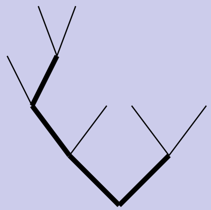
Algèbre de Loday-Ronco et tableaux alternatifs de Catalan (2/2)

J.-C. Aval, X. Viennot

GT - LaBRI - 30/01/09

Rappels – algèbre des arbres binaires plans

[Loday-Ronco]



Y_n : ensemble des arbres binaires plans

$Y = \mathbb{Q}[\sqcup Y_n]$

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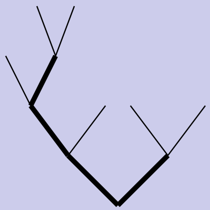

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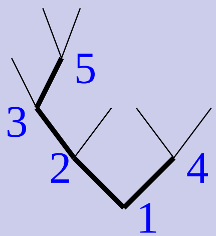

Y_n^+ : ensemble des arbres binaires plans croissants

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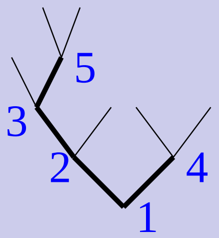

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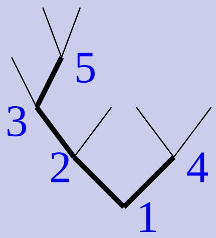

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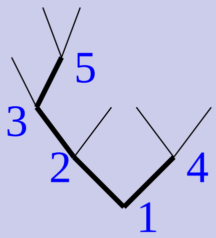

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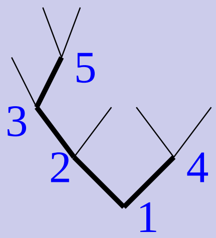

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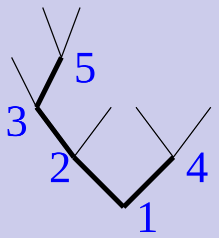

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$$\sigma * \alpha = \sum_{\substack{u,v=\{1,\dots,n\} \\ Std(u)=\sigma, Std(v)=\alpha}} uv$$

$$12 * 21 = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$$

RaAlgèbre des arbres binaires plans

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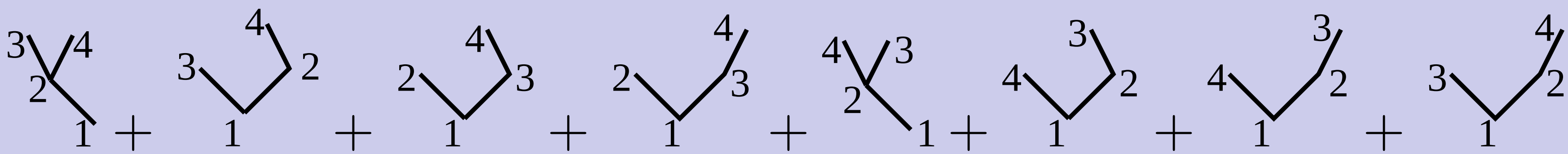
$=$ 

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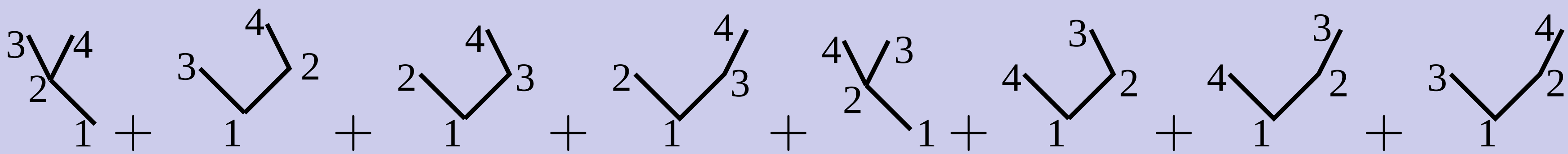
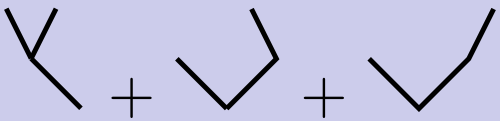
$=$ 

$=$ 

La question

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Comment multiplier deux arbres dans l'algèbre de LR ?

