# Chapter 1

## Ordinary generating functions

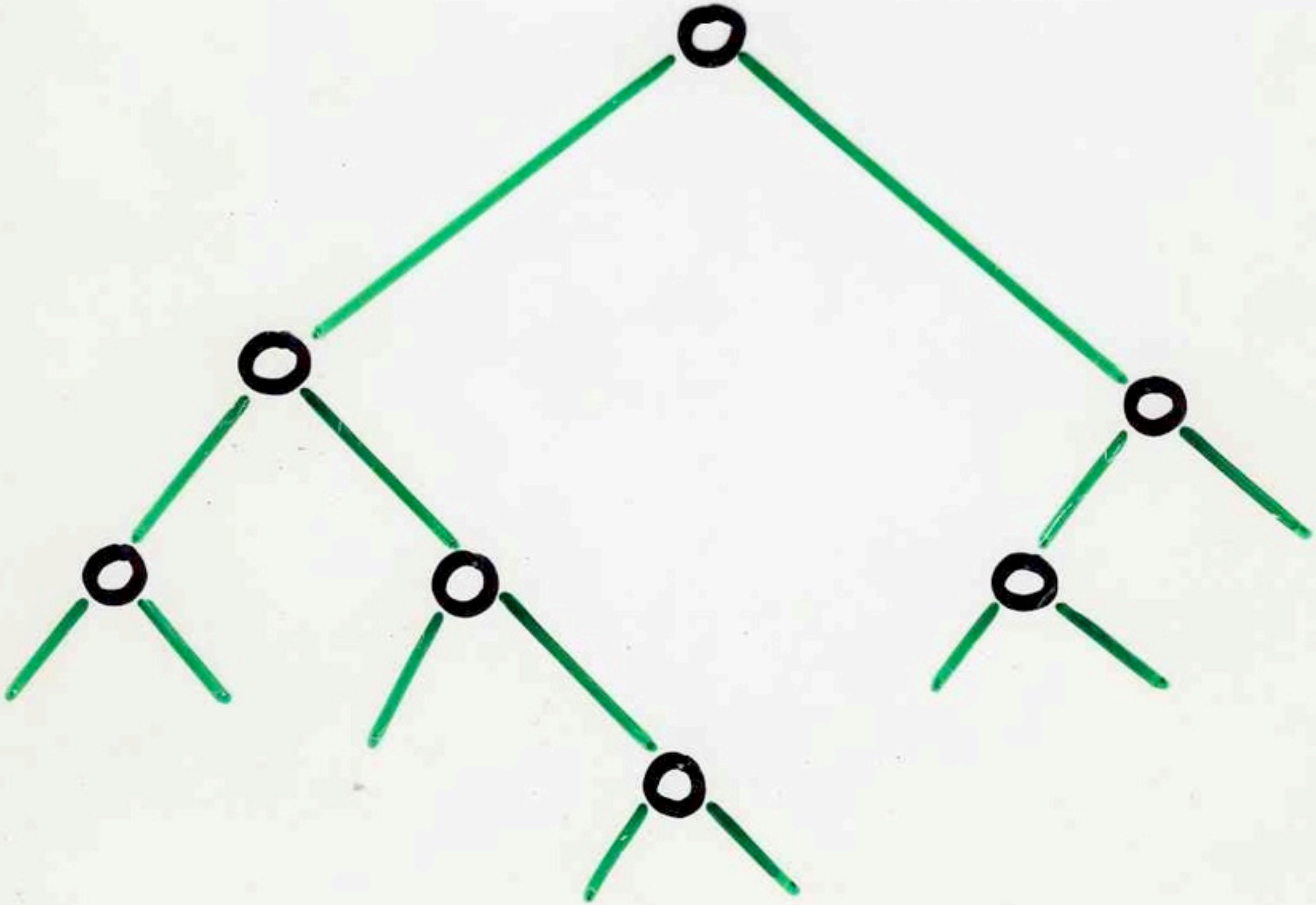
## The Catalan garden

1st december 2010

Talca

# binary tree

# Binary tree



$C_n =$ nombre
d'arbres binaires
ayant $n$
sommets internes

(et donc $n+1$ feuilles)

<u>nombre de Catalan</u>

number of binary trees
having n internal vertices
(or n+1 ) leaves (external vertices)

recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$\begin{array}{cccccc} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \\ 1, & 1, & 2, & 5, & 14, & 42, \ \dots \end{array}$$

