "jeu de taquin". A remarkable fact is the invariance of the "redressé" under the "jeu de taquin", i.e. the "redressé" is independant of the successive choices of the corner cells of each diagram F_μ. This invariance has been proved by Schützenberger [97] and Thomas [105] (see an idea of the proof below). The "redressé" of the skew Young tableau Y is denoted by $R(Y)$. We follow again Schützenberger's terminology ; "redressé" is badly translated by saying that the South-West border of skew Young tableau Y is no more a zig-zag line.

For example, the tableau $d(Y)$ defined in § 4 is nothing but $R(Y \setminus a_{11})$ where $Y \setminus a_{11}$ denotes the (skew) Young tableau obtained by deleting the value a_{11} in the $(1,1)$ cell from the Young tableau Y.

In the example displayed in figure 16, we have shortened the construction by giving only the tableaux obtained once the "hole" is outside of the diagram F_λ. We have circled the values which move for each choice of the corner cells of the successive diagrams F_μ.

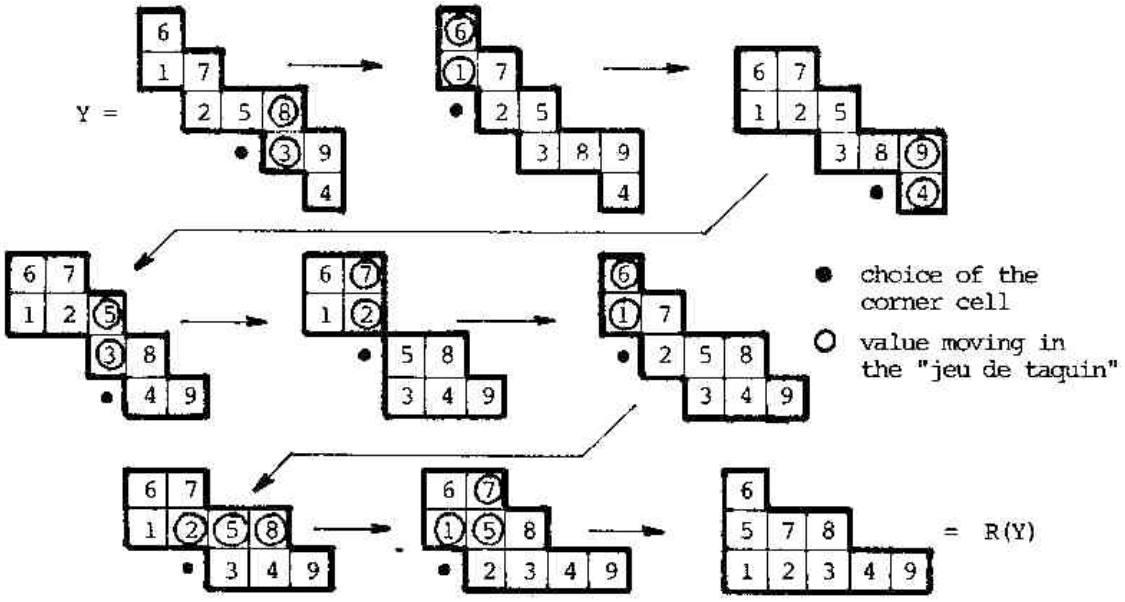


Figure 16 - The "redressé" R(Y) of a skew Young tableau Y .

Let σ be a permutation of $\mathfrak{S}_n$. We denote by $Y(\sigma)$ the skew Young tableau obtained by writting the successive values $\sigma(1), \ldots, \sigma(n)$ from North-West to South-East (see figure 17). A fundamental property of the "jeu de taquin", and giving another definition of the Robinson-Schensted correspondence is the following

PROPOSITION 8. (Schützenberger) - For any permutation σ of $\mathfrak{S}_n$, the "redressé" of the skew Young tableau $Y(\sigma)$ coding σ is identical to the P-symbol $P(\sigma)$ obtained by the Robinson-Schensted correspondence.

Define a strip to be a diagram $F_\lambda \setminus F_\mu$ such that the borders of F_λ and F_μ are disjoint and such that $F_\lambda \setminus F_\mu$ does not contain any square $F_{(2,2)}$ of

four cells. Such diagrams with n cells are coded by a word of length $n-1$ on two letters. There exists a trivial bijection between permutations of $\mathfrak{S}_n$ and skew Young tableaux with distinct entries $1, 2, \ldots, n$ and with a strip shape. From such tableaux, we associate a permutation by reading the entries from North-West to South-East following the strip. Conversely, from a permutation σ we define a skew Young tableau $Str(\sigma)$ having a strip shape and such that the cell containing $\sigma(i+1)$ is consecutive and at the East (resp. South) of the cell containing $\sigma(i)$ iff $\sigma(i) < \sigma(i+1)$ (resp. $\sigma(i) > \sigma(i+1)$). In other words, the strip is nothing but a coding of the up-down sequence of the permutation defined in § 4 .