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Comment multiplier deux TAC dans l’algèbre de LR ?

Le résultat

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T_1 et T_2 deux arbres binaires, en bijection avec les TAC A_1 et A_2

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la somme étant prise sur les arbres

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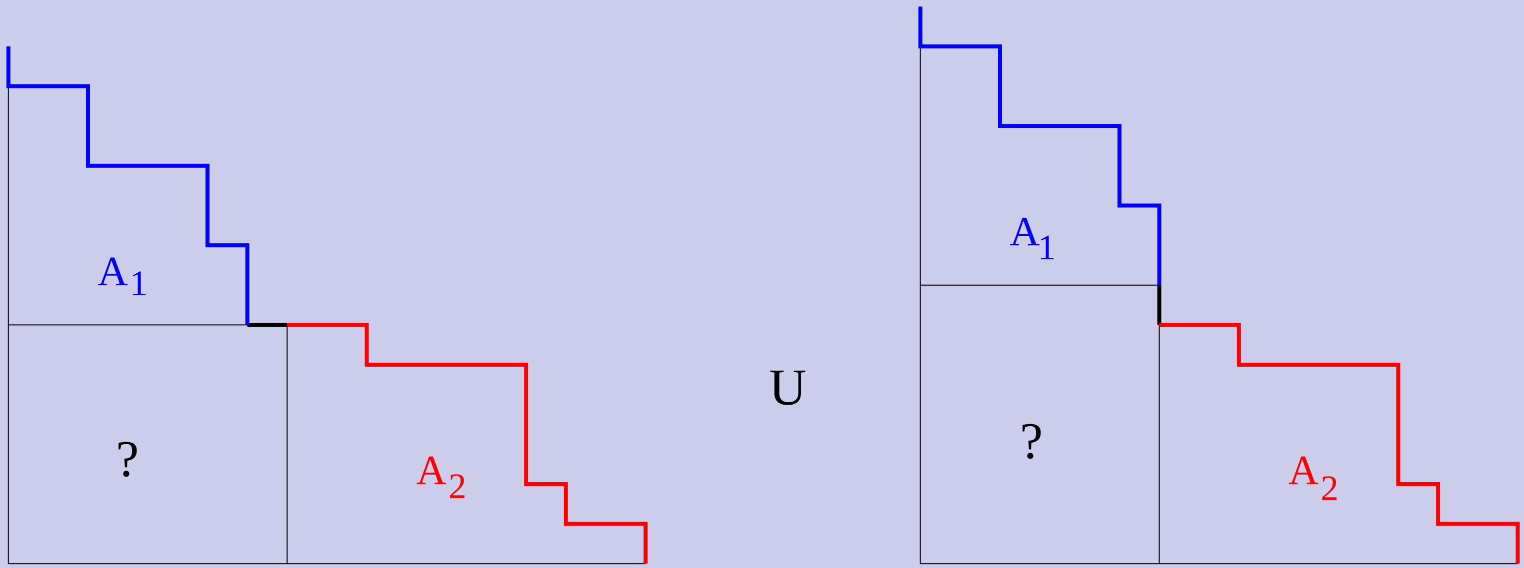
la somme étant prise sur les arbres en bijection avec les TAC A dans l'union :



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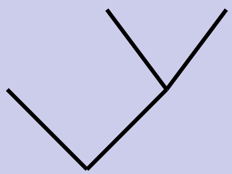
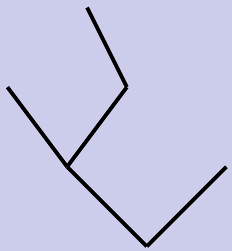
$T_1 =$ 