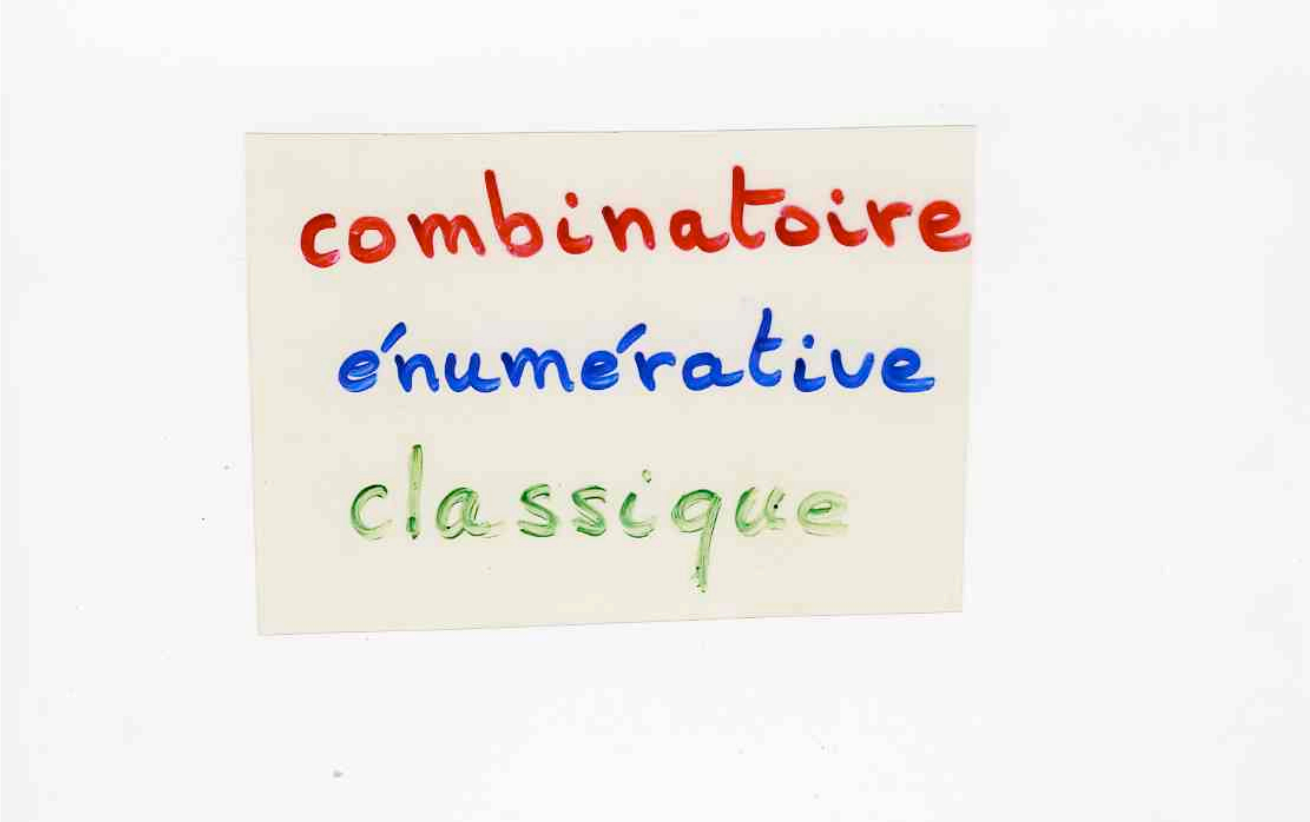
$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

$$132 \qquad 1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\ n!}$$

$$n! = 1 \times 2 \times \dots \times n$$




classical
enumerative
combinatorics

(1838)

## *Note sur une Équation aux différences finies;*

**PAR E. CATALAN.**

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \dots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\cdots\frac{4n-6}{n}P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\dots(4n-6)}{2.3.\ 4.\ 5\dots n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\dots(4n-6) = 2^{n-1}.1.3.5.7\dots(2n-3) = \frac{2^{n-1}.1.2.3.4.5\dots(2n-2)}{2.4.6.8\dots(2n-2)} = \frac{1.2.3.4\dots(2n-2)}{1.2.3\dots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\dots(2n-2)}{2.3.4\dots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de $m$ lettres, prises $p$ à $p$; et si l'on change $n$ en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1}C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{array}{l}\frac{1}{n+1}C_{2n,n} = \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1} \\ \quad + \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \dots + \frac{1}{n}C_{2n-2,n-1}.\end{array}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

1, 2, 6, 20, 70, 252, 924...;

qu'on les divise respectivement par

1, 2, 3, 4, 5, 6, 7...;

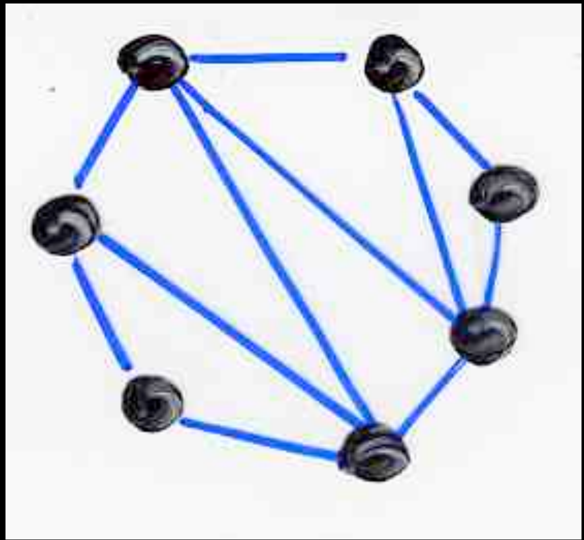
e nombres,

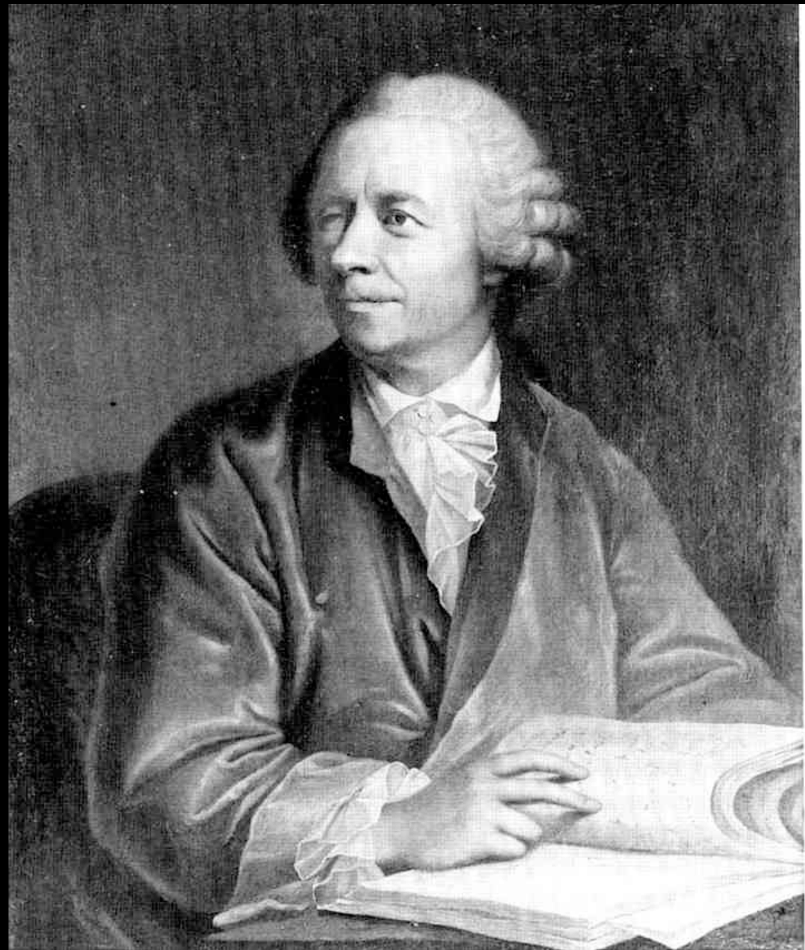
14, 42, 132..., (A)

té :