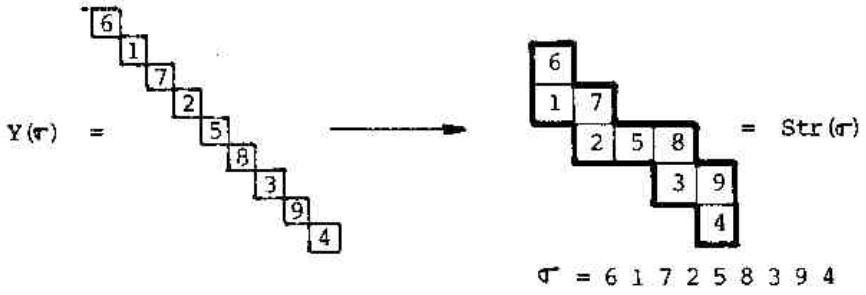


Figure 17 - The skew Young tableaux $Y(\sigma)$ and $Str(\sigma)$ coding a permutation σ .

Obviously, the tableau $Str(\sigma)$ can be obtained from $Y(\sigma)$ by applying the "jeu de taquin", and thus, by Schützenberger's theorems (invariance of the "redressé" under the "jeu de taquin" and **proposition** 8),the "redressé" $R(Str(\sigma))$ is the P-symbol $P(\sigma)$ of the permutation σ . The reader will check this fundamental fact with figure 16 for the generic permutation appearing in all the examples of this paper.

The invariance of the "redressé" under the "jeu de taquin" is proved by introducing the following functions, based on Greene's interpretation of the shape of the tableaux $P(\sigma)$ and $Q(\sigma)$ (we shall assume that the entries are distinct, the general case can be deduced from that).

Let Y be a skew Young tableau (with possibly a "hole" inside). We define an increasing subsequence of Y to be a sequence of integers $x_1 < \ldots < x_k$ such that for every i, $1 \leqslant i < k$, x_{i+1} is in a cell which is at the South-East of the cell where x_i is located. Then, for any integers k and $\ell \geqslant 1$, we define $G_{k,\ell}(Y)$ to be the maximum cardinality of subsets of entries of Y that are unions of k increasing subsequences with values $\leqslant \ell$.

It can be shown that the function $G_{k,\ell}$ is invariant under the elementary sliding of the "jeu de taquin". The fact mentioned above is proved from the remark that a standard Young tableau is characterized by the function $G_{k,\ell}$.

Then one can show directly that the correspondence $\sigma \to (R(\sigma), R(\sigma^{-1}))$ is a bijection. The "bumping process" can be seen as particular case of the slidings of the "jeu de taquin", and thus we have proposition 8. From there, Greene's interpretation and properties of Knuth's transpositions can be deduced. Other properties mentioned in § 4 are also deduced. For example, the property relating the up-down sequence and the "line of route" is easily shown as a property invariant under the "jeu de taquin". Also, taking the dual of a tableau corresponds to reversing the "jeu de taquin", that is reverse the order between integers and move the entries to the North-East (instead of the South-West).

Products of Young tableaux.

The above combinatorial considerations can be expressed in a more algebraic way by introducing the plactic monoid.

Let Y and Z be two Young tableaux. Denote by $Y \cdot Z$ the skew Young tableau obtained by putting Z at the South-East of Y as shown on figure 18. The product of the two Young tableaux Y and Z is defined to be the "redressé" $R(Y \cdot Z)$. By the invariance of the "redressé" under the "jeu de taquin", this product is associative. The empty tableau is the identity element. The set of Young tableaux with entries in the set of integers A, and with the product just defined, is the plactic monoid generated by A.

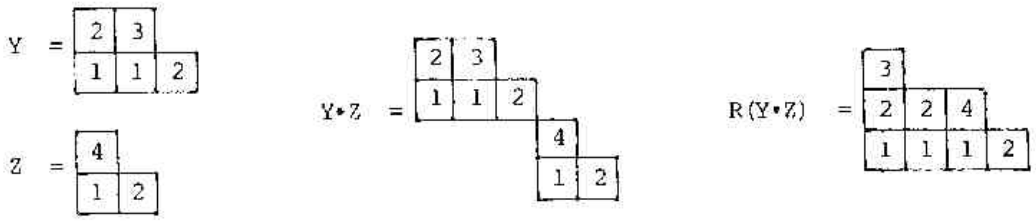


Figure 18 - The product of two Young tableaux.

Proposition 8 says nothing but that the P-symbol $P(\sigma)$ is the plactic product of the n Young tableaux reduced to one cell $\sigma(1), \sigma(2), \ldots, \sigma(n)$. The insertion process $I(T,x)$ defined in § 3 is in fact the plactic product of T with the single-cell x placed at the South-East.

