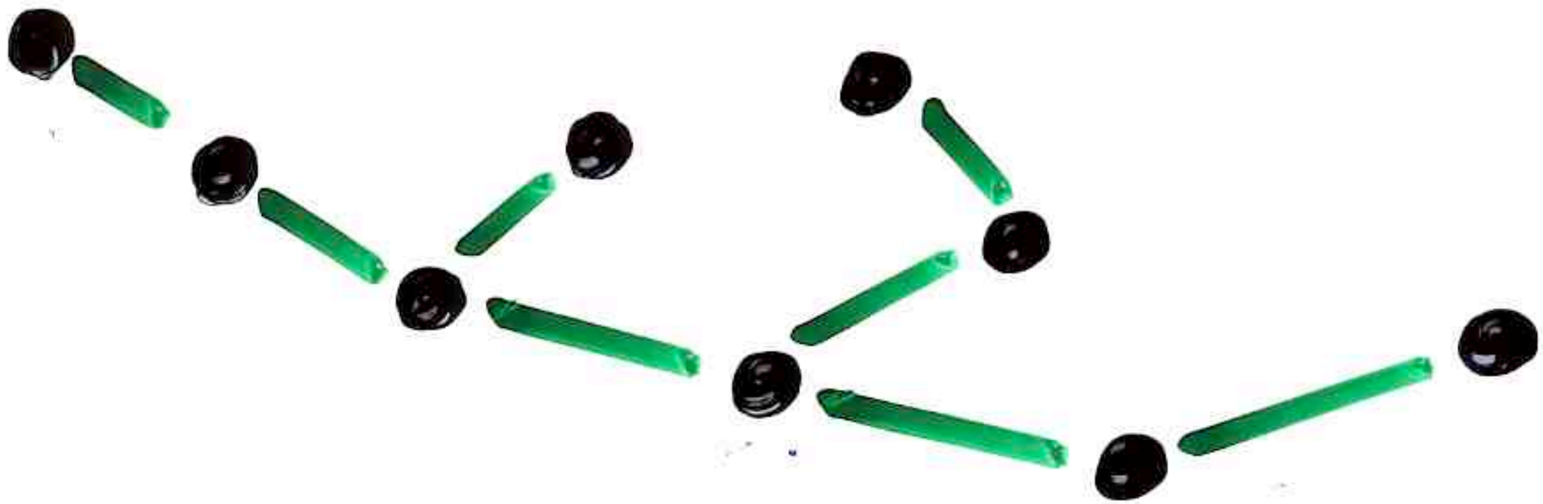
Chapter 6b

tableaux and Increasing / alternative binary trees

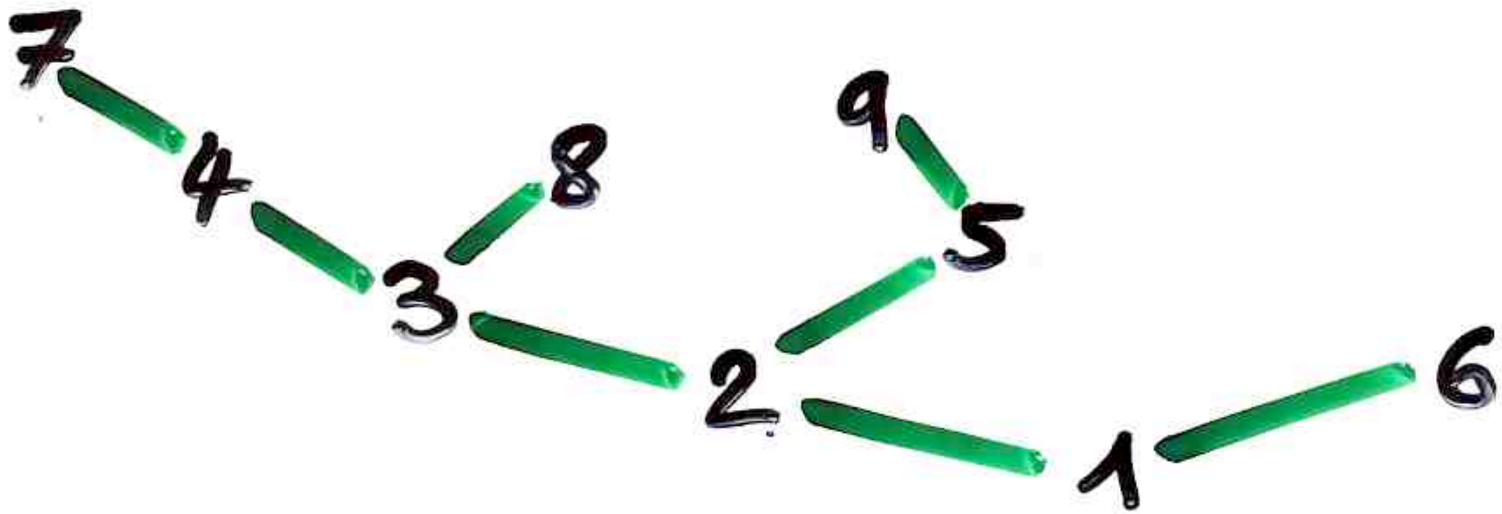
24 january 2011
Talca

increasing binary trees: canopy and up-down sequences