*ite* (A) *est égal à la somme des*
*ivant au-dessous d'elle-même, et*
*s termes précédents, et en multi-*
*les deux séries.*

5 + 5.2 + 14.1 + 42.1.





$$2(2n+1)C_n = (n+2)C_{n+1}$$

$$\frac{1}{n+1}\binom{2n}{n}$$

ordinary generating functions

formal power series


Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynomial

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots$$

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \dots + C_n t^n + \dots$$

Wilf

corde à linge

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots\ldots$$
$$\ldots + a_n t^n + \ldots\ldots$$

generating function

# formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$

# a little exercise

$$\frac{1}{1-(t+t^2)} = \; ?$$

$$\frac{1}{1-(t+t^2)} = ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4 + 8t^5 + 13t^6 + 21t^7 + 34t^8 + 55t^9 + \dots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \ldots$$

$$+ (t^5 - - - -$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \dots$$
$$+ (t^5 \text{ — — —}$$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$$f(t) = \sum_{n \geq 0} a_n t^n$$

$t + t + t + \ldots t + \ldots$

$1 + 1 + 1 + \ldots$

$$t + t + t + \ldots\ t + \ldots$$

$$1 + 1 + 1 + \ldots$$

formal power series algebra

formalisation

# Formal power series

## Formal power series algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \ldots]$

$\mathbb{K}[t]$ polynomials algebra

$\deg(P)$ degree

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable t and coefficients in $\mathbb{K}$)

$$a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$$a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n + \cdots$$

generating power series

of the coefficients (numbers) $a_n$

$$\sum_{n \geqslant 0} a_n t^n = f(t)$$

s.g. ordinaire

exponential $\sum a_n \frac{t^n}{n!}$

convergence

- formal
- real (complex)

ultrametric topology

# algebra

- sum $\qquad f + g = h, \quad a_n + b_n = c_n$
- product $\qquad fg = h, \quad c_n = \sum_{\substack{p+q=n \\ p,q \geq 0}} a_p b_q$
- product (by a scalar) $\qquad \lambda f = h, \quad c_n = \lambda a_n$

$$f = \sum_{n \geq 0} a_n t^n, \quad g = \sum_{n \geq 0} b_n t^n, \quad h = \sum_{n \geq 0} c_n t^n$$

summable family

$$\sum_{i \in I} f_i(t)$$

infinite product

$$\prod_{i \in I} (1 + g_i(t))$$

# other operations

- substitution

$$f(t) = \sum_{n \geq 0} a_n t^n \; , \quad g(t) = \sum_{n \geq 0} b_n t^n$$

$b_0 = 0$

$$f \circ g(t) \; ; \; f(g(t)) = \sum_{n \geq 0} a_n (g(t))^n$$

- Inverse

$$\frac{1}{1-f} = 1 + f + f^2 + \ldots + f^n + \ldots$$

(si $\mathrm{ord}(f) \geq 1$ )

- derivative

$$f' \qquad \frac{df}{dt} = \sum_{n \geq 1} n\, a_n t^{n-1}$$

exponential

logarithm

binomial power series

$$\exp(t) = \sum_{n \geq 0} \frac{t^n}{n!}$$

$$\log(1-t)^{-1} = \sum_{n \geq 1} \frac{t^n}{n}$$

$$(1+t)^{\alpha} = \sum_{n \geq 0} \binom{\alpha}{n} t^n$$

$$= \sum_{n \geq 0} \alpha(\alpha-1)\cdots(\alpha-n+1)\frac{t^n}{n!}$$

$\mathrm{ord}(f) \geq 1$ $\quad \exp(f) \quad \log(1+f) \quad (1+f)^{\alpha}$

formal power series
in several variables

$$f(t_1, t_2, \ldots, t_p) = \sum_{n_1,\ldots,n_p} a_{n_1,\ldots,n_p} t_1^{n_1} t_2^{n_2} \cdots t_p^{n_p}$$