$T_2 =$ 

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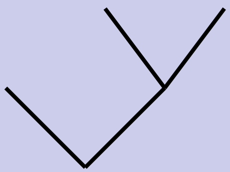
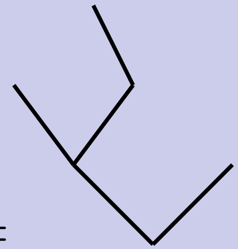
$T_1 =$ 

$T_2 =$ 

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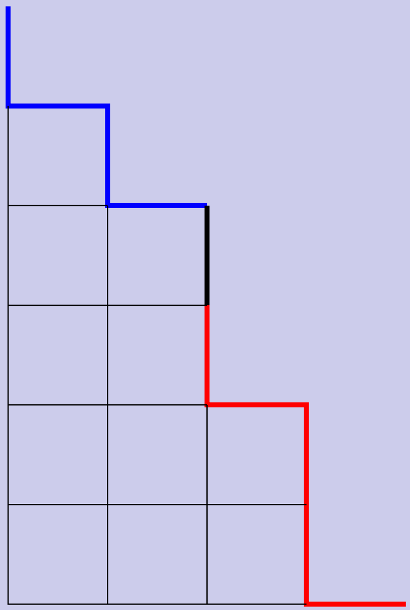

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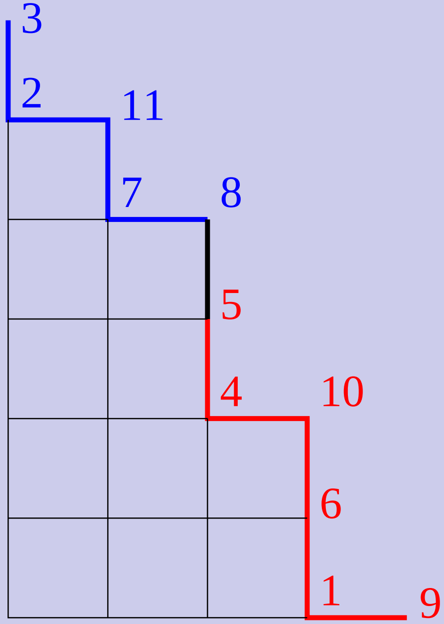

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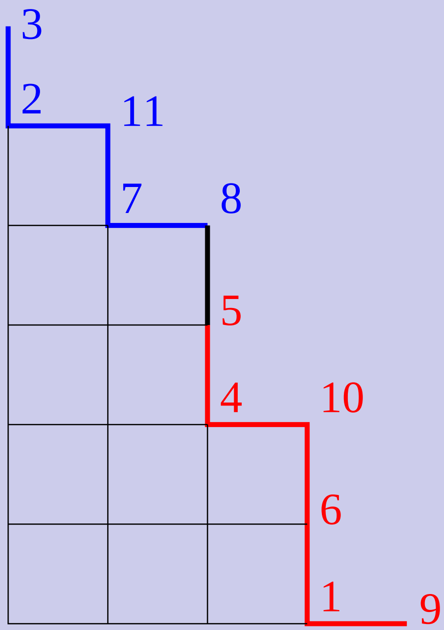

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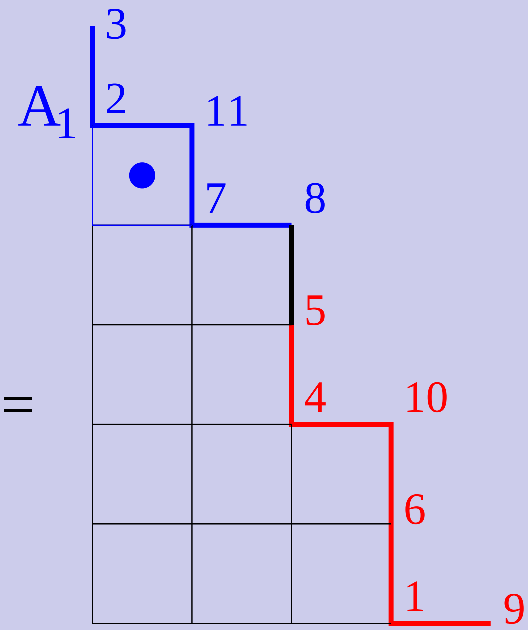