Def- Increasing binary tree

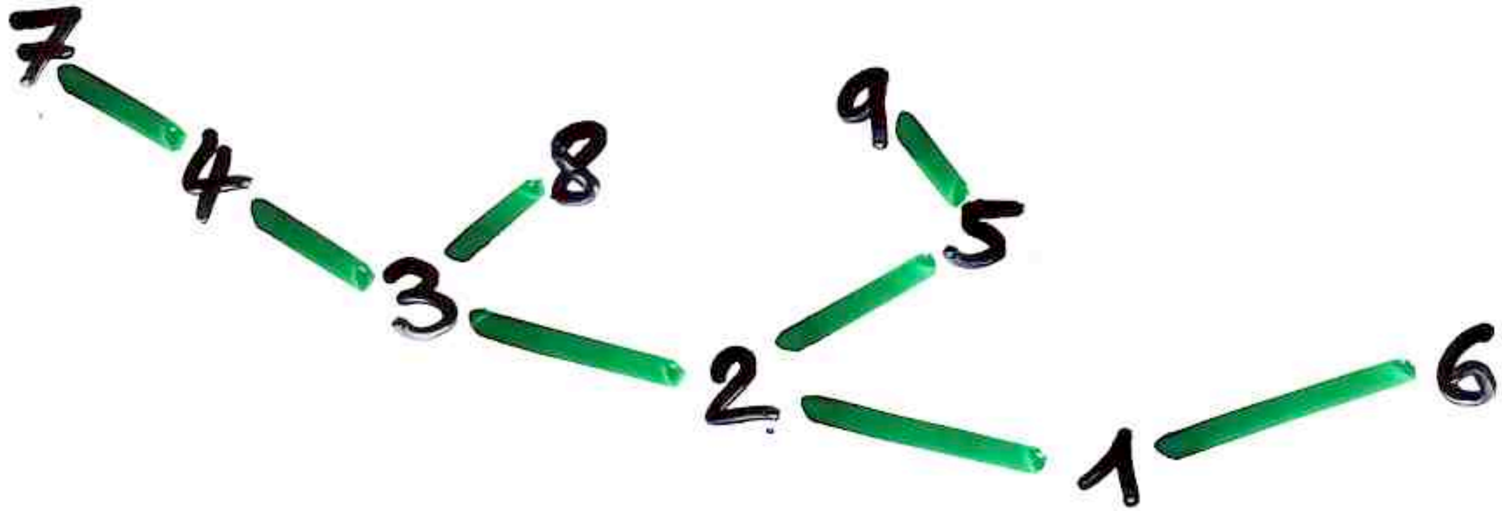


Def- Increasing binary tree



Bijection increasing binary tree ⟷ permutation

T ⟷ σ



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6$

Bijection