Another way to define the plactic monoid comes from Knuth's transpositions. Let A be a totally ordered alphabet. Let A^* be the free monoid generated by A, that is the set of words $w = w_1 \ldots w_n$ with letters w_i in A and with product the concatenation product : for $u = u_1 \ldots u_p$ and $v = v_1 \ldots v_q$, $uv = u_1 \ldots u_p v_1 \ldots v_q$. We define the plactic congruence $\equiv$ to be the congruence of A^* generated by the relations

$$\text{for any letters } x,y,z \text{ of } A \text{ such that } x \leq z \leq y \text{ or } y \leq z \leq x,\quad x\,y\,z \equiv y\,x\,z \text{ and } z\,x\,y \equiv z\,y\,x, \tag{6}$$

$$\text{for any letters } x,y \text{ of } A \text{ such that } x \leq y,\quad x\,y\,x \equiv y\,x\,x \text{ and } y\,x\,y \equiv y\,y\,x.$$

In each equivalence class, there is one and only one "tableau" ; that is, a word that can be obtained by reading a Young tableau row by row, from left to right in each row, the rows being ordered up-down (this corresponds to the row-canonical permutations defined in § 4). It can be shown that the quotient monoid $A^*/\equiv$ is isomorphic to the plactic monoid generated by A defined above.

§ 7 - Grids.

This section presents a summary of a forthcoming paper of the author [127], where complete proofs of the new results announced will be given.

A word σ with pq distinct letters is called a (p,q)-grid iff there exists a Young tableau Y with distinct entries and $p \times q$ rectangular shape such that the pq entries of Y are exactly the letters of σ, and the rows and columns of Y are (monotone) subsequences of σ.

It is easily seen that such a tableau Y is unique and is in fact the P-symbol $P(\sigma)$. Also we have the following lemma

LEMMA 9 - Let σ be a word with distinct letters (or permutation). The following conditions are equivalent

(i) σ is a grid,

(ii) the shape of $P(\sigma)$ is rectangular,

(iii) for any $i \geq 1$, the i^{th} column of $P(\sigma)$ is composed of the y-coordinates of the points of $\tilde{\sigma}$ lying on the i^{th} outstanding line of $\tilde{\sigma}$.

Permutations with rectangular shape for the associated Greene diagram have many other nice properties. The number of such permutations, with a fixed $p \times q$ rectangle associated Greene diagram, has been considered by Hiller [44], in term of the "Schubert calculus" describing the "cohomology of the complex grassman manifold".

In this section, we are mostly interested in subgrids of permutations, that is subsequences that are grids (see example in figure 19). When a permutation σ is viewed as a poset $Pos(\sigma)$, the concept of subgrids is closely related to the concept of "orthogonal families" introduced by Frank [23] in his proof of Greene's Theorem (see definition § 8 below).

First, we have the following

LEMMA 10 - Let σ be a permutation for which the associated poset $Pos(\sigma)$ has Greene diagram $G(Pos(\sigma))$ (shape of $P(\sigma)$). For any $p \times q$ rectangle contained in $G(Pos(\sigma))$, there exists a (p,q)-subgrid of σ.

The proof relies on Greene's interpretation of $G(Pos(\sigma))$ with chains and antichains k-families of $Pos(\sigma)$ (proposition 6) : first one takes a subsequence τ of σ that is a union of p chains (increasing subsequences). The Greene diagram $G(Pos(\tau))$ is not necessarily the same as the diagram $G_p(Pos(\sigma))$, obtained from $G(Pos(\sigma))$ by deleting all rows indexed by $i > p$, but the number of elements of the p^{th} row of $G(Pos(\tau))$ is not less than the number of elements of the p^{th} row of $G_p(Pos(\sigma))$. Applying the same reasoning on τ with antichains (decreasing subsequences) leads to lemma 10.

As shown in figure 19, the converse of lemma 10 is not true. A major problem is to give a characterization of subgrids giving exactly the Greene diagram $G(Pos(\sigma))$ and thus obtain an interpretation of $G(Pos(\sigma))$ symmetric in rows and columns (i.e. chains and antichains).