$\mathbb{K}[t_1, \ldots, t_p]$

$\mathbb{K}[[t_1, \ldots, t_p]]$

algebras

operations

$\partial / \partial t_i$

rational power series $\qquad \sum_{n\geq 0} a_n t^n = \frac{N(t)}{D(t)}$

$\cap\!|$

algebraic power series $\qquad P(y,t) = 0$

$\cap\!|$

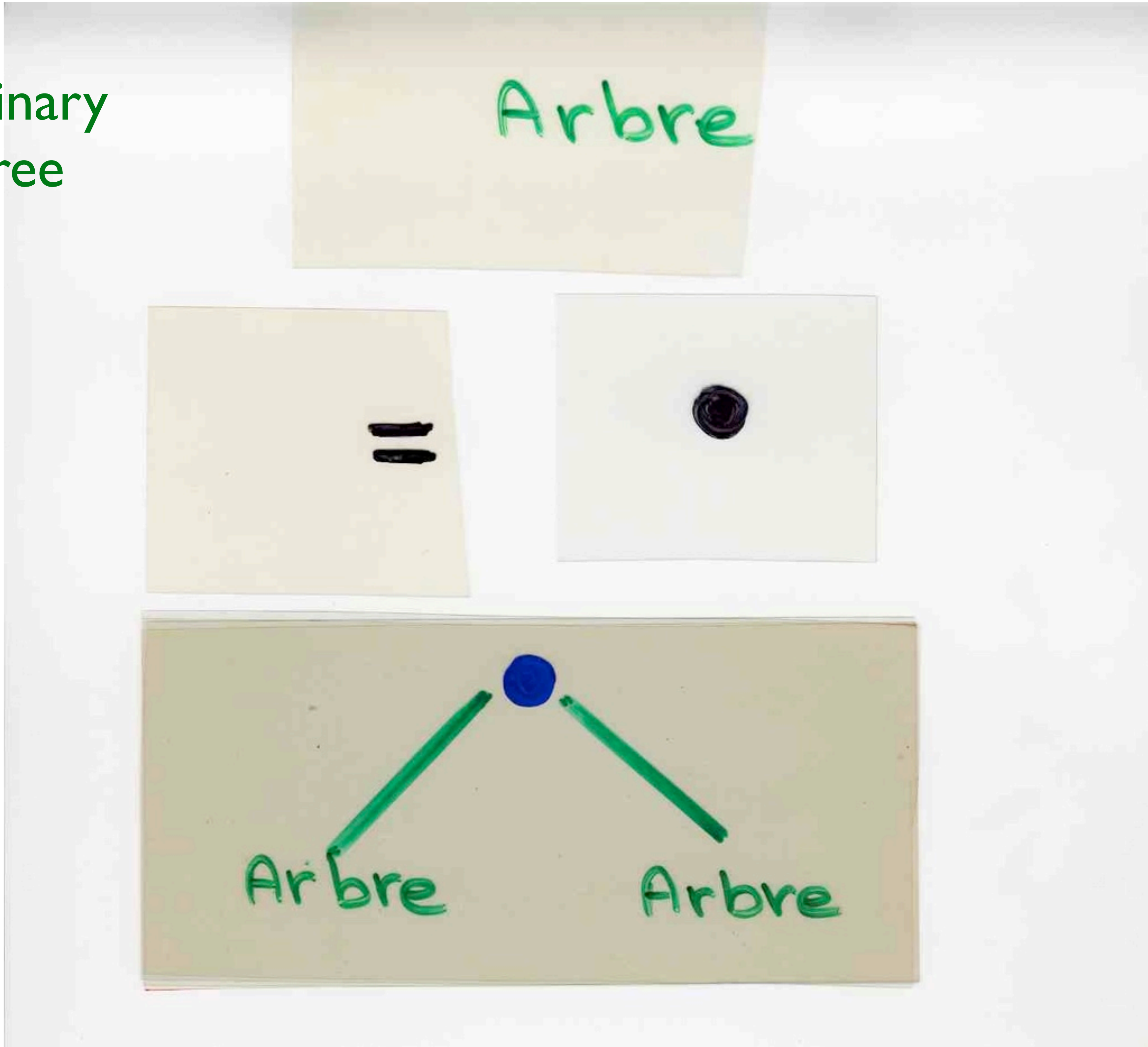
P-recursive (D-finite) power series

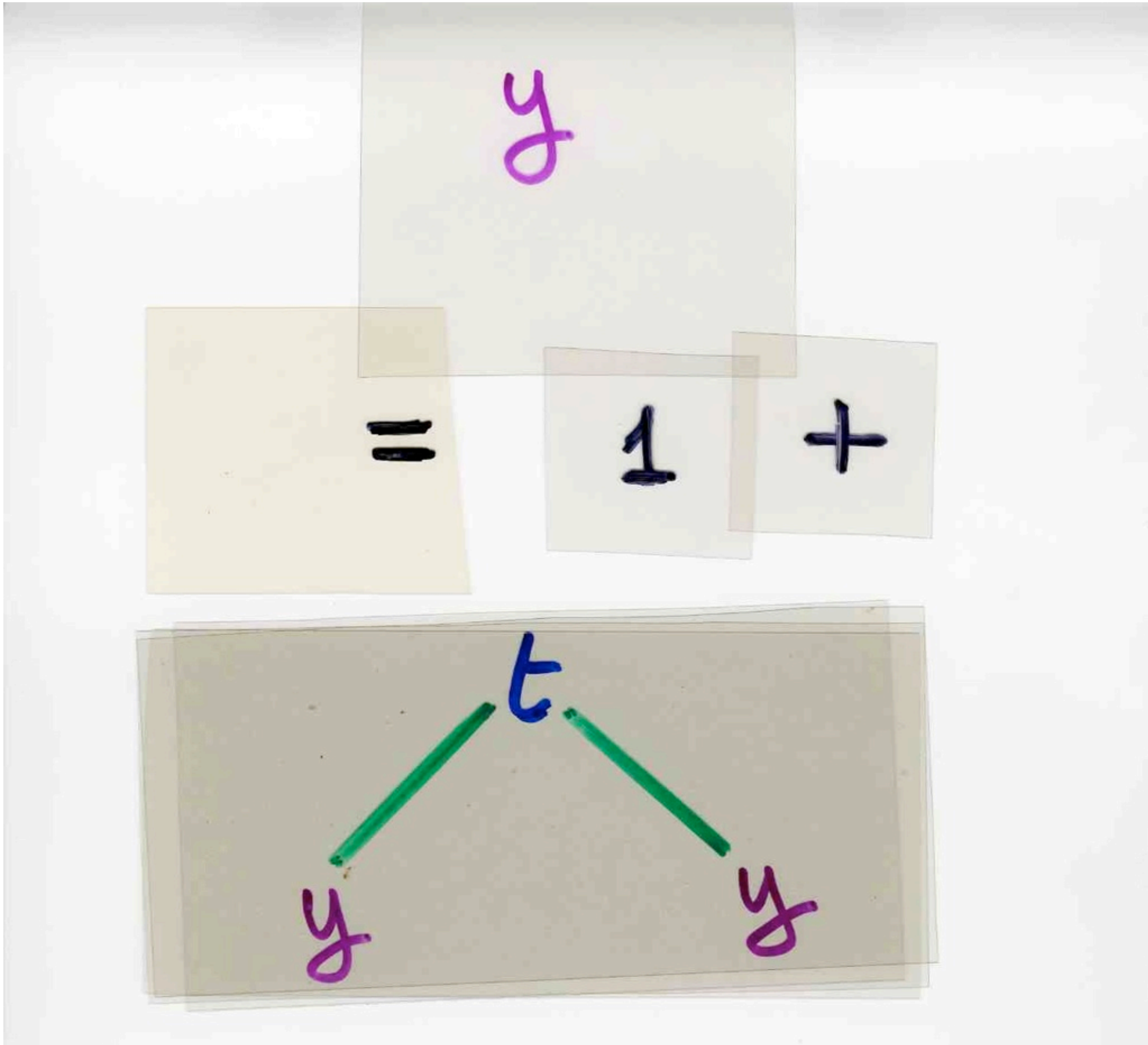
$$P_k(n)a_{n+k} + P_{k-1}(n)a_{n+k-1} + \cdots + P_0(n)a_n = 0$$