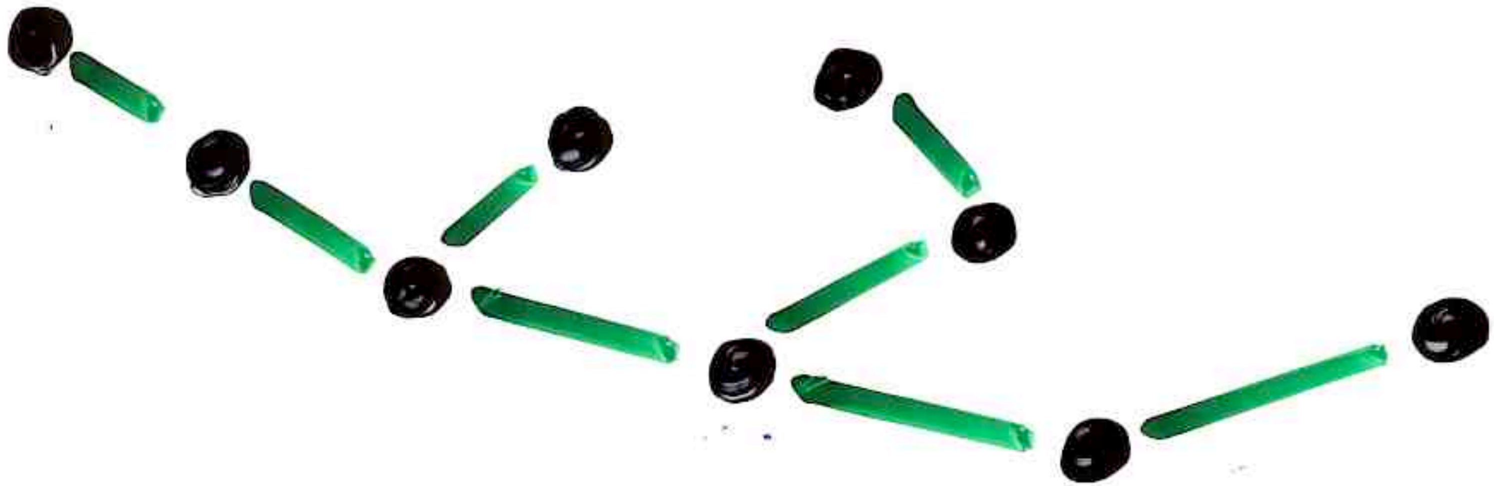
increasing binary tree T $\longleftrightarrow$ permutation σ

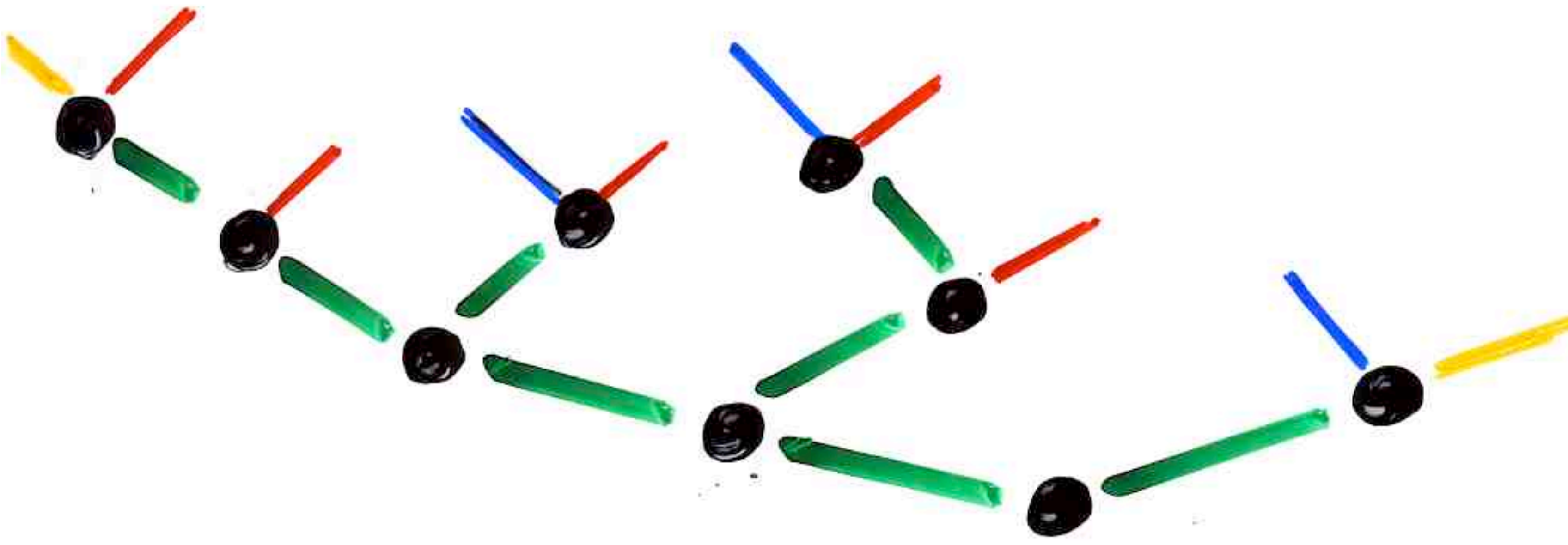
$T \xrightarrow{\ \pi\ } \sigma$ symmetric order of vertices (or "projection")