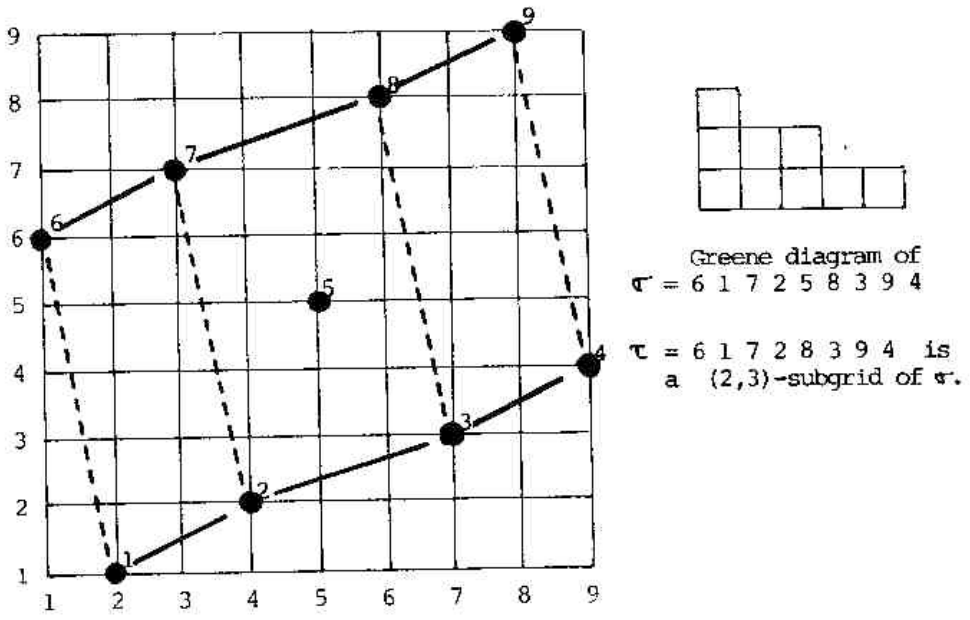


Figure 19 - A (2, 4)-subgrid of a permutation with no corresponding 2×4-rectangle.

Let σ be a permutation of $\mathfrak{S}_n$ and τ be a subgrid of σ. We define the interior of the grid τ to be the intersection of the four shadows (see § 5) of $\hat{\tau}$ related to the four possible directions of the light : North-West, South-West, South-East and North-East. In other words, this is also the set of points (x, y) of $\mathbb{R}\times\mathbb{R}$ such that there exists at least one point of $\hat{\tau}$ in each of the four quadrants with origin at (x, y).

The interior of a grid is a union of elementary cells having the property that for every pair of cells on the same vertical (resp. horizontal), all the cells between them are in this union. Such objects are known in combinatorics as convex polyominoes (see [9] and the references therein).

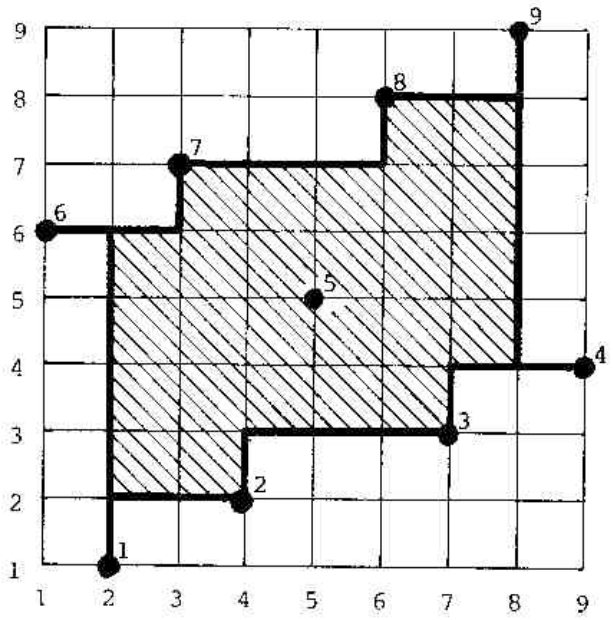


Figure 20 - The interior of a grid.

Let τ be a (p, q)-subgrid of the permutation σ of $\mathfrak{S}_n$ and $P(\tau) = Y$ the unique corresponding tableau in the definition of grids. We shall say that the subgrid τ is extendable (in σ) iff there exists increasing subsequences $\alpha_1, \ldots, \alpha_p$ of σ and decreasing subsequences $\beta_1, \ldots, \beta_q$ of σ

satisfying the three following conditions (see figure 21).

(i) for any i, $1 \leq i \leq p$, the i^{th} row of Y is a (increasing) subsequence of α_i,

