Chapter 2b

Exponential generating functions

Permutations

4 january 2011

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año

$H = \exp F$

$\{A_1, \ldots, A_k\}$ partition of $\{1, 2, \ldots, n\}$

$\{\alpha_1, \ldots, \alpha_k\}$ α_i F-structure on A_i

log

$h(t) = \exp(f(t))$

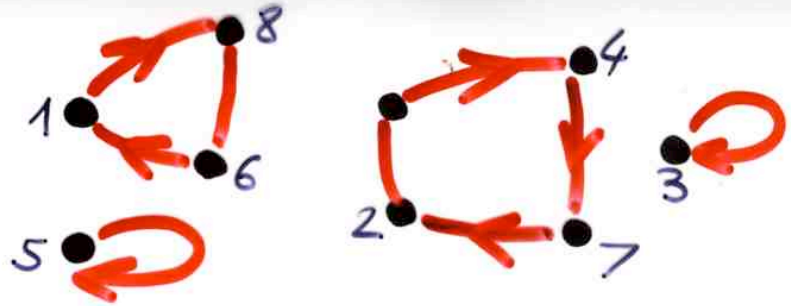
$H = \exp F$

$\{A_1, \dots, A_k\}$ partition of $\{1, 2, \dots, n\}$

$\{\alpha_1, \dots, \alpha_k\}$ α_i F-structure on A_i

log $\qquad h(t) = \exp(f(t))$

Permutation



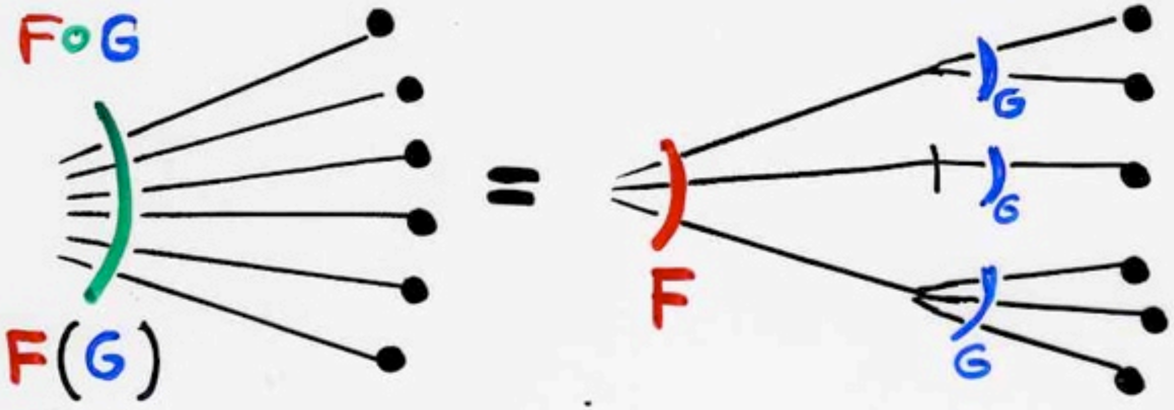
cyclic permutation

$$\sum_{n \geq 0} \frac{n! t^n}{n!} = \frac{1}{1-t} \qquad \sum_{n \geq 1} \frac{(n-1)! t^n}{n!} = \sum_{n \geq 1} \frac{t^n}{n}$$

permutation = exp (cycle)

Def. F G $G[\emptyset] = \emptyset$

Substitution of G into F



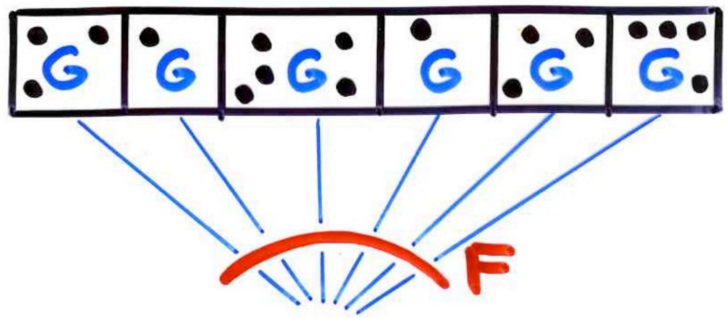
$\gamma \in F(G)[U]$

γ
- partition $\{U_1, \dots, U_k\}$ de U, classes $\neq \emptyset$
- $\beta_i \in G[U_i]$, $i = 1, \dots, k$
- $\alpha \in F[U/\equiv]$

F- assemblée of G-structures

ex- permutation = assemblée of cycles

$H =$



$$H = F(G)$$

Prop. $(F \circ G)(t) = F(G(t))$

$$c_n = \sum_{\substack{k=0 \\ n_1+\dots+n_k=n \\ n_1,\dots,n_k \geq 1}}^{n} \frac{n!}{k!\, n_1! \dots n_k!} a_k b_{n_1} \dots b_{n_k}$$

Cor $F = E$

$$(E \circ G)(t) = \exp(G(t))$$

assemblée of G-structures $\quad E^G$

ex. • Permutations $\quad S = E \circ C$

$$\log(1-t)^{-1} = \sum_{n\geq 1} \frac{t^n}{n}$$

(set of cycles)

• Partitions $\quad B = E \circ E^*$

(set of non-empty blocks)

$$B(t) = \exp(e^t - 1)$$

• Graphs $\quad G = E \circ GC$

(set of connected graphs)