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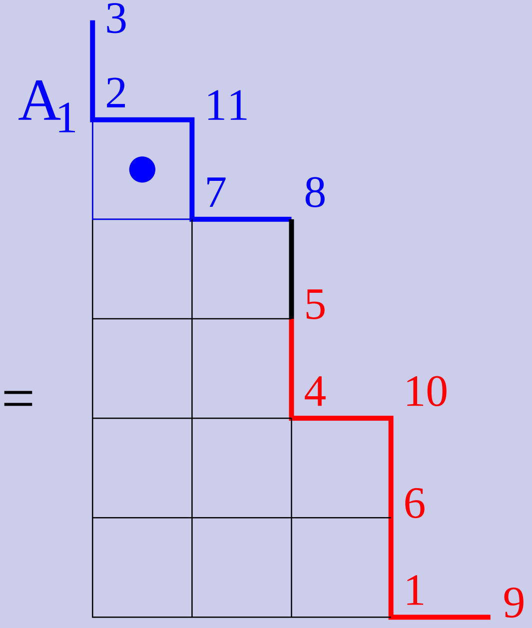

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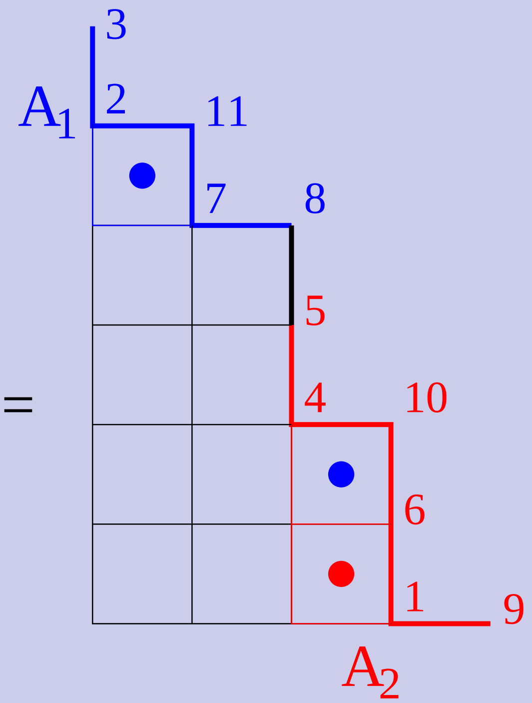

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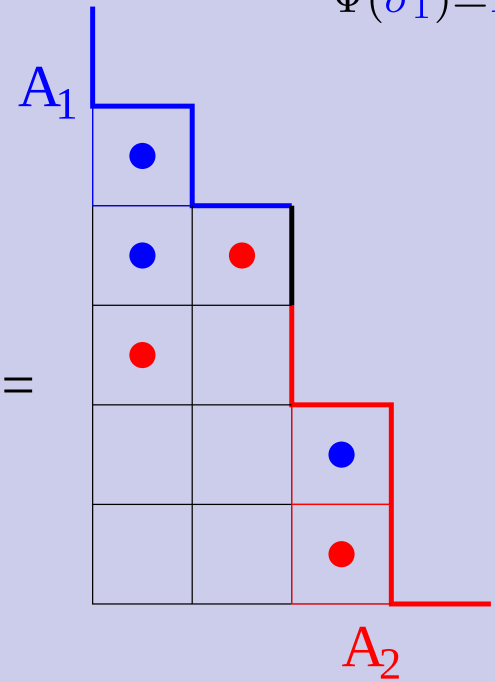