operations on combinatorial objects

example: binary trees

# Binary Tree




y
=
1
+
t
y
y

$$y = 1 + t(y)^2$$

# algebraic equation

$$y = 1 + t\,y^2$$

equation algébrique

$$y = \frac{1 - (1-4t)^{1/2}}{2t}$$

$$(1+u)^m =$$

$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^2 + \frac{m(m-1)(m-2)}{3!}u^3 +$$

$$+ \dots$$

$$m = \frac{1}{2}$$

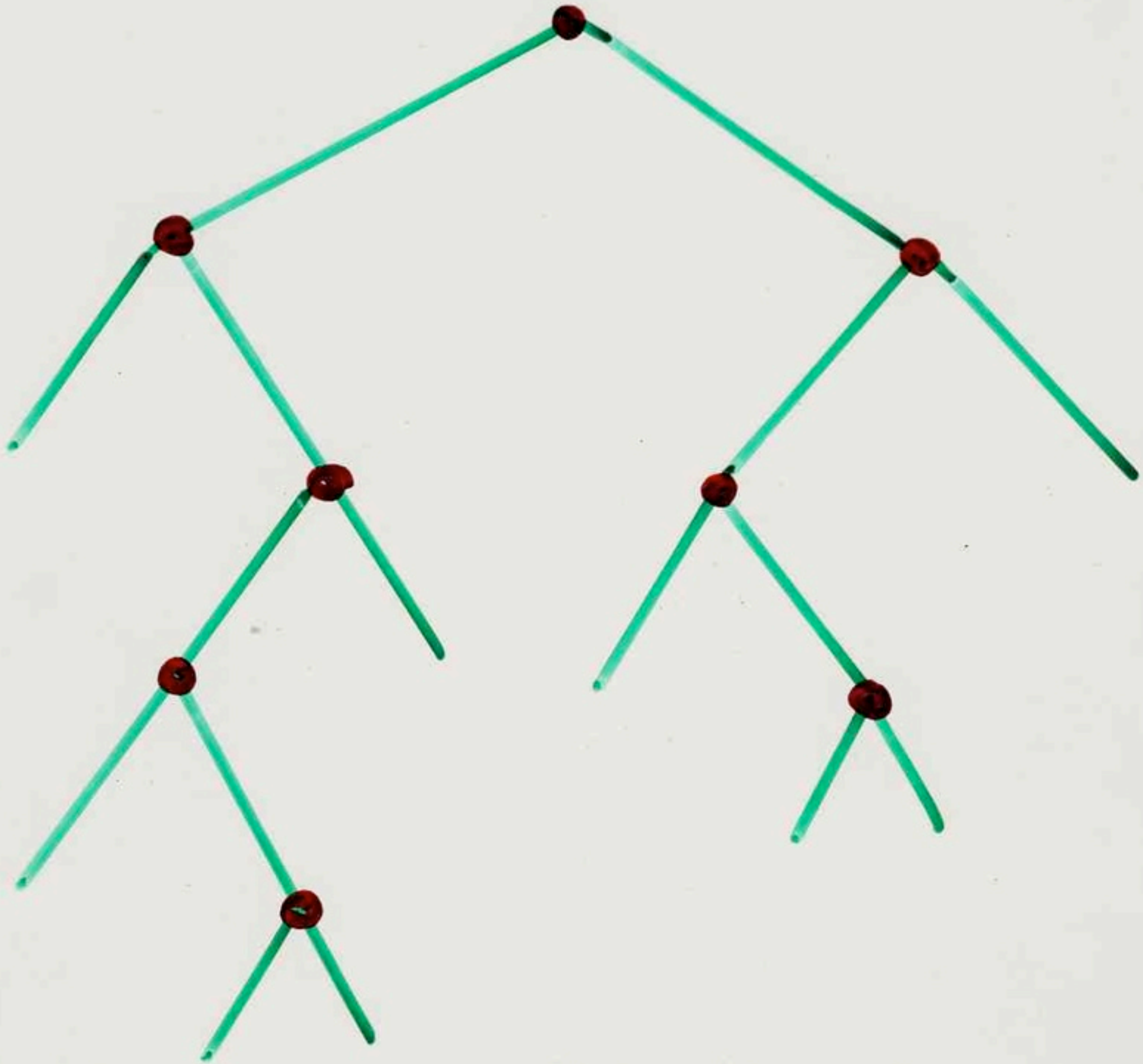
$$u = -4t$$