Stirling numbers
and
binomial type polynomials

ex.-

Stirling numbers

$s_{n,k}$

$$s_n(x) = \sum_{1 \le k \le n} s_{n,k}\, x^k$$

cycles

$$\sum_{n \geq 0} s_n(x) \frac{t^n}{n!} = \exp\left(x \log(1-t)^{-1}\right)$$

$$\sum_{n \geq 0} s_n(x) \frac{t^n}{n!} = (1-t)^{-x}$$

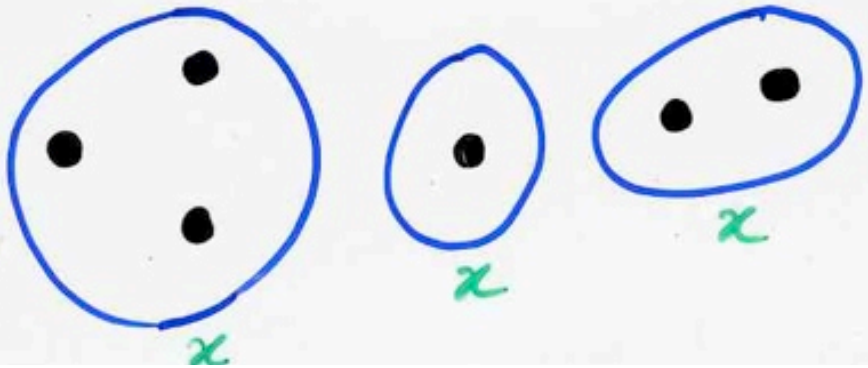
$$s_n(x) = x(x+1)\dots(x+n-1)$$

ex. Stirling numbers $S_{n,k}$

$$S_n(x) = \sum_{1\le k\le n} S_{n,k}\, x^k$$

partitions

$$\sum_{n\ge 0} S_n(x)\,\frac{t^n}{n!} = \exp\big(x\,(e^t-1)\big)$$



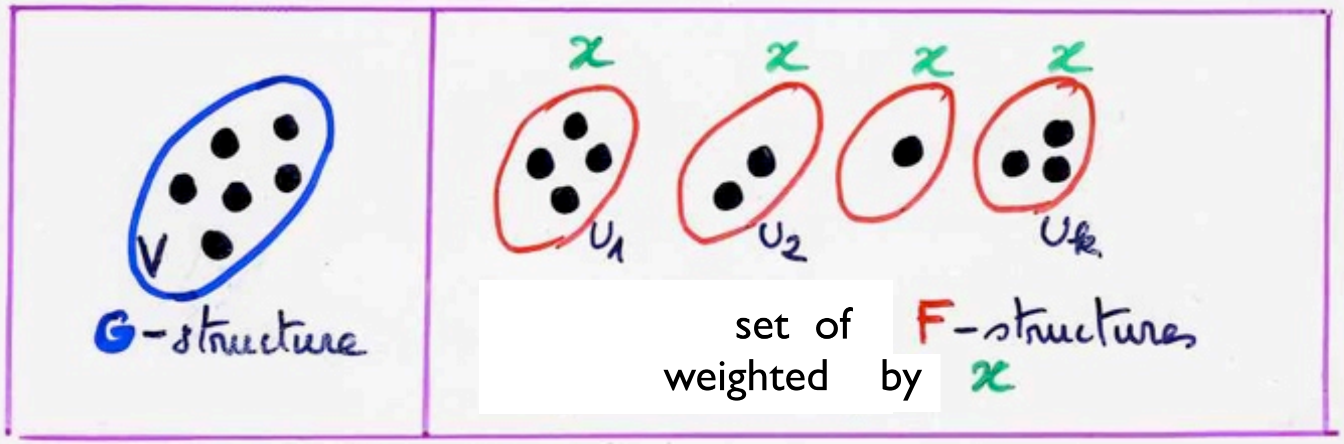
$$B = E \circ E^*$$

combinatorial interpretation

F $\quad$ G $\quad$ $H = G \cdot (E \circ F)$

$f(t)$ $\quad$ $g(t)$

$$P_n(x) = \sum_{0 \le k \le n} a_{n,k}\, x^k$$

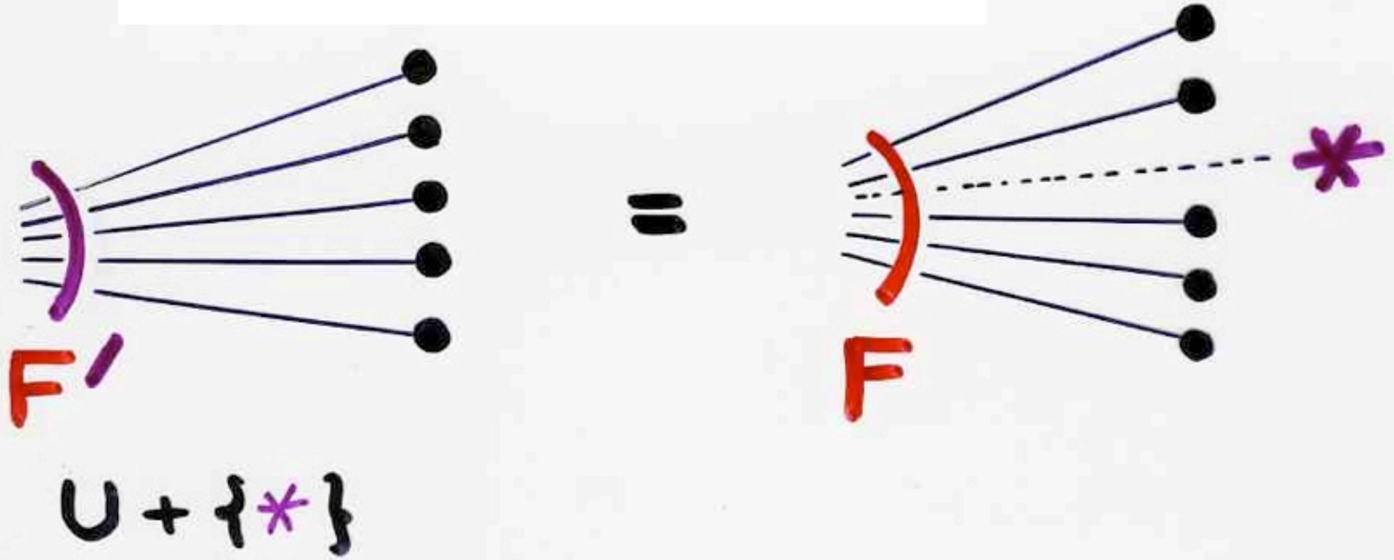


H_V - structure

F_{V_1} $\quad$ G_{V_2} $\quad$ $\mathbb{K}$

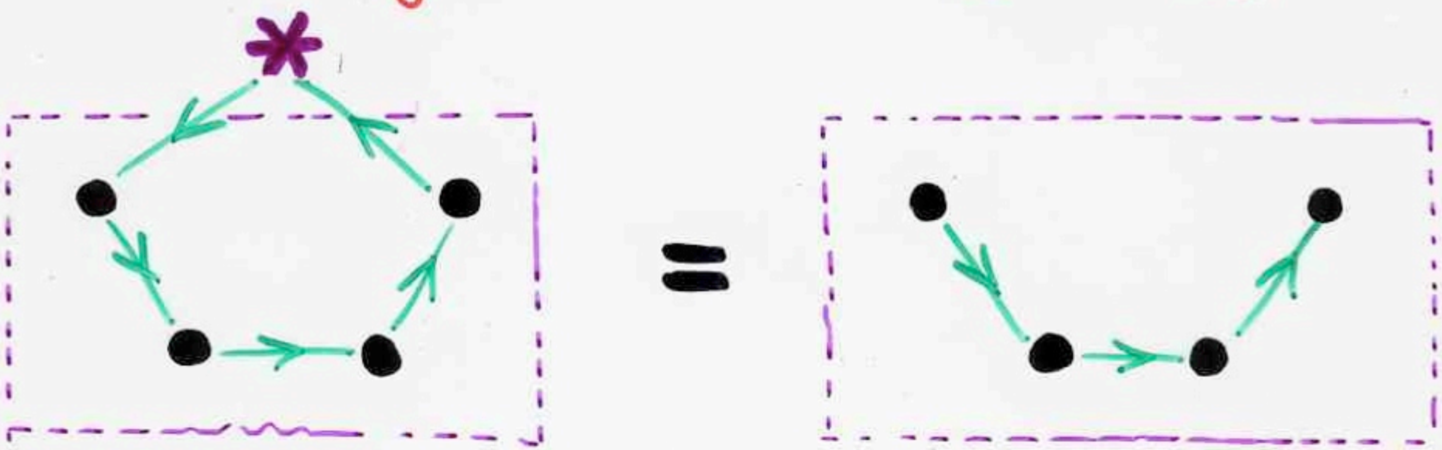
Differential equations

Def. derivative of species F'



Prop. $(F')(t) = \frac{d}{dt} F(t)$

ex – C cycles $C' = L$

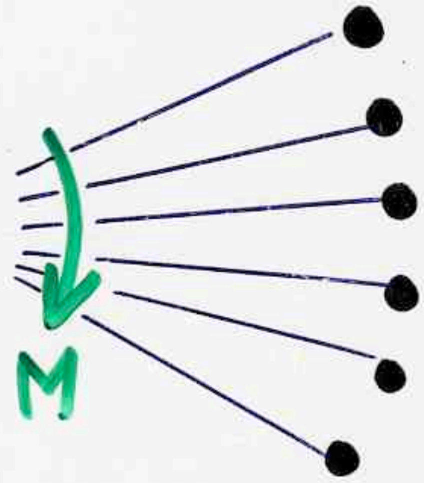


$$C(t) = \int_0^t L(u)\,du$$

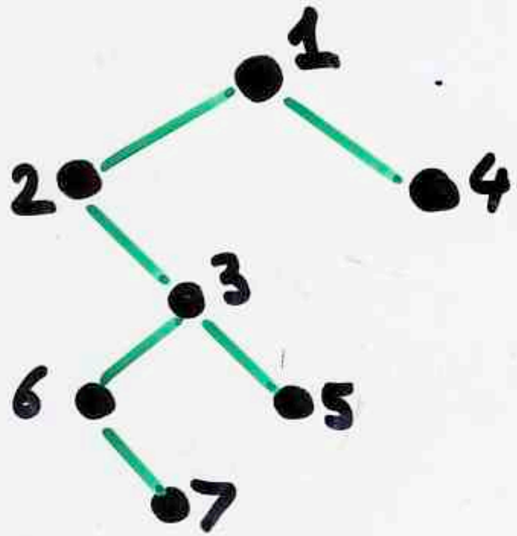
$$\log(1-t)^{-1} = \int_0^t \frac{du}{1-u}$$

(well known!)

B - species
L - species



ex. increasing binary trees



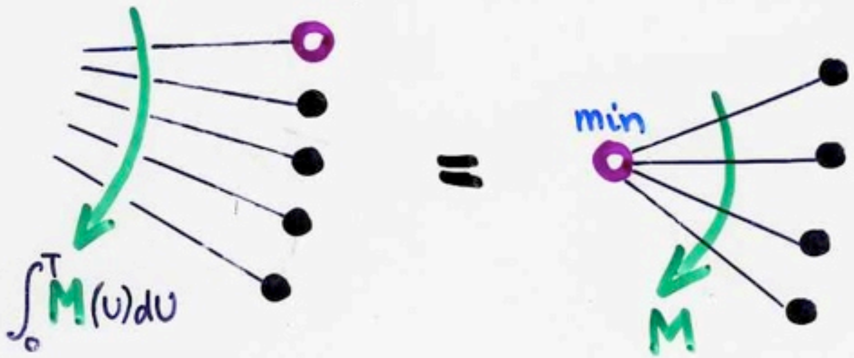
n!

Def. Integral of an $\mathbb{L}$- species M

$$F = \int_0^T M(U)\,dU$$

$$F[\emptyset] = \emptyset$$

$$F[U] = M[U \setminus \min(U)] \qquad U \neq \emptyset$$



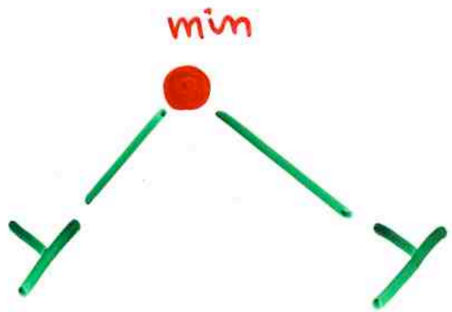
Prop

$$F(t) = \int_0^t M(u)\,du$$

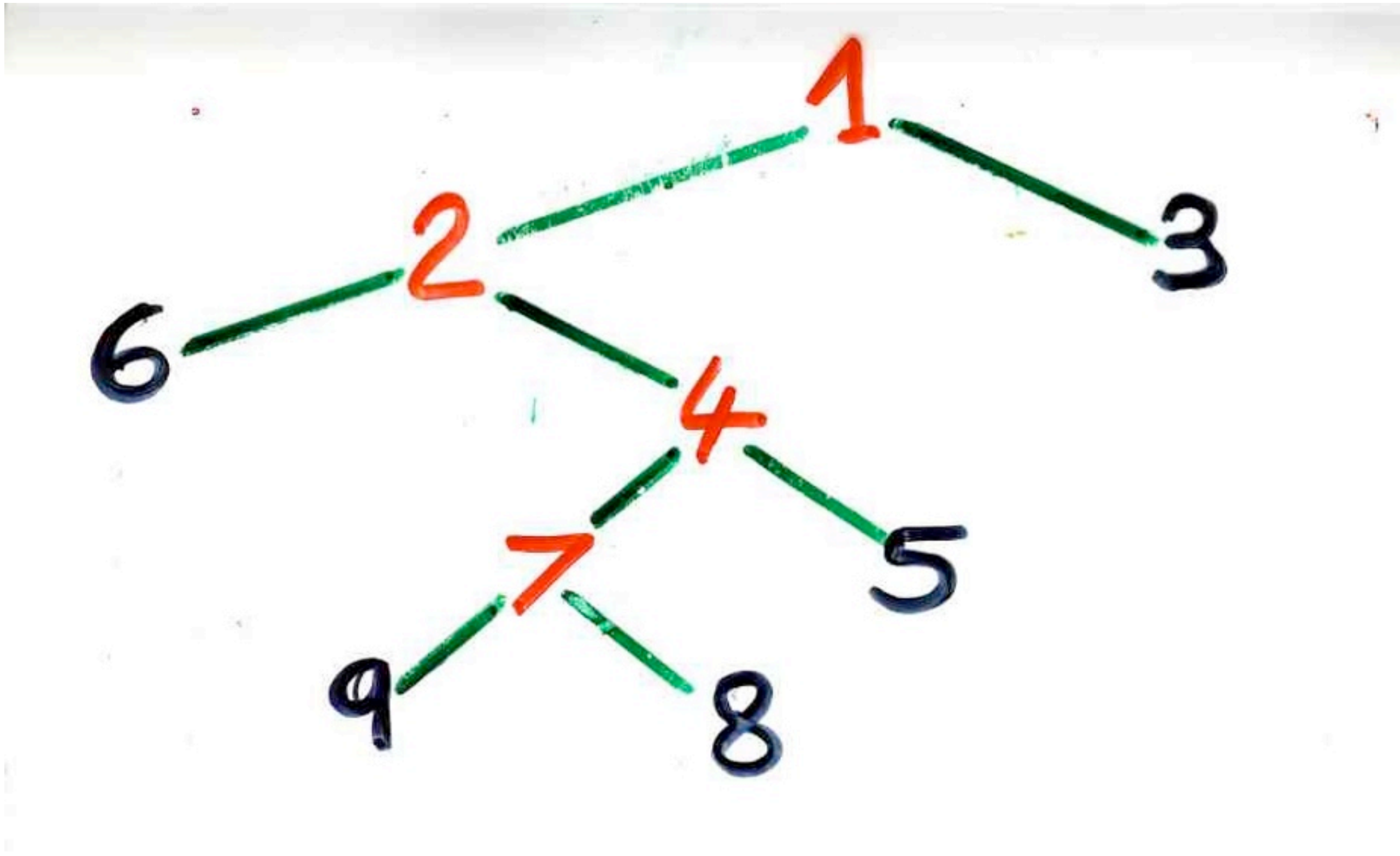
$$y = t + \int_0^t y^2 \, dt$$

$$Y = T + \int_0^T Y^2(T) \, dT$$

$Y = \bullet$ ou



complete increasing binary trees



6 2 9 7 8 4 5 1 3

Leonhard
Euler
(1707-1783)