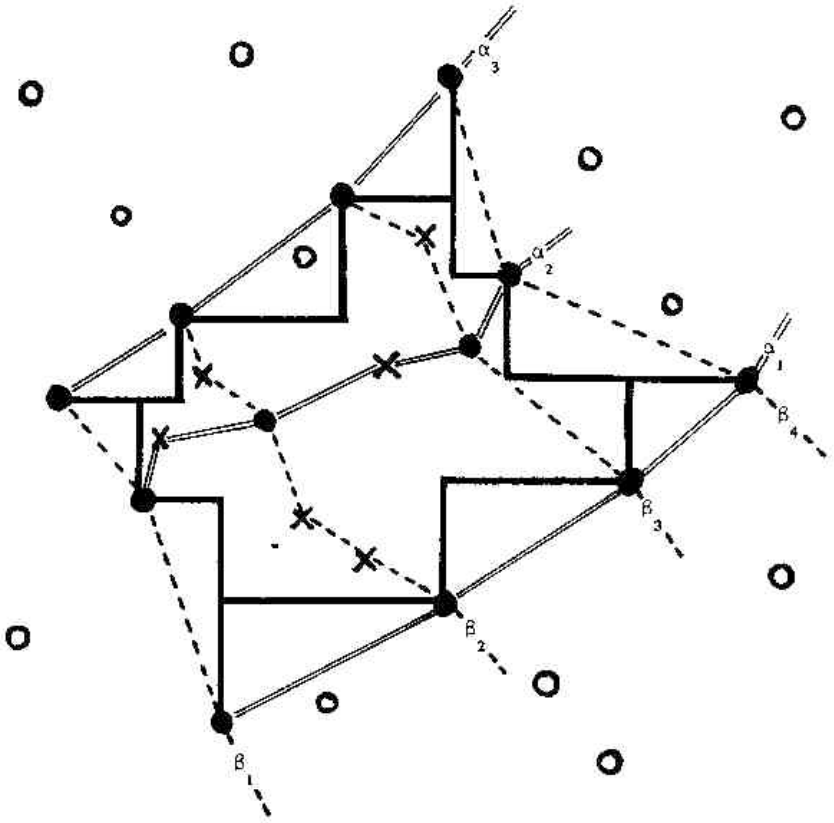
(ii) for any j, $1 \leq j \leq q$, the j^{th} column of Y is a (decreasing) subsequence of β_j,

(iii) every point of $\hat{\sigma}$ that is in the interior of τ belongs to one of the subsequences $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$.

The definition of extendable subgrids is not changed if we require that the subsequences $\alpha_1, \ldots, \alpha_p$ are pairwise disjoint, and similarly for $\beta_1, \ldots, \beta_q$.

The (2, 4)-subgrid displayed in figures 19 and 20 is not extendable. This comes from the fact that (2, 5, 8) and (7, 5, 3) are respectively increasing and decreasing subsequences of σ, with the point labeled 5 in the interior of σ. More generally, a point x of the permutation σ is said to be a <u>critical</u> point for the subgrid τ iff there exists entries of the Young tableau $P(\sigma)$ such that $(a_{i,j}, x, a_{i+1,j+1})$ is an increasing subsequence and $(a_{i+1,j}, x, a_{i,j+1})$ is a decreasing subsequence of σ. Obviously, such critical point x is in the interior of the (2, 2)-subgrid (of τ) $\{a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}\}$, and thus is also in the interior of the subgid τ. Now one can see that x cannot be "incorporated" in one of the (monotone) subsequences of σ formed with the rows and columns of the Young tableau $Y = P(\sigma)$, (see figure 22).

Thus, a subgrid having a critical point is not extendable. Note that the converse is not true (see figure 23). Note also that the shape (4, 2) of the subgrid of figure 23 is not contained in the Greene diagram.



● points of the subgrid τ

× points of σ in the interior of τ but not not in τ

○ other points of the permutation σ

—— border of the interior of the subgrid τ

═══ increasing subsequences (chains of $\tilde{\sigma}$)

- - - - decreasing subsequences (antichains of $\tilde{\sigma}$)

Figure 21 - An extendable (3,4)-subgrid.

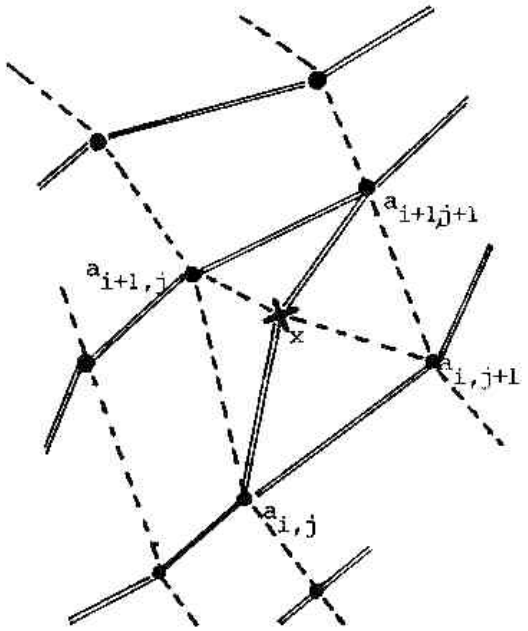


Figure 22 - A critical point of a subgrid.