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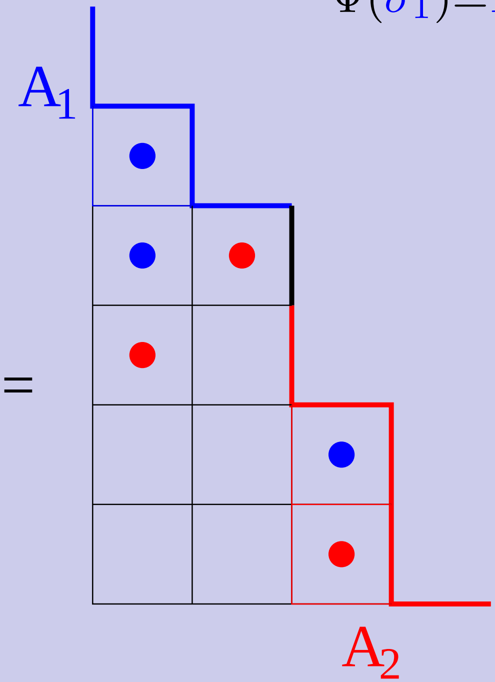

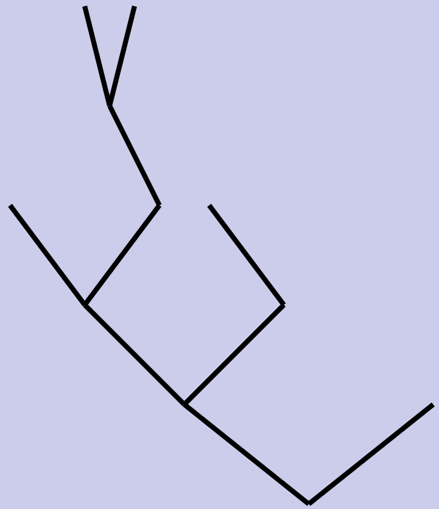

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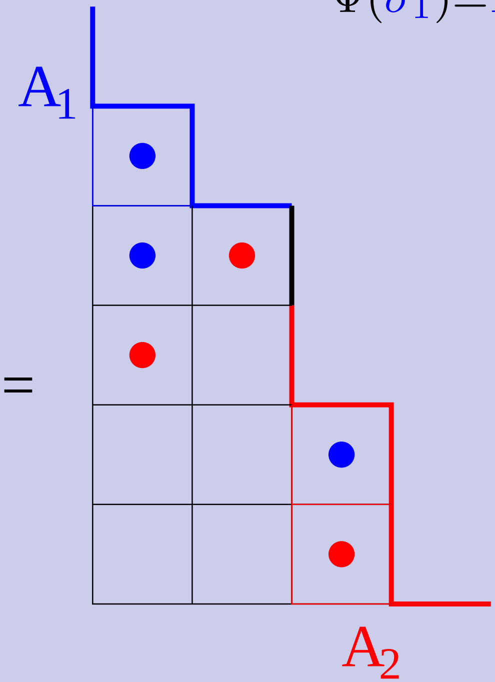