erit: $\alpha = 1$, $\beta = 1$, $\gamma = 5$, $\delta = 61$, $\varepsilon = 1385$, $\zeta = 50521$, $\eta = 2702765$, $\theta = 199360981$, $\iota = 19391512145$, $\varkappa = 2404879661671$ &c.

ex hisque valoribus obtinebitur:

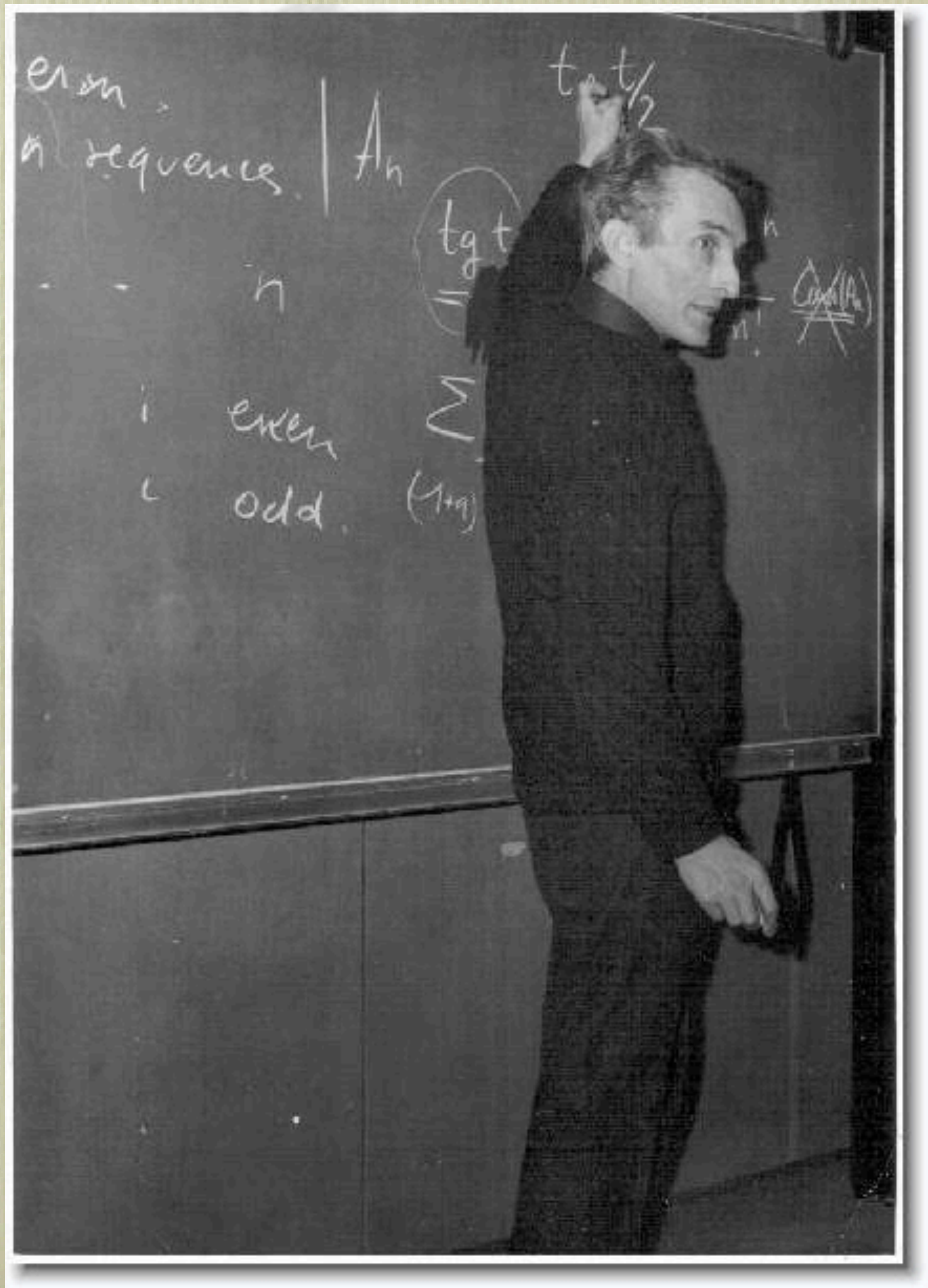
$$\sec x = \alpha + \frac{\beta}{1.2} xx + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2 \ldots 6} x^6 + \frac{\varepsilon}{1.2 \ldots 8} x^8 + \&c.$$

erit hanc feriem ab illa fubtrahendo:

$$\operatorname{tg} x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots.6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots.8} + \&c.$$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1.2} - \frac{2^4\mathfrak{B}x^3}{1.2.3.4} - \frac{2^6\mathfrak{C}x^5}{1.2.3\ldots6} - \frac{2^8\mathfrak{D}x^7}{1.2\ldots.8} - \&c.$$

Si ergo hic introducantur numeri A, B, C, &c. §. 182. inventi; erit: $\operatorname{tang} x = \frac{2Ax}{1.2} + \frac{2^3Bx^3}{1.2.3.4} + \frac{2^5Cx^5}{1.2\ldots6} + \frac{2^7Dx^7}{1.2\ldots8} + \&c.$



our Master

Marcel Paul
Schützenberger

1920 - 1996

André permutations,

non-commutative
differential equations

combinatorial theory
of differential equations
and integral calculus

P. Leroux, X.G.V.

control theory
non-linear

$$y' = y^2 + u(t)$$

differential equations
with forced terms

Weighted species

$\mathbb{K}$ commutative ring

Def. weighted species F_v

$$\alpha \in F[U] \longrightarrow v(\alpha) \in \mathbb{K}$$

weight of valuation F-structure α

$$f : U \to V \qquad \alpha \in F[U] \xrightarrow{F[f]} \beta \in F[V] \qquad v(\alpha) = v(\beta)$$

Def. generating power series F_v

$$F_v(t) = \sum_{n \geq 0} P_n \frac{t^n}{n!}$$

element of $\mathbb{K}[[t]]$

$$P_n = \sum_{\alpha \in F[U]} v(\alpha)$$

$|U| = n$

Operations on weighted species

F_{v_1} $\quad G_{v_2}$

Sum $\quad F_{v_1} + G_{v_2} \qquad (F+G)_v$

Produit $\quad F_{v_1} \cdot G_{v_2} \qquad (F\cdot G)_v$

$$v(\gamma) = v_1(\alpha)\, v_2(\beta) \qquad \gamma = (U_1, U_2, \alpha, \beta) \in F\cdot G[U]$$

Substitution $\quad F_{v_1} \circ G_{v_2} \qquad (F\circ G)_v$

$$\gamma = (\{U_1, \ldots, U_k\} ; \alpha_1, \ldots, \alpha_k ; \beta) \in F\circ G[U]$$