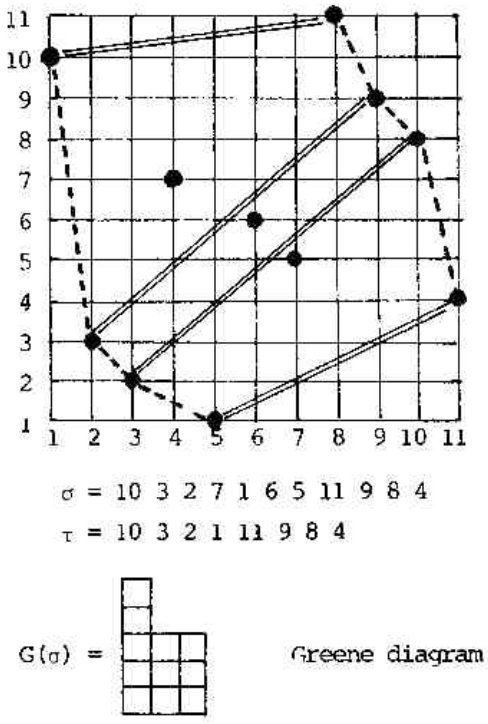


Figure 23 - A non-extendable subgrid having no critical point.

The main result of this section is

PROPOSITION 11 - Let σ be a permutation and $P(\sigma)$ its P-symbol obtained by the Robinson-Schensted correspondence. The value $P_{ij}(\sigma)$ located in the (i,j) cell of $P(\sigma)$ is equal to

$$P_{ij}(\sigma) = \min_{\tau} (\max \tau) ,$$

where the minimum is taken over all extendable (i,j)-subgrids τ of σ.

In other words, if σ is a permutation in $\mathfrak{S}_n$ and x an integer $1 < x \leq n$, there exists a unique pair (i,j) such that x is the maximum value of an extendable (i,j)-subgrid and such that no extendable (i,j)-subgrid can be extracted from the subsequence of σ obtained by deleting all values $\geq x$. Furthermore, this (i,j) is precisely the position of x in the P-symbol of σ. This kind of "topological" property of points displayed in the plane is a "local" definition of the Robinson-Schensted correspondence (and a fourth one after the "bumping process" of § 3, the shadows and skeletons of § 5 and the "jeu de taquin" and plactic monoid of § 6).

Proposition 11 is trivial in the case of row (or column) canonical permutations (see § 4). Thus proposition 11 is proved by showing the invariance of the min-max quantity under Knuth's transpositions.

In fact, more can be said. One can introduce analogously "extendable" subgrids of skew Young tableaux (with "holes"). There is no more "interior" of a subgrid but an analagous definition is possible from the next section. The corresponding min-max function plays the role of the function $G_{k,\ell}$ introduced in § 6. Again one can deduce most of the properties of the Robinson-Schensted correspondence from the invariance of this min-max function under the elementary slidings of the "jeu de taquin". The proofs are more complicated, but one has the satisfaction of keeping the symmetry between rows and columns, that is between chains and antichains.

§ 8 - Extensions to arbitrary posets.

In conclusion, we discuss (very briefly) possible extension of the Robinson-Schensted correspondence and of the grid concept to arbitrary posets. This section is mainly based on work of Greene [35], Fomin [16], Gansner [27], Frank [23] and a forthcoming paper of the author [128].

Let P be a poset of cardinality n. Let φ be a linear extension in $[n] = \{1, \ldots, n\}$. The problem is to associate a standard Young tableau $Y(P,\varphi)$ such that this tableau is $P(\sigma)$ (resp. $Q(\sigma)$) in the case $P = Pos(\sigma)$ with σ permutation of $\mathfrak{S}_n$, and φ is the labeling of the vertices of $Pos(\sigma)$ by the values $\sigma(i)$ (resp. indices i).

As shown by Fomin [16], a solution comes from Greene's theory (theorem 1 and proposition 6) and the following lemma : if P' is the poset obtained by

deleting from P the vertex labeled n (maximum value), then $G(P') = G(P)$. Here $G(P')$ and $G(P)$ are the Greene's diagrams of P' and P as defined in § 2.

For any $i \in [n]$, let P_i be the subposet obtained by taking the vertices labeled $1, \ldots, i$. The Ferrers diagrams $\{G(P_i),\ 1 \leqslant i \leqslant n\}$ form a chain under inclusion. We can define a standard Young tableau $Y(P,\varphi)$ by placing i in the unique cell of $G(P_i) \setminus G(P_{i-1})$ (for any $i \in [n]$, with the convention $G(P_0) = \emptyset$). Figure 24 gives an example with the poset $P = Pos(\sigma)$ of figure 1, but with φ different from the two natural lalelings giving the P and Q-symbol of the generic permutation $\sigma = 6\ 1\ 7\ 2\ 5\ 8\ 3\ 9\ 4$.