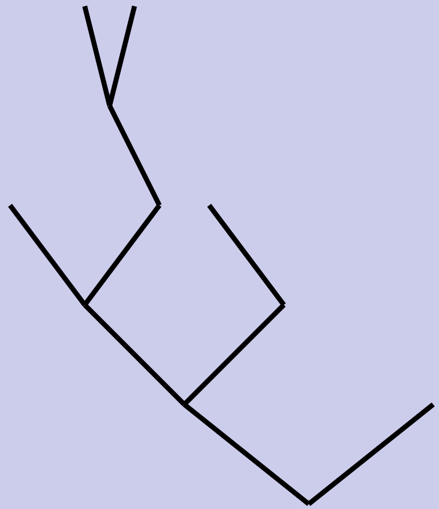

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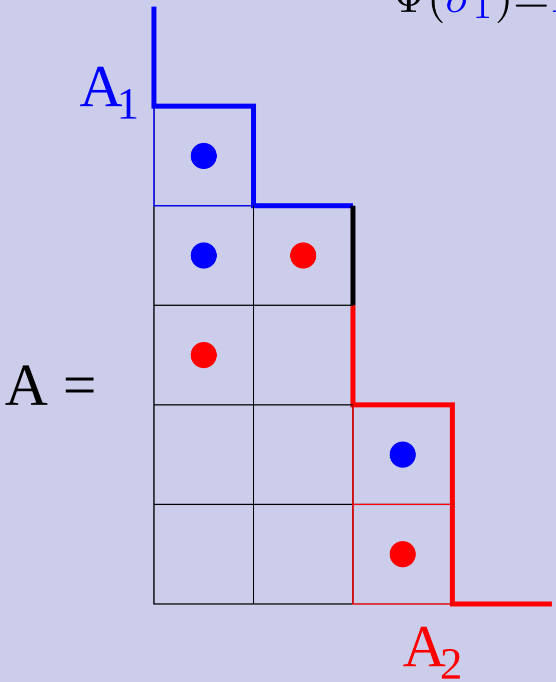

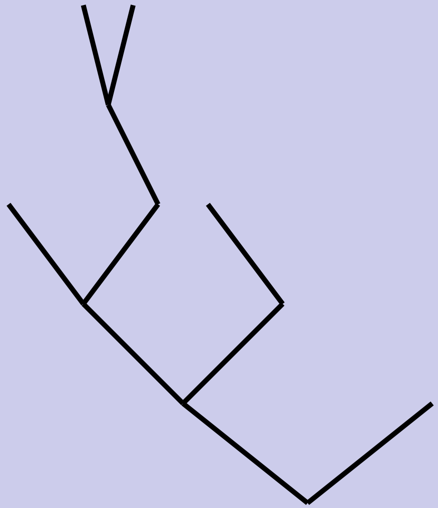

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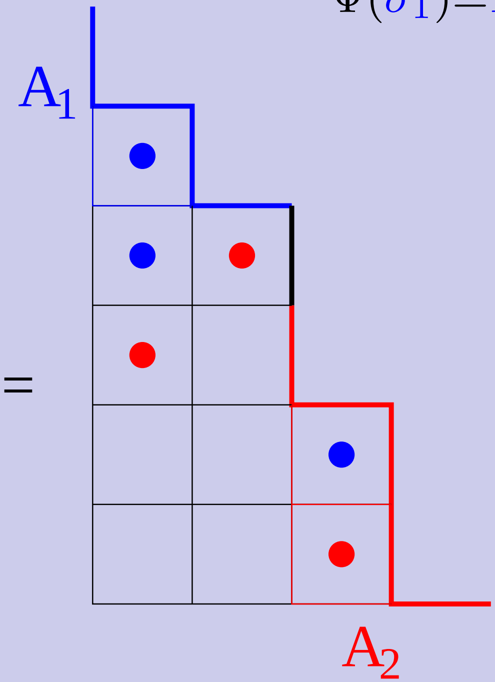