$$v(\gamma) = v_1(\beta)\, v_2(\alpha_1) \ldots\ldots v_2(\alpha_k)$$

Pointed

$F^\bullet_{v_1}$ $\qquad$ $(F^\bullet)_v$

$\gamma = (\alpha, x) \in F^\bullet[U]$ $\qquad$ $v(\gamma) = v_1(\alpha)$

Derivative

F'_{v_1} $\qquad$ $(F')_v$

$\gamma \in F'[U]$ $\qquad$ $\gamma \in F[U + \{*\}]$

$v(\gamma) = v_1(\gamma)$

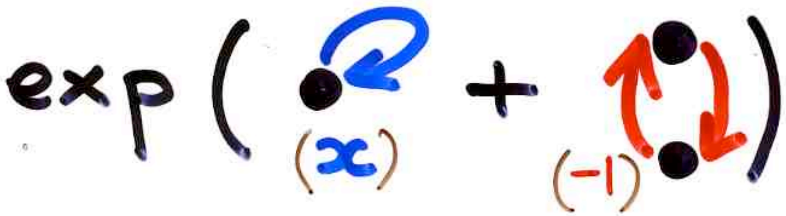
Integral

($\mathbb{L}$- species) $\qquad$ $F_v = \int_0^T M_{v_1}(U)\,dU$

$F = \int_0^T M(U)\,dU$ $\qquad$ $\gamma \in M[U \setminus \min(U)]$

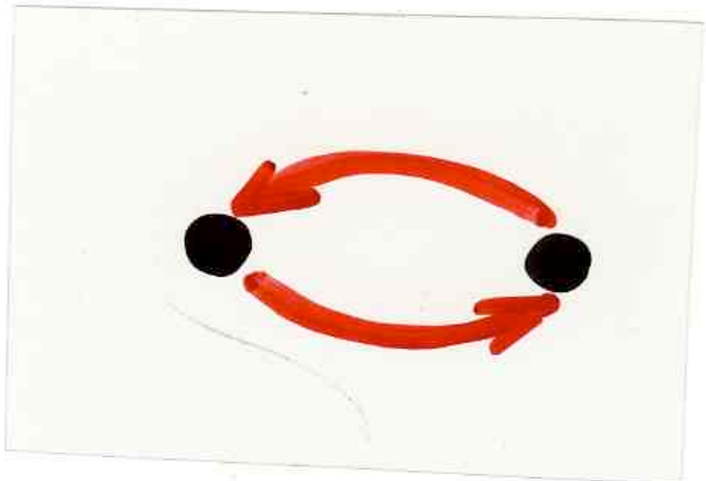
$v(\gamma) = v_1(\gamma)$

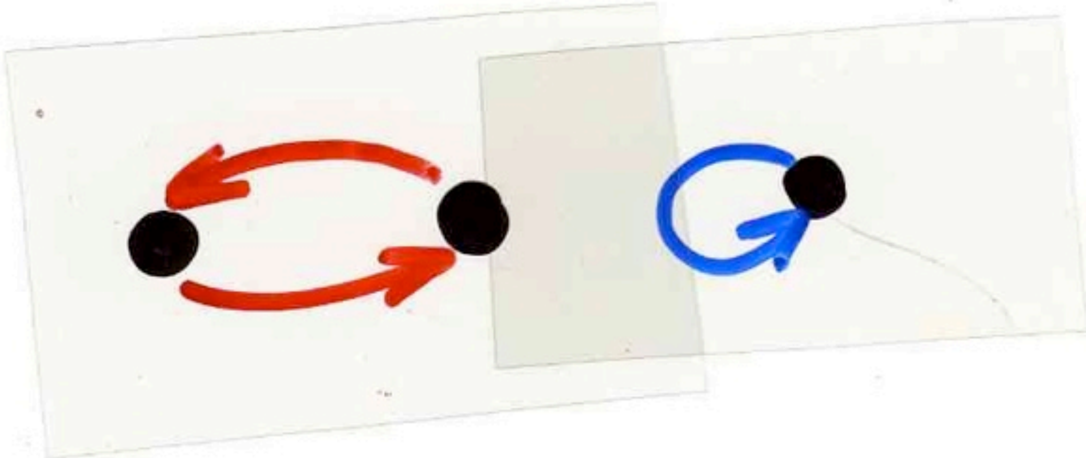
Examples:
orthogonal polynomials

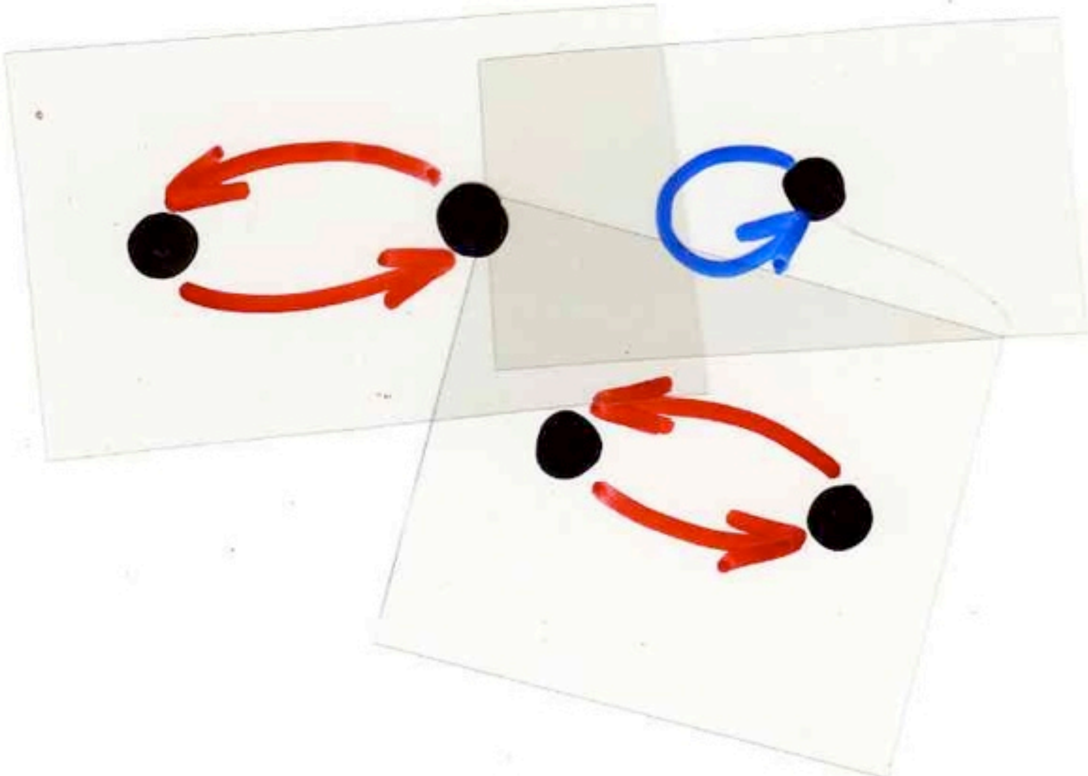


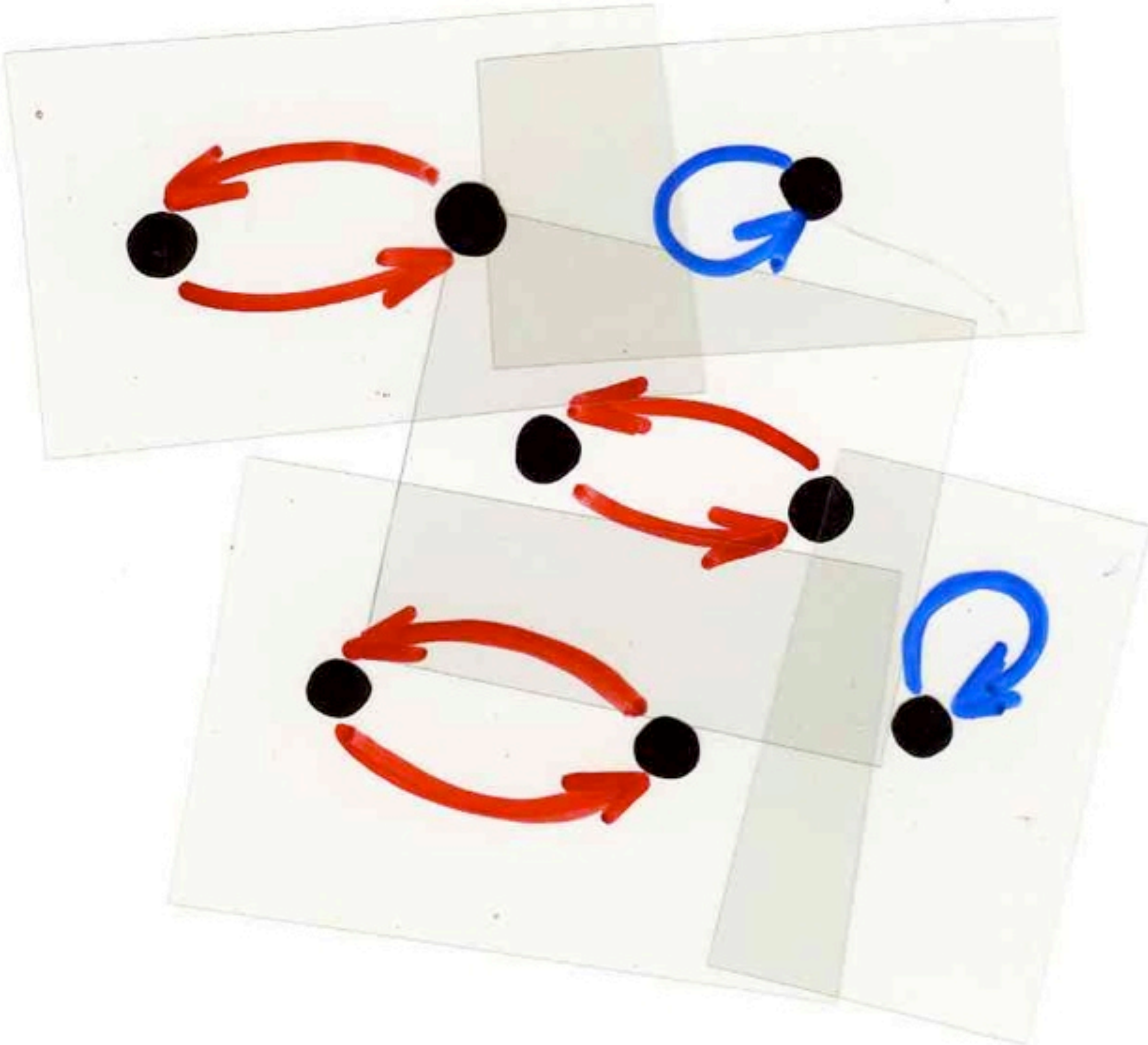
$$\sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = \exp\left(xt - \frac{t^2}{2}\right)$$