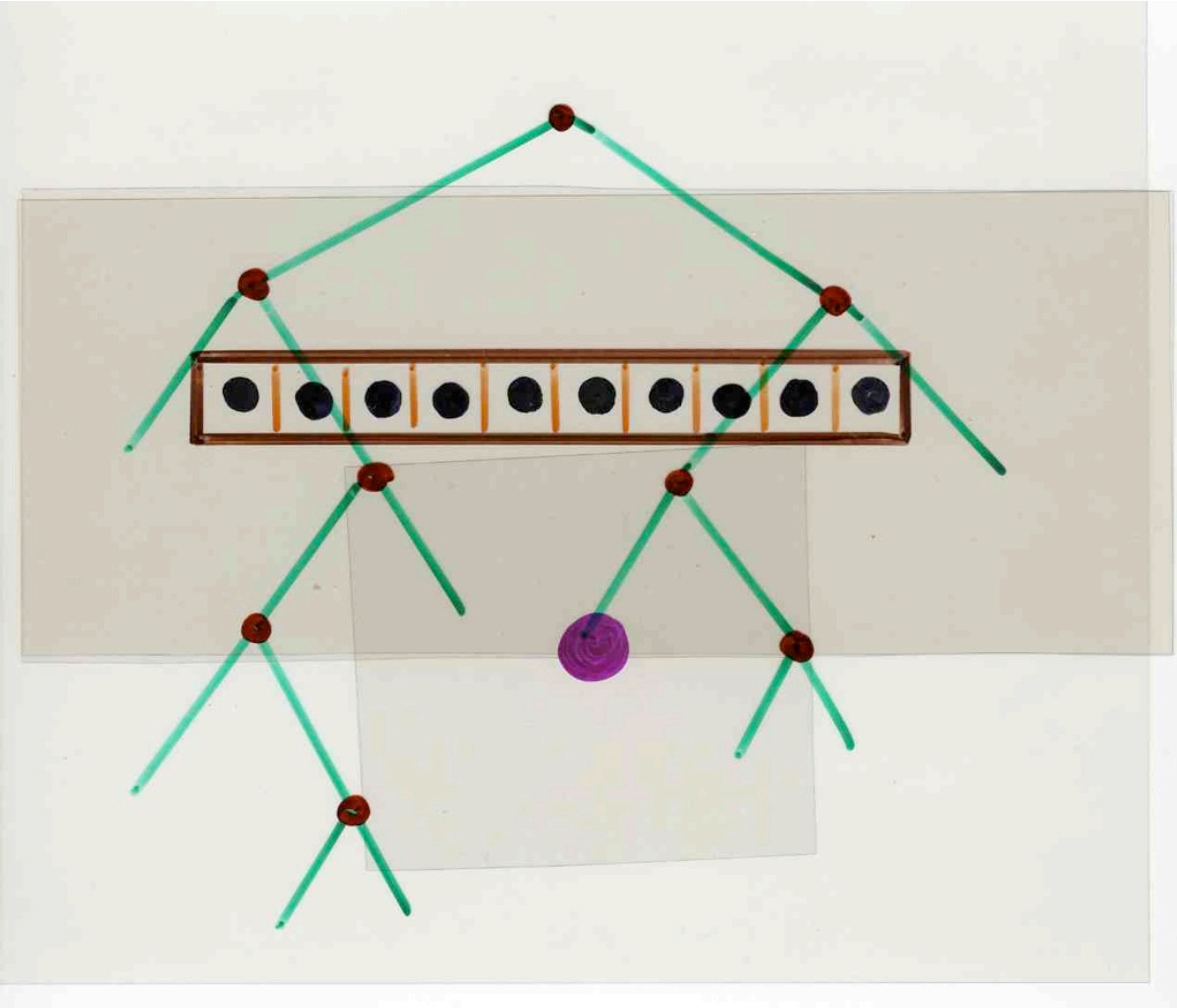
bijective combinatorics

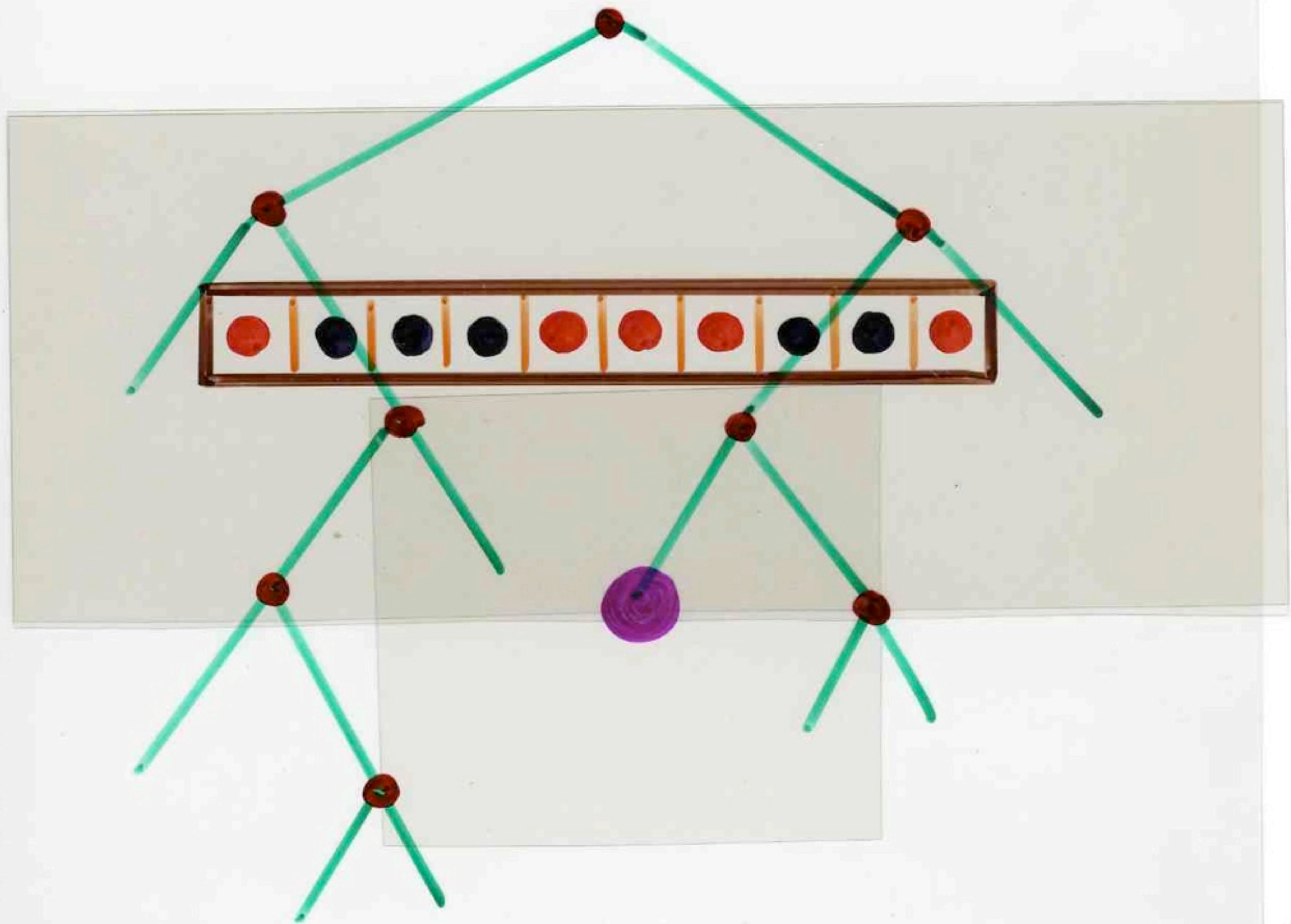
example: Catalan numbers

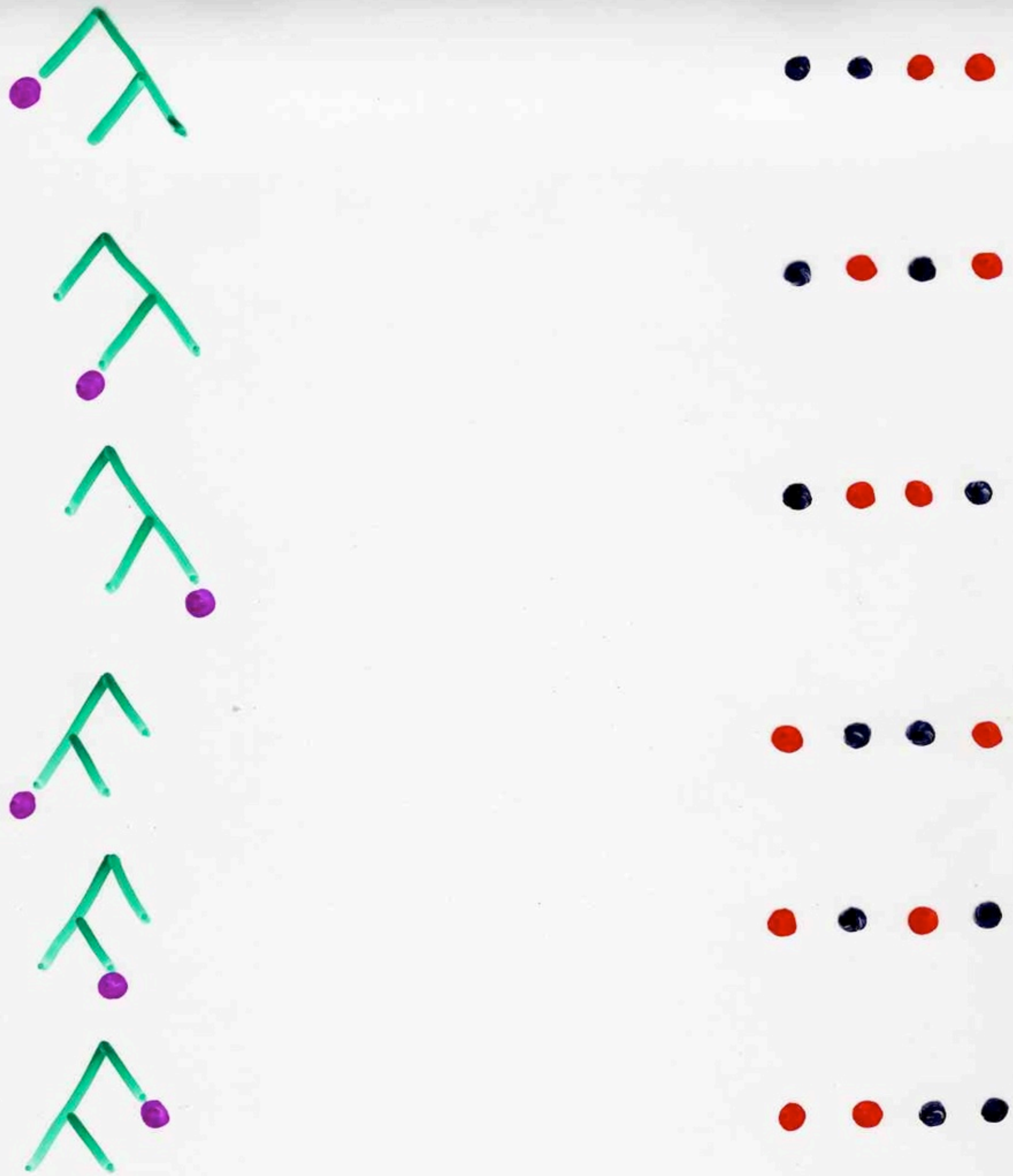
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