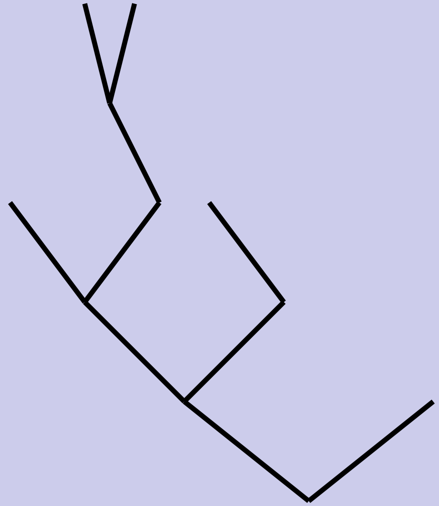

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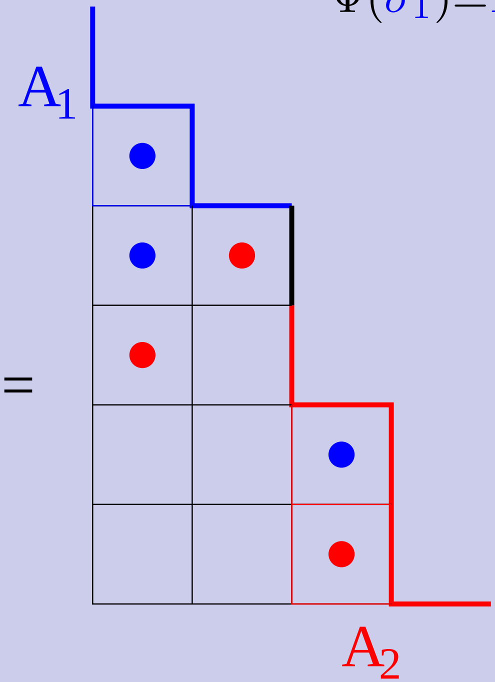
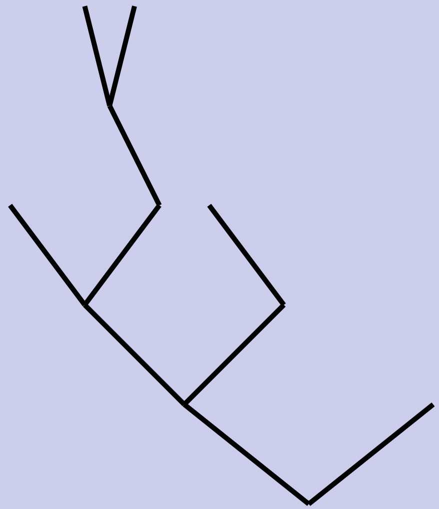
A = 



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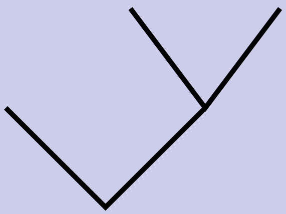
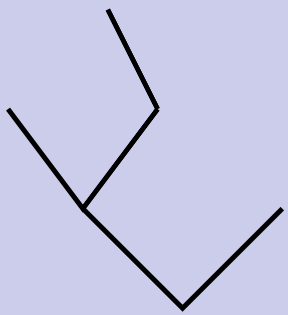


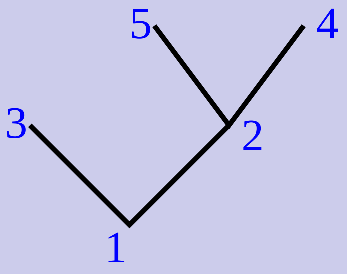
$\sigma_2 = 4\,2\,5\,3\,1\,6$

$\Psi(\sigma_2) = T_2$

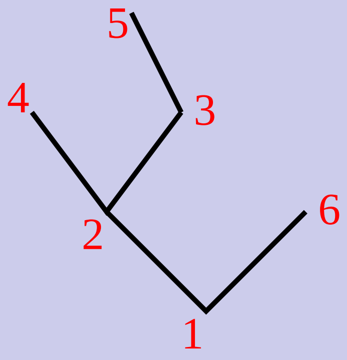


La preuve (⊃)

$\Psi(\sigma_1) = T_1$

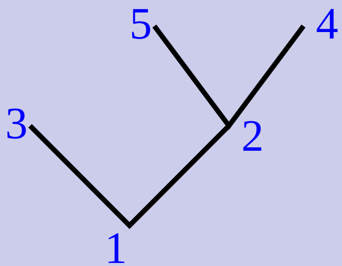


$\Psi(\sigma_2) = T_2$

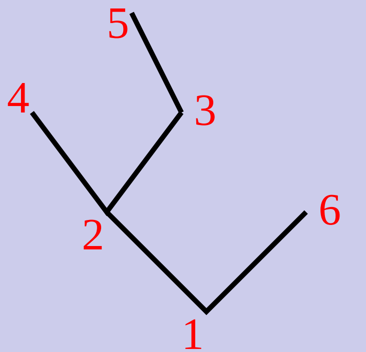


La preuve ($\supset$)

$\Psi(\sigma_1) = T_1$



$\Psi(\sigma_2) = T_2$



en effet :

A =