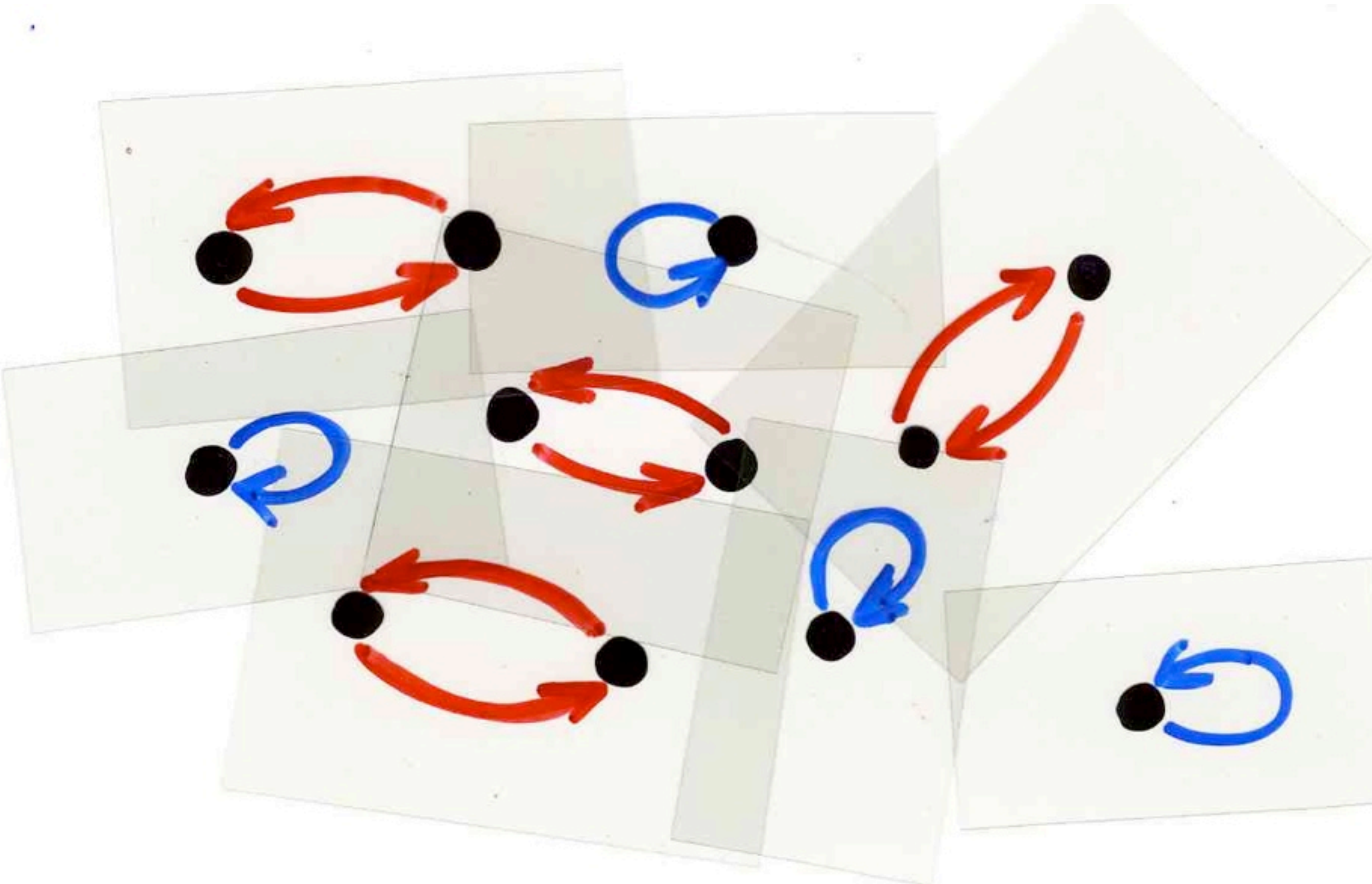
Charles Hermite
1822 - 1901

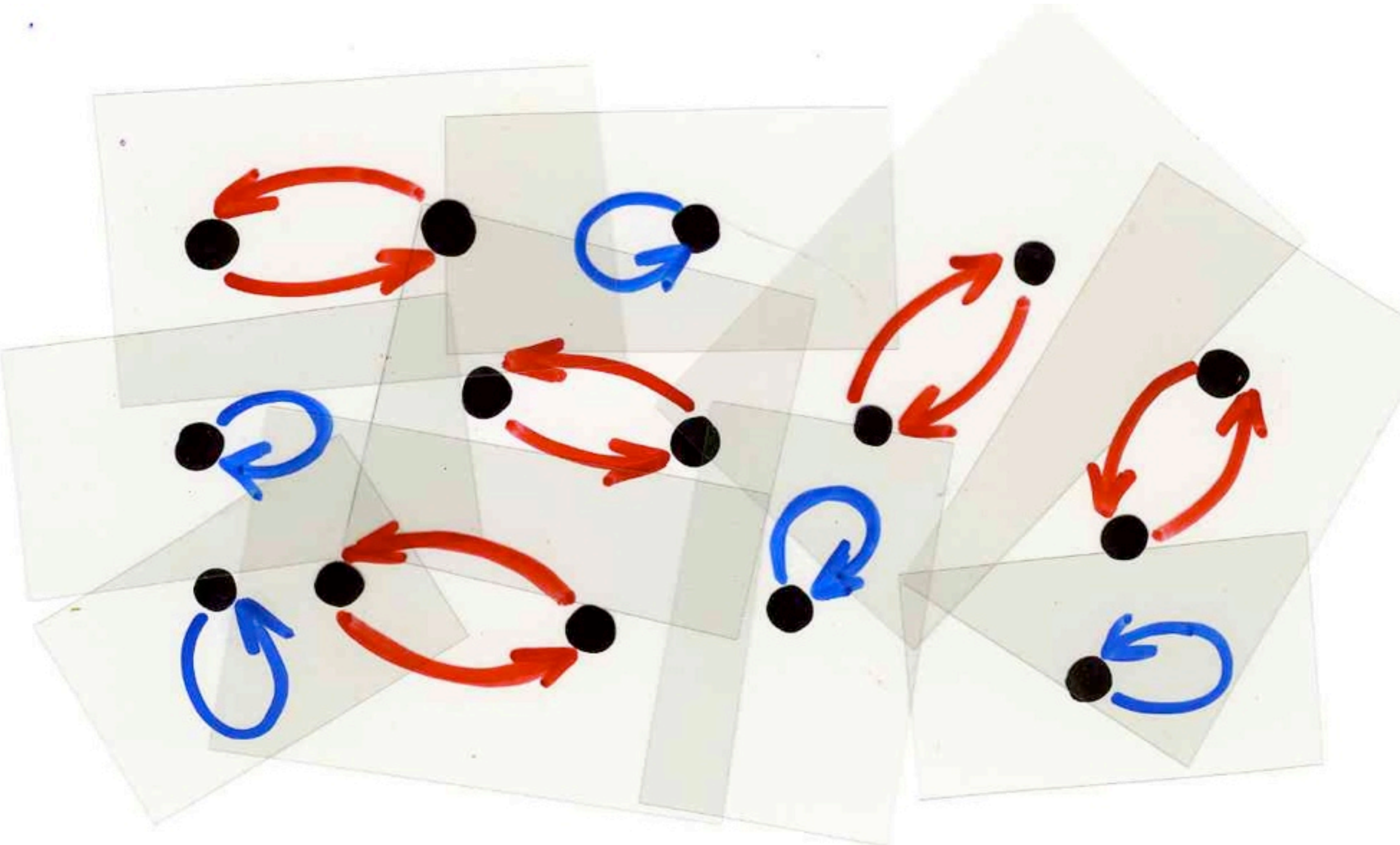




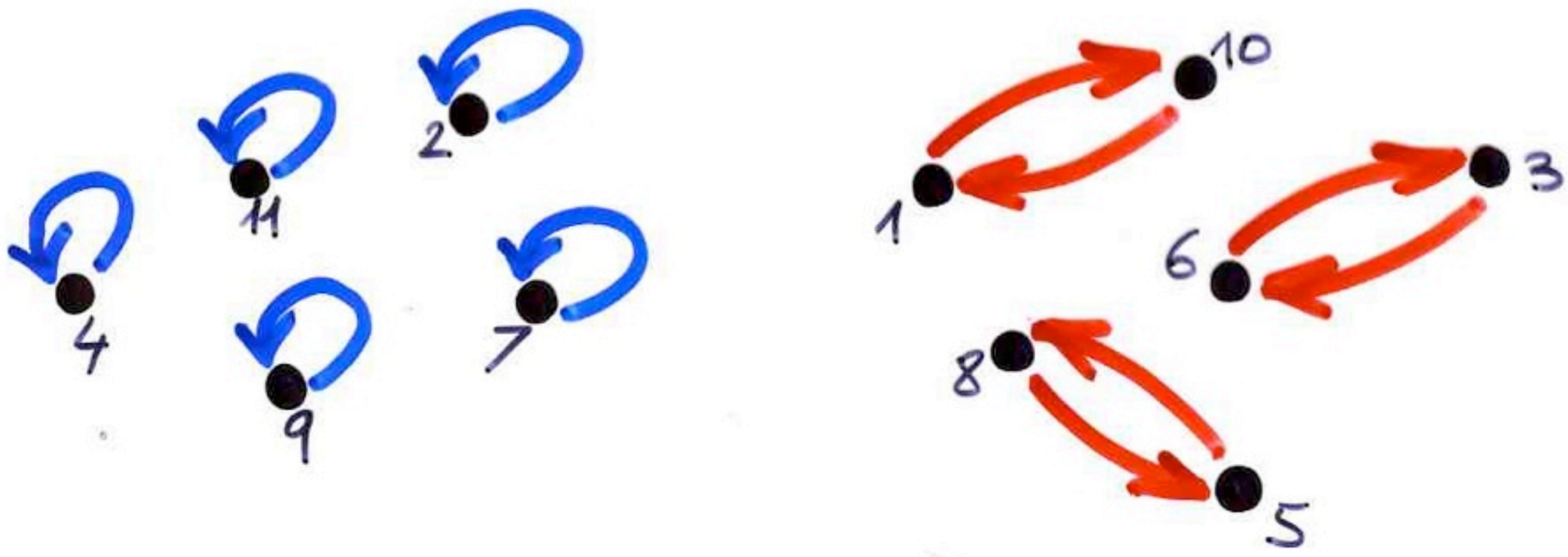




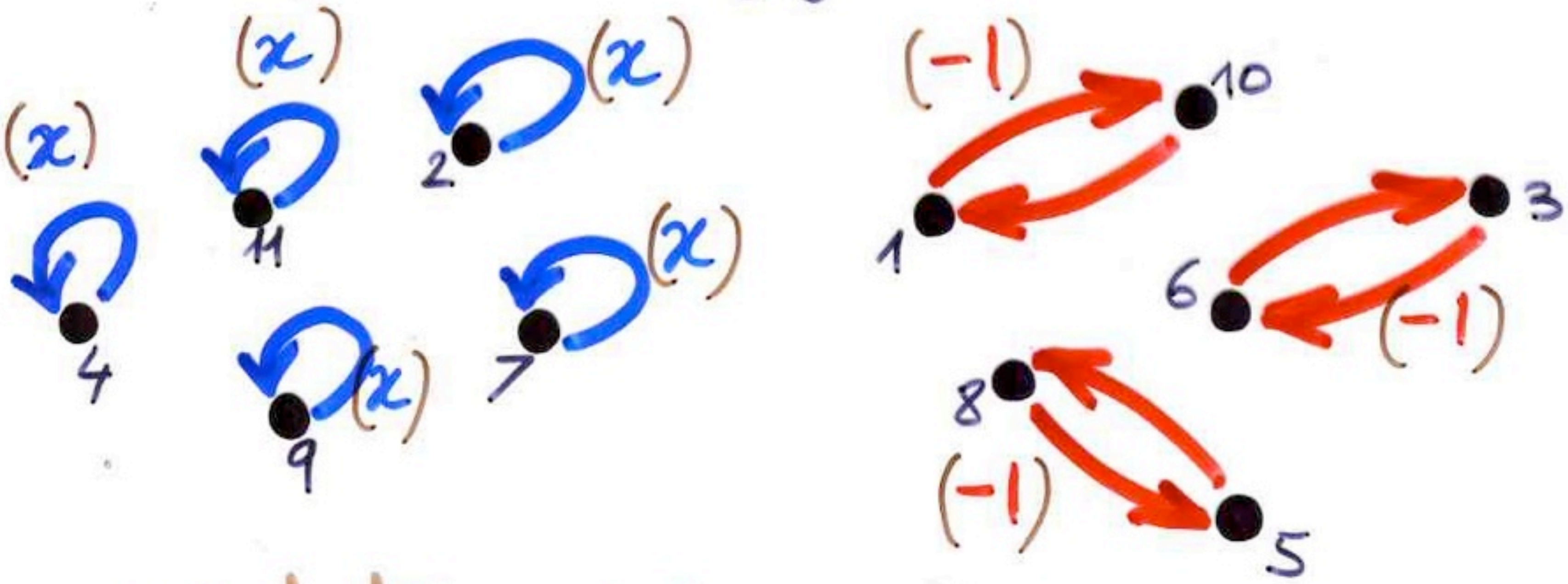




Hermite configurations



Hermite configurations



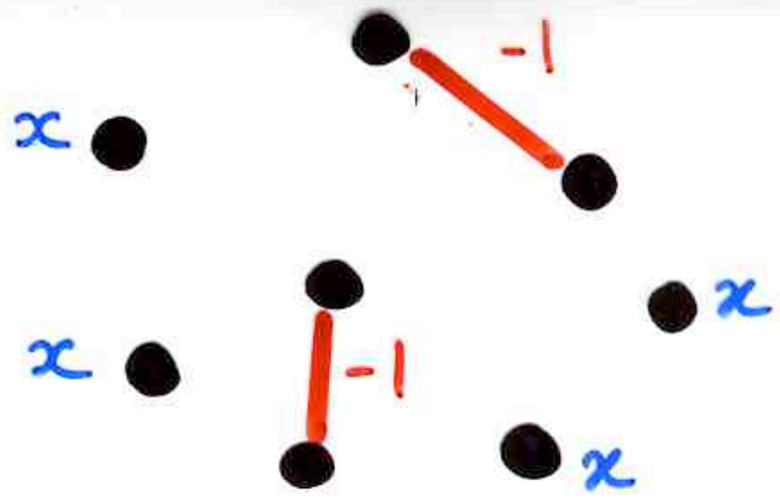
weight (x)

(-1)

ex: Hermite

$$H_n(x) = \sum_{\substack{\text{matching } \gamma \\ \text{of } K_n}} (-1)^{|\gamma|} x^{\mathrm{fix}(\gamma)}$$

$$\mu_{2n+1} = 0 \qquad \mu_{2n} = \underbrace{1 \times 3 \times \ldots \times (2n-1)}_{\text{number of perfect matchings of } K_{2n}}$$



matching

$$H_n(x) = \sum_{0 \leq 2k \leq n} (-1)^k \frac{n!}{2^k\, k!\, (n-2k)!} x^{n-2k}$$

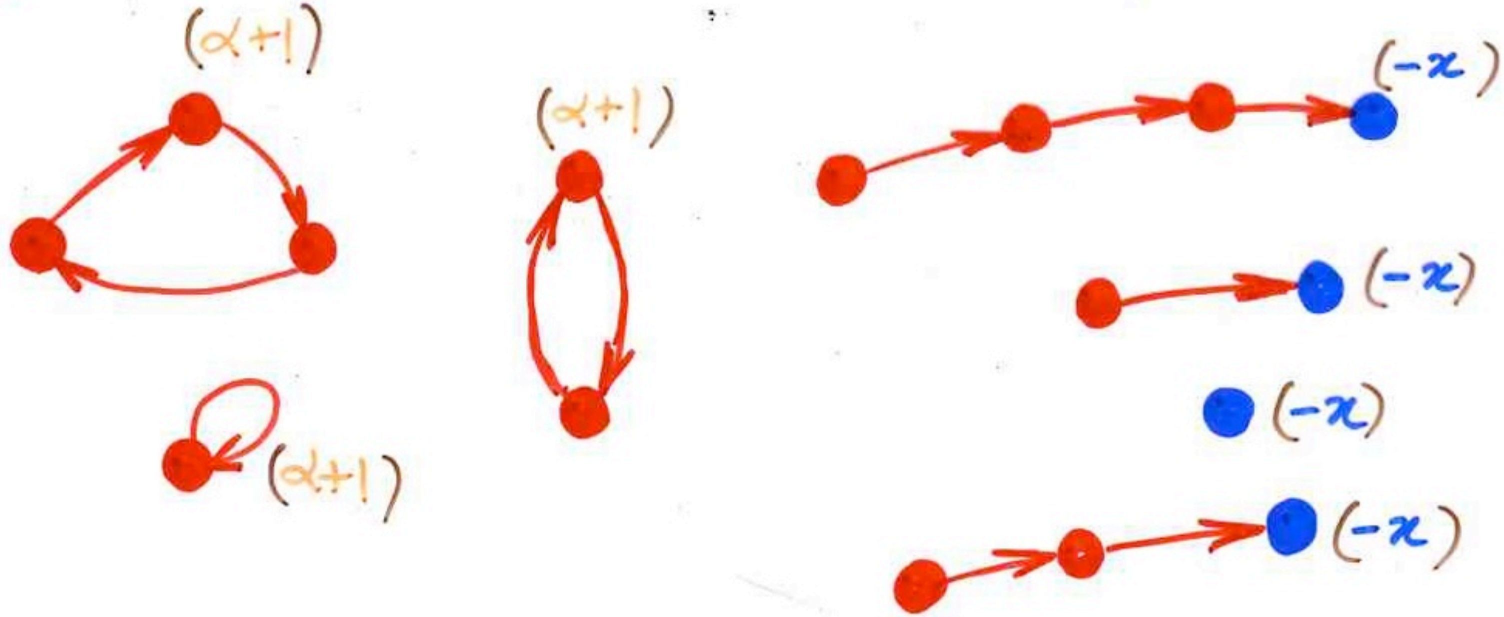