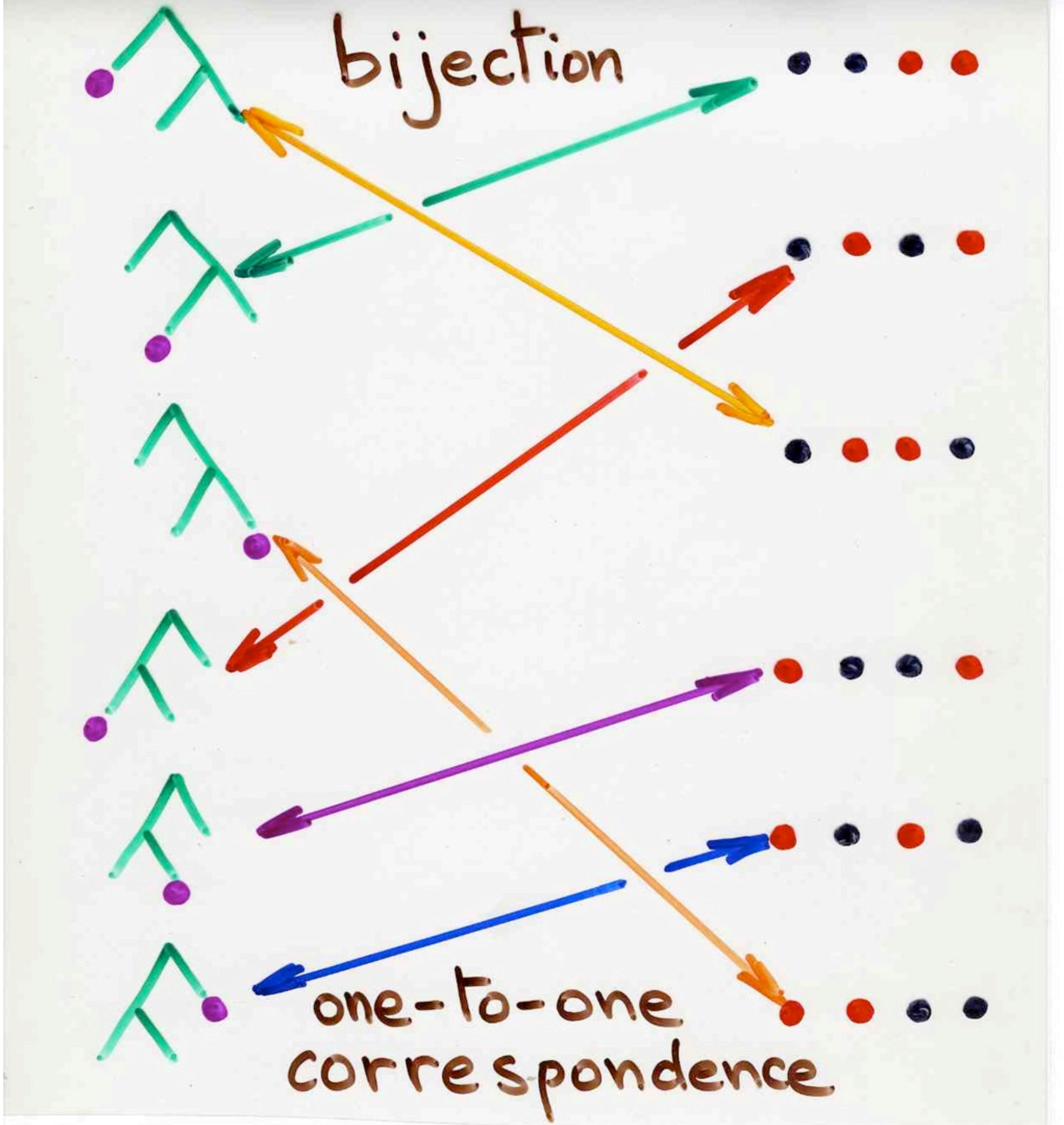
$$(n+1)\,C_n = \binom{2n}{n}$$



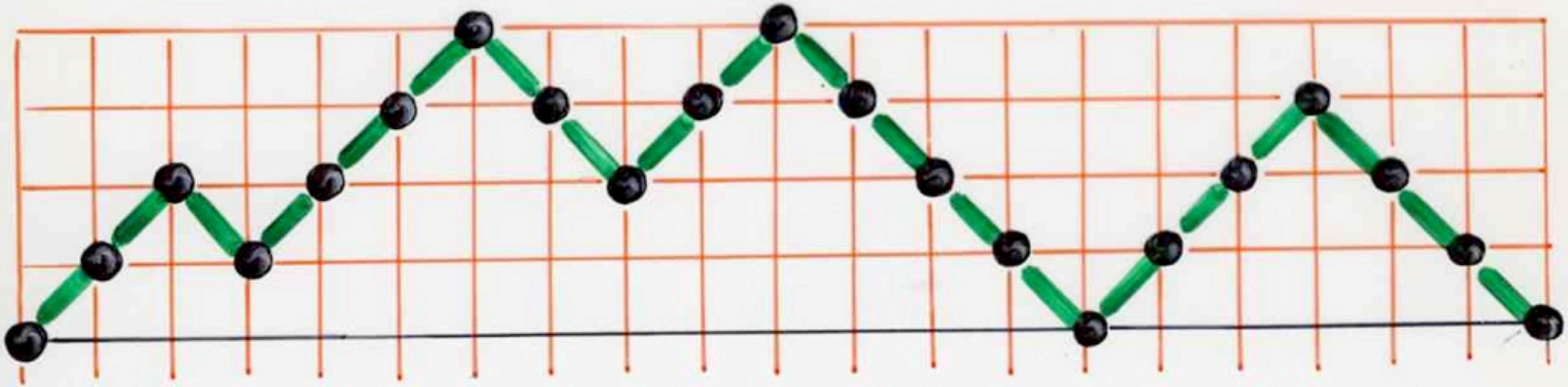







bijection
one-to-one
correspondence

# Dyck paths



binary trees

generating power series

power series algebra

operations on combinatorial objects

bijective combinatorics

Dyck paths