Very little is known about this correspondence $(P,\varphi) \to Y$. Greene's properties about k-matching of permutations (see § 4) have been extended by Gansner [27], where he gives also a dual analogue with k-scatters.

Altough we are going to give an extension of proposition 11 to arbitrary posets, no simple algorithm, analogous to the Robinson-Schensted algorithm, has been found for the construction of $Y(P,\varphi)$.

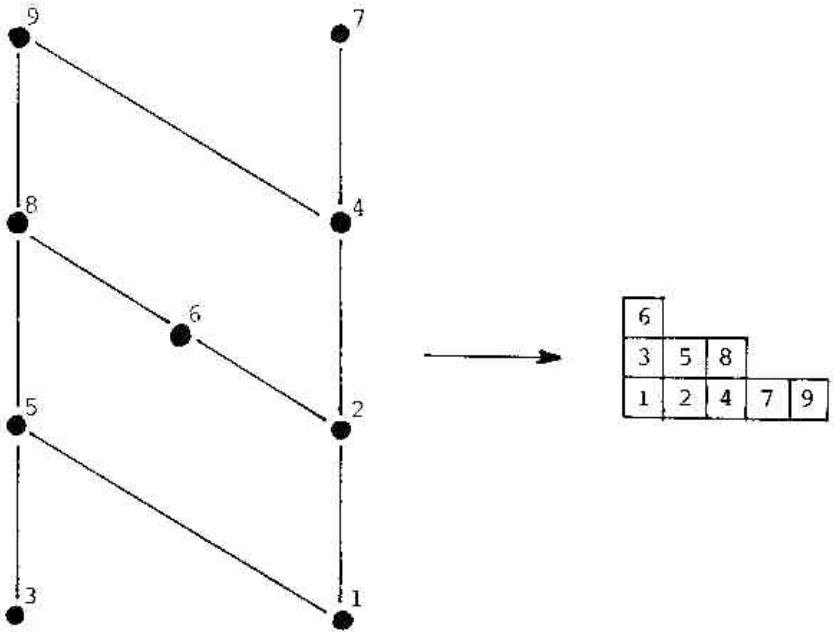


Figure 24 - The correspondence $(P,\varphi) \longrightarrow Y$.

Let P be a poset. A subset γ of P is said to be a (p,q)-grid iff γ can be partitioned into p chains $\{c_{i,1} \leqslant \ldots \leqslant c_{i,q}\}$, $1 \leqslant i \leqslant p$, such that for any j, $1 \leqslant j \leqslant q$, $\{c_{i,j}, \ldots, c_{p,j}\}$ is an antichain of P.

With this extension of the grid definition, it is easy to see that lemma 10 is still valid for any poset P. But figure 24 gives an example of a $(2,4)$-grid in a poset whose Greene's diagram does not contain a 2×4 rectangle. As in § 7, we give a characterization of grids with shape contained in the Greene diagram. For that, we use the following notion introduced by Frank [23].

Let $\{\alpha_1, \ldots, \alpha_p\}$ be a chain family and $\{\beta_1, \ldots, \beta_q\}$ be an antichain family of the poset P, with the chains (resp. antichains) supposed pairwise disjoint. These families are said to be orthogonal iff they satisfy the two conditions :

(i) $P = (\bigcup_i \alpha_i) \cup (\bigcup_j \beta_j)$,

(ii) for any i, $1 \leqslant i \leqslant p$ and j, $1 \leqslant j \leqslant q$, $\alpha_i \cap \beta_j \neq \emptyset$.

It is easy to see that the chain family $\{\alpha_1, \ldots, \alpha_p\}$ (resp. antichain family $\{\beta_1, \ldots, \beta_q\}$) has maximum cardinality. Also, the subset $\gamma = \{\alpha_i \cap \beta_j, 1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q\}$ is obviously a (p,q)-grid of P.

We shall say that a grid γ of P is completely extendable iff there exist orthogonal families of chains $\{\alpha_1, \ldots, \alpha_p\}$ and antichains $\{\beta_1, \ldots, \beta_q\}$ and subsets $I \subseteq [p]$, $J \subseteq [q]$ such that

$$\gamma = \{\alpha_i \cap \beta_j, i \in I, j \in J\}.$$

The concept of orthogonal families is essential in Frank's proof of Greene's Theorem. Recall that this proof relies on linear programming techniques.

From Frank's proof of Greene's Theorem (see theorem 3 of [23]), we can deduce

PROPOSITION 12 - The Greene diagram of the poset P contains a $p\times q$ rectangle iff there exists a (p,q)-completely extendable grid in P.

Also, using Greene's Theorem and results of Fomin [16], we can deduce from proposition 12.