Laguerre

polynomial

Laguerre $L_n^{(\alpha)}(x)$

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}}\,\exp\left(\frac{-xt}{1-t}\right)$$

Laguerre configuration



Def- $\{P_n(x)\}_{n\geq 0}$ $\quad P_n(x) \in \mathbb{K}[x]$

sequence of Sheffer polynomials

$$\sum_{n\geq 0} P_n(x)\frac{t^n}{n!} = g(t)\ \exp(x f(t))$$

$f(t),\ g(t) \in \mathbb{K}[[t]]$, $\quad f(0)=0$, $f'(0)\neq 0$, $g(0)\neq 0$

$\Rightarrow \quad \deg(P_n(x)) = n$

Def- binomial type polynomials

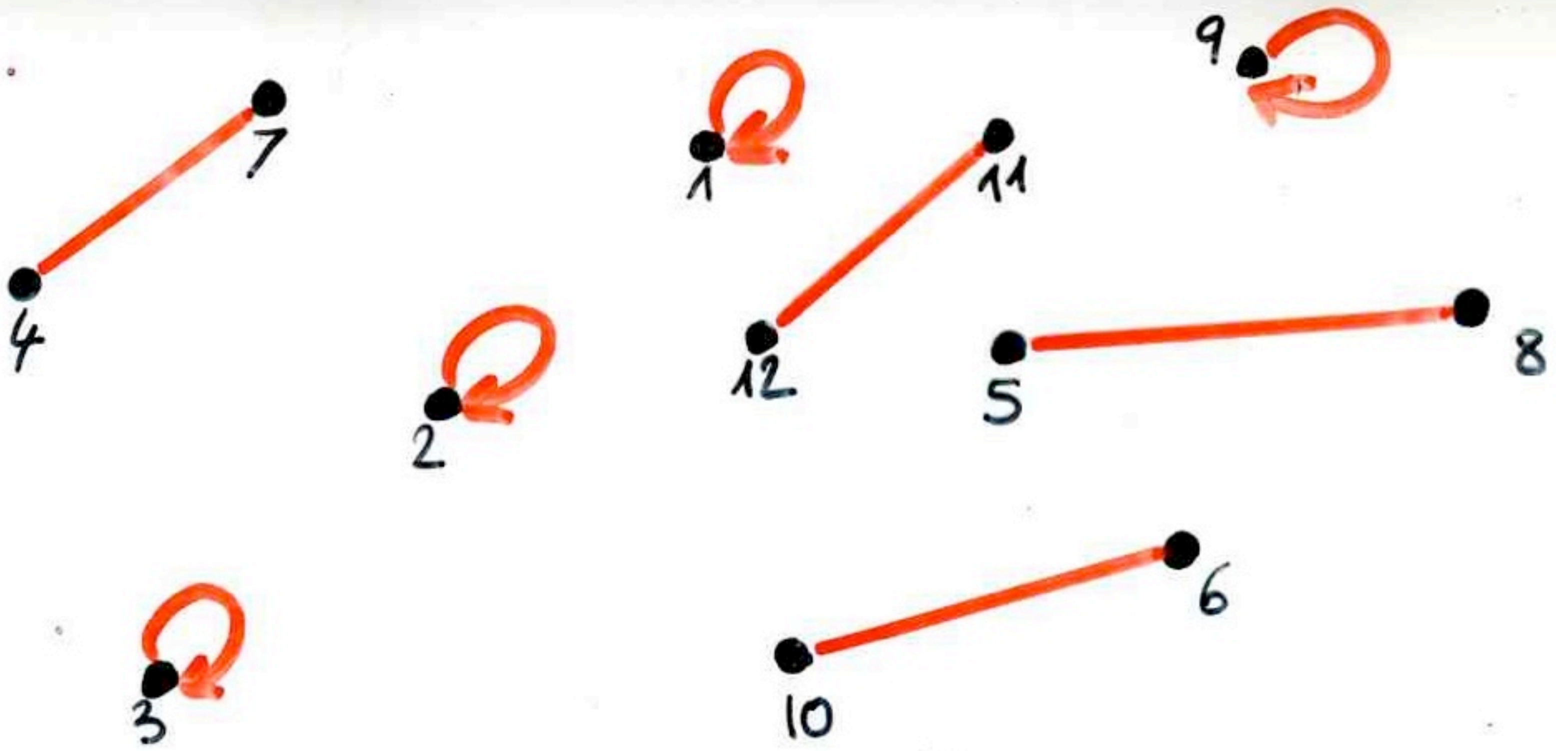
$g(t) = 1$

Mehler identity
for Hermite polynomials

$$\sum_{n\geq 0} H_n(x)H_n(y)\frac{t^n}{n!} = (1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\sum_{n\geq 0} H_n(x) \quad \frac{t^n}{n!} =$$

$$H_n(y)$$



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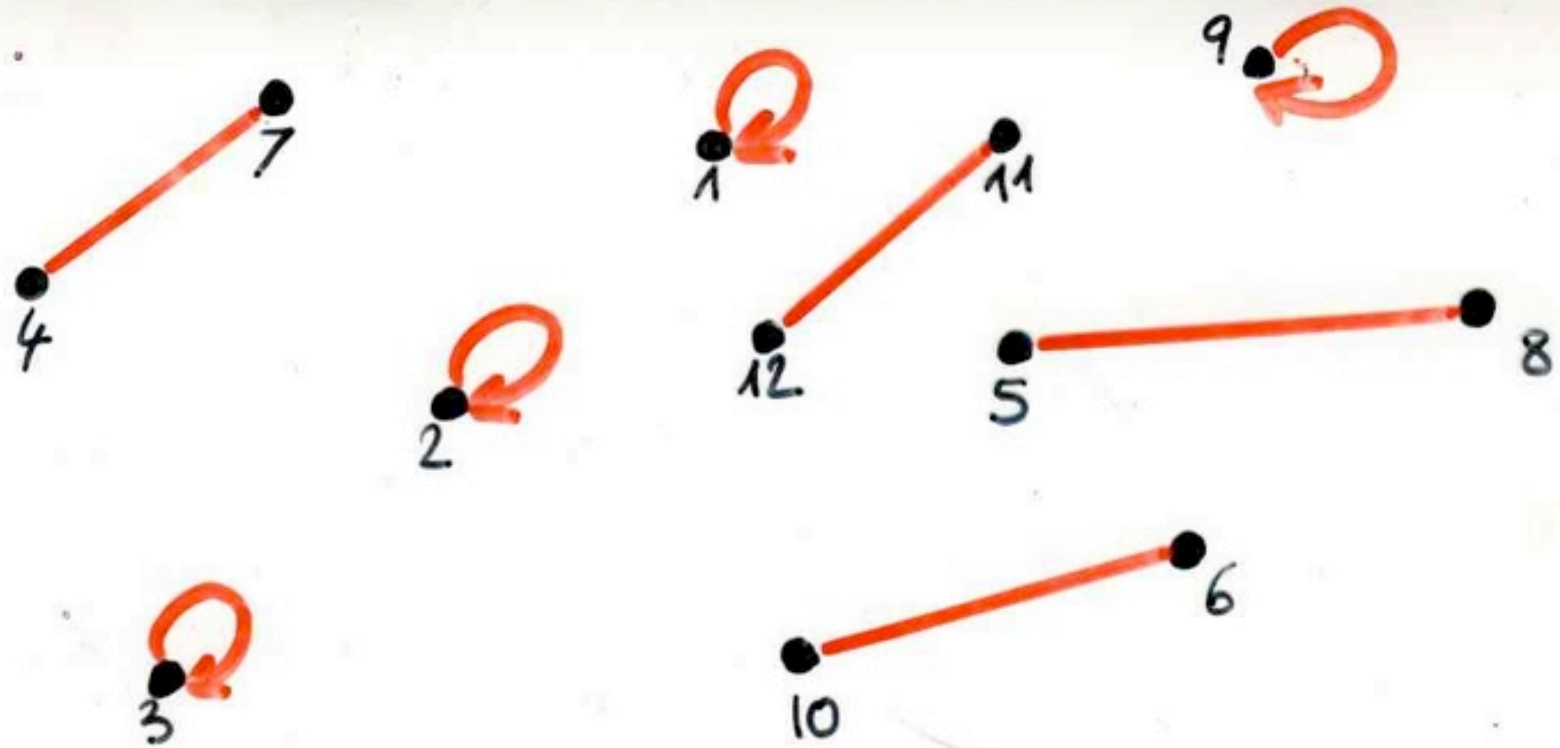
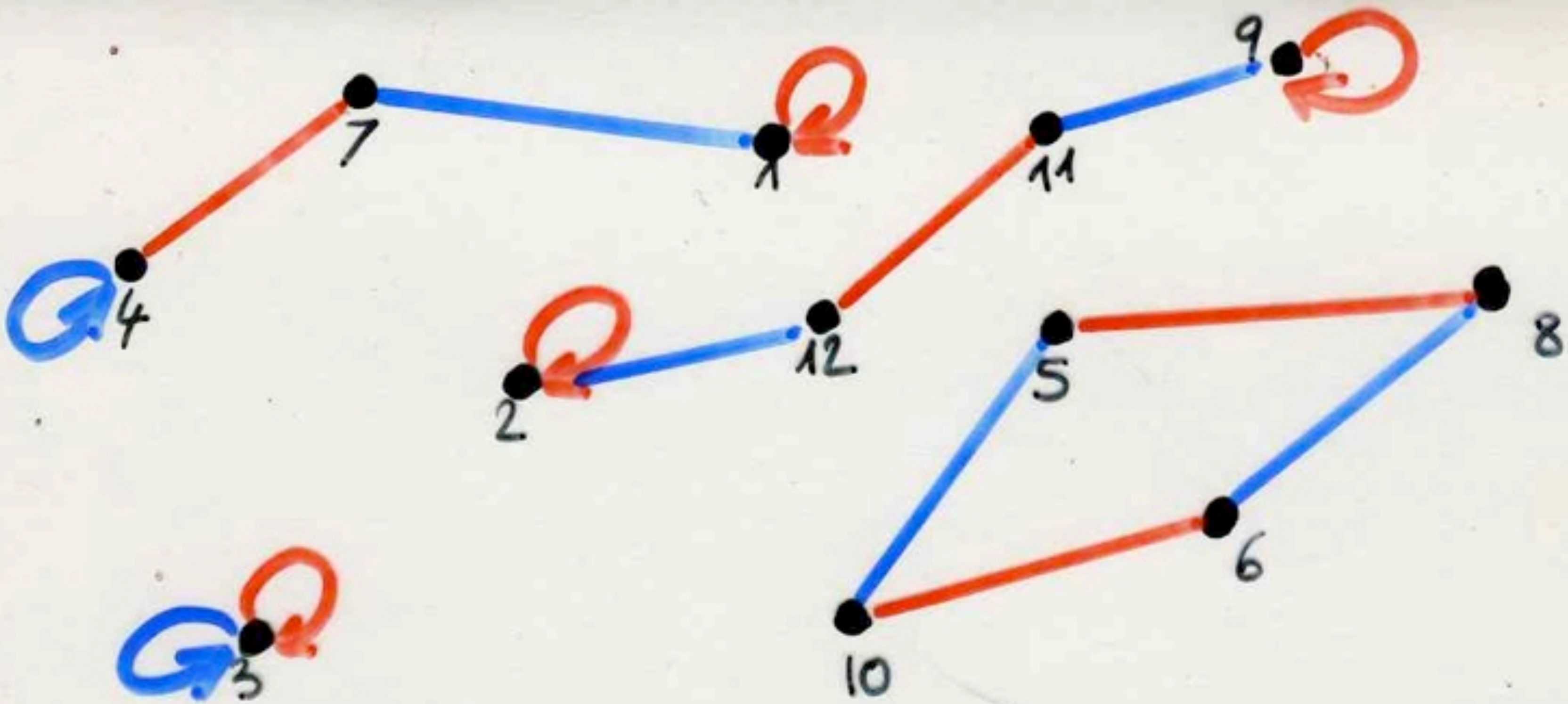


$$\sum_{n\geq 0} H_n(x) \quad \frac{t^n}{n!} =$$



$$\sum_{n\geq 0} H_n(x) H_n(y) \frac{t^n}{n!} =$$

$$(1-4t^2)^{-1/2}\exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\exp\left[\frac{1}{2}\log\frac{1}{(1-4t^2)}+\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$