PROPOSITION 13 - Let P be a poset, $\varphi : P \to [n]$ be a natural labeling of P and Y be the standard Young tableau defined by the above correspondence. The value $Y_{ij}(P)$ located in the (i,j) cell of Y is equal to

$$Y_{ij}(P) = \min_{\gamma} (\max \gamma)$$

where the minimum is taken over all completely extendable (p,q)-grids, and $\max(\gamma)$ denotes the greatest label of the vertices of γ.

I have not been able to find a direct combinatorial proof of Proposition 12 and 13, not using linear programming techniques.

This proposition can be applied in the case $P = \mathrm{Pos}(\sigma)$ where $\sigma \in \mathfrak{S}_n$. It is possible to define the concept of completely extendable grid extracted from a skew Young tableau, show its invariance under the "jeu de taquin" and after § 6 make an ultimate synthesis of the Robinson-Schensted correspondence and the plactic monoid [127].

Instead of extending the Robinson-Schensted correspondence to arbitrary posets, one can also look for generalizations involving other kinds of tableaux than the Young tableaux used above. In § 3 we have mentioned the shifted Young tableaux. An analog of the "bumping process" has been defined by Sagan [83], where he describes a bijection between permutations and pairs of "colored shifted Young tableaux" having the same shape. This bijection proves the analog of relation (1) for projective representations of the symmetric group. Other generalizations have been made by Zelevinsky [121] to certain kind of diagrams called pictures, in connection with the intertwining number of representations of the symmetric group, by Stanton and White [101], [115] to rim hook tableaux giving a combinatorial proof of the orthogonality of the characters of $\mathfrak{S}_n$ and "rim hook" version of the "bumping process" and of the "jeu de taquin", and by Berele and Regev [4] to "semi-standard tableaux" in two sets of variables in connection with the representation theory of the general linear group $GL(k,F)$, Weyl's "Strip" theorem [114] and the theory of P.I. algebras.

Also we mentioned above in § 4 the Grassl-Hillman correspondence, related to the theory of plane partitions, a subject that we do not touch here but closely related to the Robinson-Schensted correspondence (see for example Andrews [1] chapter 11, Mac Mahon [73], [74] collected papers, chapters 11 and 12,

Stanley [98]). Gansner [26] showed that this correspondence, when viewed through the Frobenius correspondence (between reverse plane partitions and certain pairs of tableaux), is an extension of column insertion, and closely related to the "Burge correspondence" [6]. In [111] Vo and White showed that (a slight variation, equivalent to the original, of) the Grassl-Hillman correspondence is just row insertion in disguise. This correspondence has been extended to the two other families of posets with hooklengths quoted in § 4 by Sagan [85].

We conclude this paper by mentioning briefly some very recent and interesting work of Edelman and Greene [123], [124]. In [125], Stanley conjectured that the number of maximal chains in the (weak) Bruhat order of $\mathfrak{S}_n$ is equal to the number f_ν of standard Young tableaux having the "staircase" shape $\nu = (n-1, n-2, \ldots, 1)$. (The weak Bruhat order on $\mathfrak{S}_n$ is the order generated by the relations $\sigma \lessdot \theta$ if θ is obtained from σ by a transposition of adjacent increasing elements). An algebraic proof is given by Stanley [126]. A bijective proof is given by Edelman and Greene. In fact, three bijections are needed, using Schützenberger's operators [94], [97] (analogous to the dual $Y \to Y^J$ described in § 4), and the new concept of balanced tableaux. Let λ be a partition of n and λ^* the conjugate partition. A balanced tableau of shape λ is a labeling t_{ij} of the cells (i,j) of the Ferrers diagram F_λ with distinct entries $1, 2, \ldots, n$, such that the value t_{ij} is the r_{ij}^{th} largest element of its "hook" H_{ij}, where $r_{ij} = \lambda^*_j - i + 1$ and where H_{ij} is the multiset $\{t_{ik}\}_{k \geq j} \cup \{t_{kj}\}_{k \geq i}$.

For example,

		3		
	2	1		
	6	10		
5	7	4	8	9

is a balanced tableau of shape $\lambda = (5, 2, 2, 1)$.

One of the most surprising fact is that the number of balanced tableaux of shape λ is f_λ, the number of standard Young tableaux of shape λ (see formulae (3a), (3b) and (3c)).

Again there is deep combinatorics behind this relation between tableaux and chains in the Bruhat order, related to some recent work of Lascoux and Schützenberger [65], [66] about what they called Schubert polynomials and

the nilplactic monoid.

This monoid is defined by a variation of the relation (6), (mixing plactic and Coxeter congruences), and corresponds to a modified Robinson-Schensted insertion algorithm. Stanley's conjecture can also be deduced using the nilplactic monoid.

Acknowledgements.

I thank one of the referees for numerous detail corrections of the original manuscript.

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