$$\frac{-4\;y^2\;t^2}{1-4t^2}$$

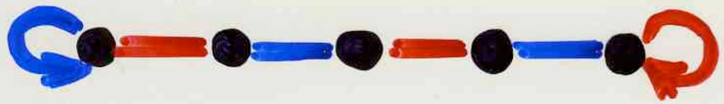
$$\frac{-4\;x^2\;t^2}{1-4t^2}$$

$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$

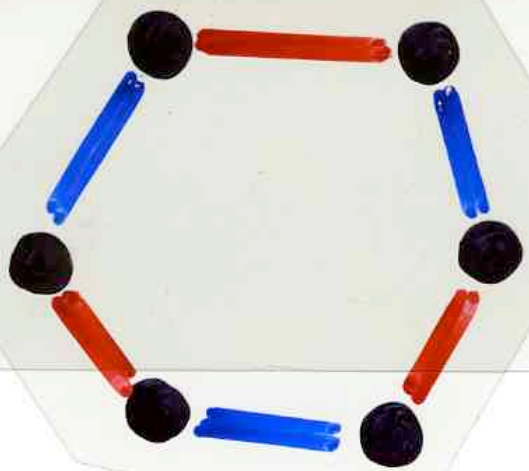
$$\frac{4xyt}{1-4t^2}$$

$$\frac{-4x^2t^2}{1-4t^2}$$

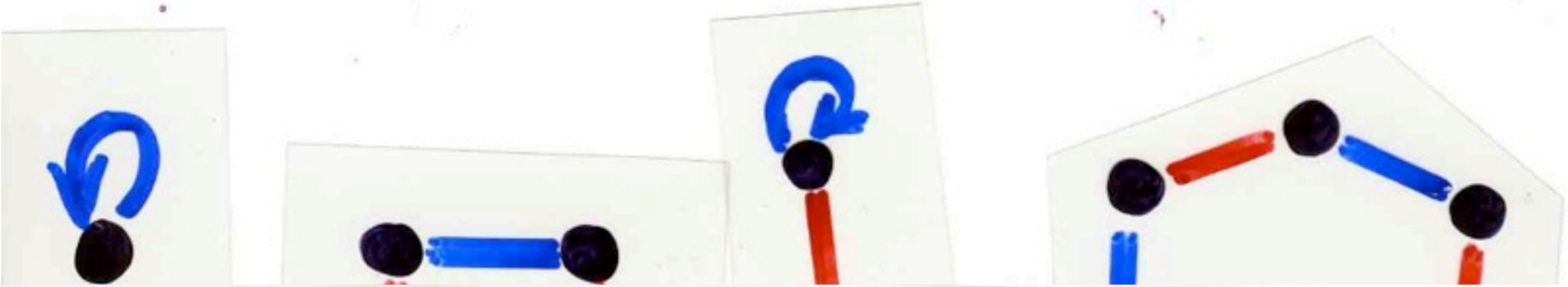
$$\frac{-4y^2t^2}{1-4t^2}$$



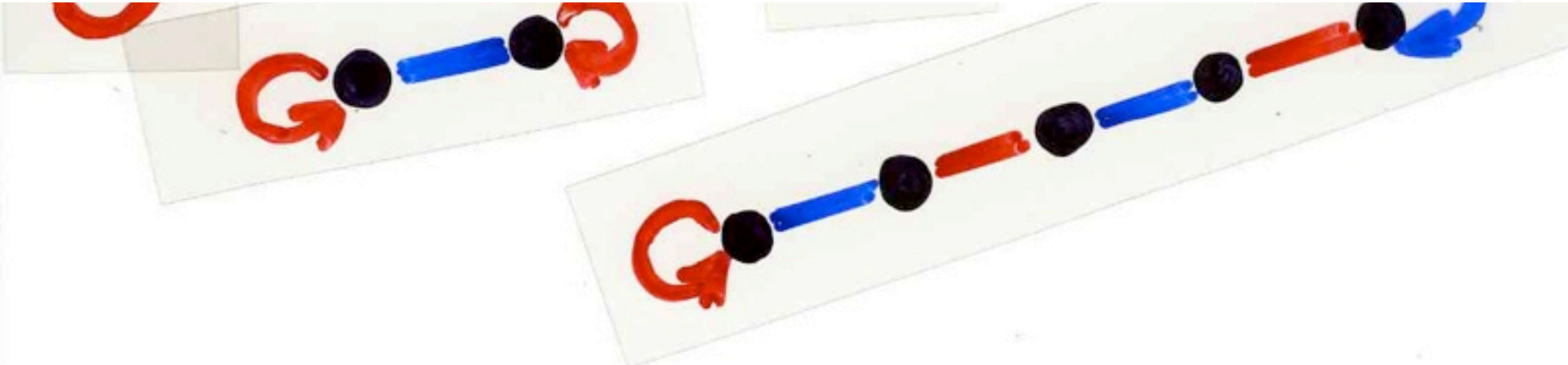
$$\frac{4xyt}{1-4t^2}$$



$$\frac{1}{2}\log\frac{1}{(1-4t^2)}$$



$$\sum_{n\geq 0} H_n(x)\, H(y)\, \frac{t^n}{n!} = (1-4t^2)^{-1/2} \exp\left[\frac{4xyt-4(x^2+y^2)t^2}{1-4t^2}\right]$$



Permutations: classical combinatorics

Inversion table, q-analogue

Increasing binary trees

exp generating functions, species and structures

Operations on species

Stirling numbers and binomial type polynomials

Differential equations

Weighted species

Examples: orthogonal polynomials

Mehler identity for Hermite polynomials

complements

1 - Introduction to enumerative, algebraic and bijective combinatorics.

- ordinary generating functions
  - the Catalan garden: trees, paths, triangulations, ...
- q-series and q-analogues
- exponential structures and generating functions
  - permutations
  - bijections: inversion tables, «Laguerre histories» (=weighted paths),
  - RSK (Robinson-Schensted-Knuth correspondance)

2 - Some topics in algebraic combinatorics

- Paths, determinants and Young tableaux

- RSK «local» from the algebra $UD=DU+Id$

- Combinatorial theory of orthogonal polynomials and continued fractions

3 - Combinatorics and physics: the cellular Ansatz

- PASEP (partially asymetric exclusion process)
  the PASEP algebra $DE = qED + E + D$

- Alternative tableaux, permutation tableaux

- TASEP and the Catalan garden
  Loday-Ronco algebra

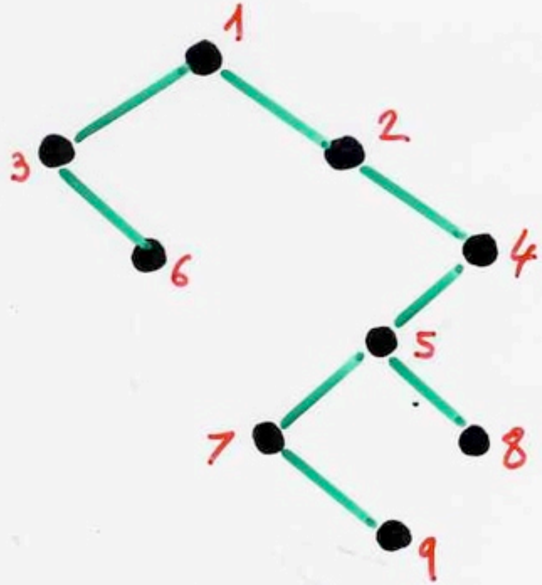
- ASM (alternating sign matrices), FPL (Fully packed loops model),
  6-vertex model, RS (the ex-Razumov-Stroganov conjecture)

- Q-tableaux and planar automata
  8-vertex model

complements

ex -

increasing binary trees



$$Y' = Y^2$$

$$Y[\emptyset] = \{\emptyset\}$$

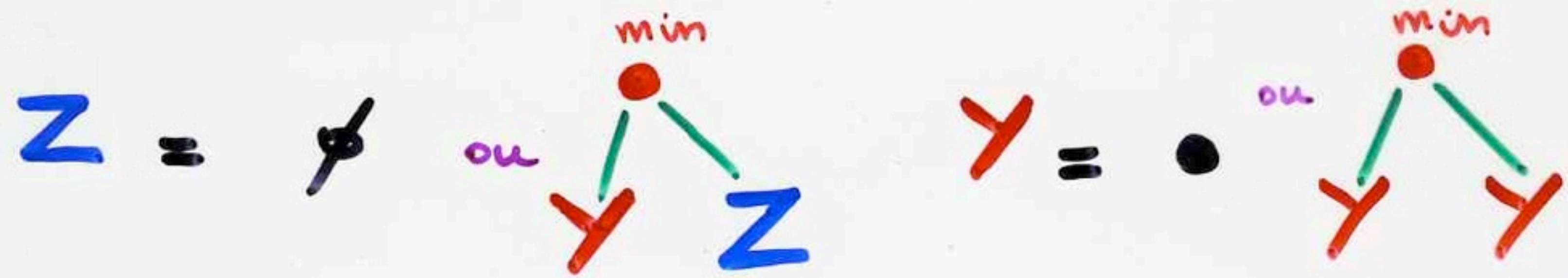
$$Y = \int_0^T Y^2(u)\,du + 1$$

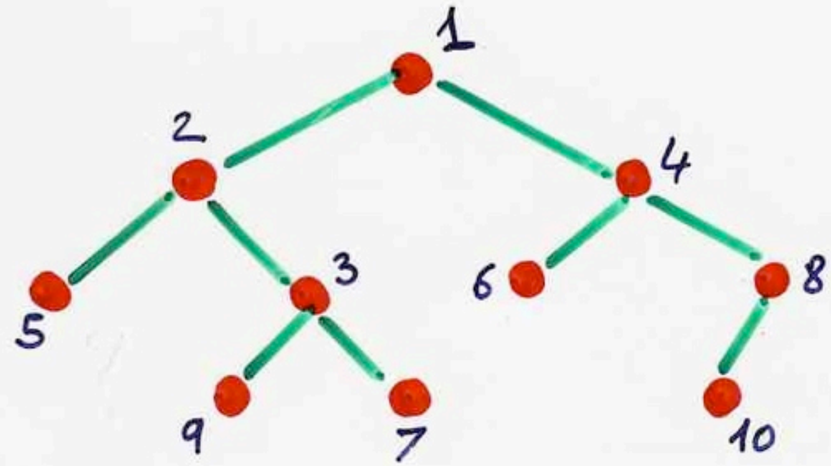
ex - secant numbers

$$\frac{1}{\cos t} = \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!}$$

$$\begin{cases} z' = yz , & z(0) = 1 \\ y' = 1 + y^2 , & y(0) = 0 \end{cases}$$

$$\begin{cases} Z' = YZ , \\ Y' = 1 + Y^2 , \end{cases} \qquad \begin{array}{l} Z[\emptyset] = \{\emptyset\} \\ Y[\emptyset] = \emptyset \end{array}$$


Z = ∅ ou
min
Y Z
Y = ● ou
min
Y Y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Def- $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{E}_n$, $x \in [1, n]$

x-factorisation $\quad \sigma = u\lambda(x)\, x\, \rho(x)\, v$

$\begin{cases} \bullet \text{ lettres de } \lambda(x) \text{ et } \rho(x) \quad > x \\ \bullet\ |\lambda(x)| \text{ et } |\rho(x)| \quad \text{maximum} \end{cases}$

...

Lemme $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{E}_n$, $x \in [1, n]$

$(u\lambda(x)\, x\, \rho(x)\, v)$ $\quad$ x-factorisation

alors le sous-arbre gauche [droit]

du sommet x dans $\delta(\sigma)$

est $\delta(\lambda(x))$ $\quad [\delta(\rho(x))]$

Cor. $\sigma \in \mathfrak{S}_n$, $\delta(\sigma) \in \mathcal{L}_n$, $x \in [1, n]$

x : Pic, Creux, Double Montée, Double Descente,

ssi dans $\delta(\sigma)$: x est

Pic	Creux	Double Montée	Double Descente
Feuille	Sommet double	Sommet simple à droite	Sommet simple à gauche