$\sigma \xrightarrow{\ \delta\ } T$ "déployé"

word $w = u\, m\, v$ m (unique) minimum letter

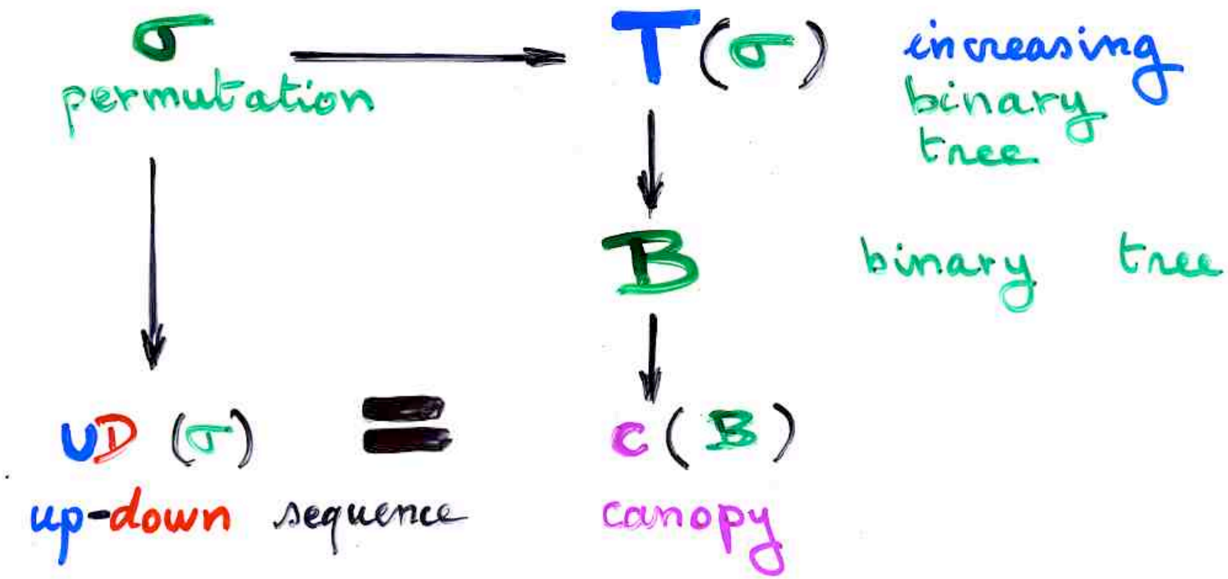
$\delta(w) =$ $\delta(u)$ $\delta(v)$
m

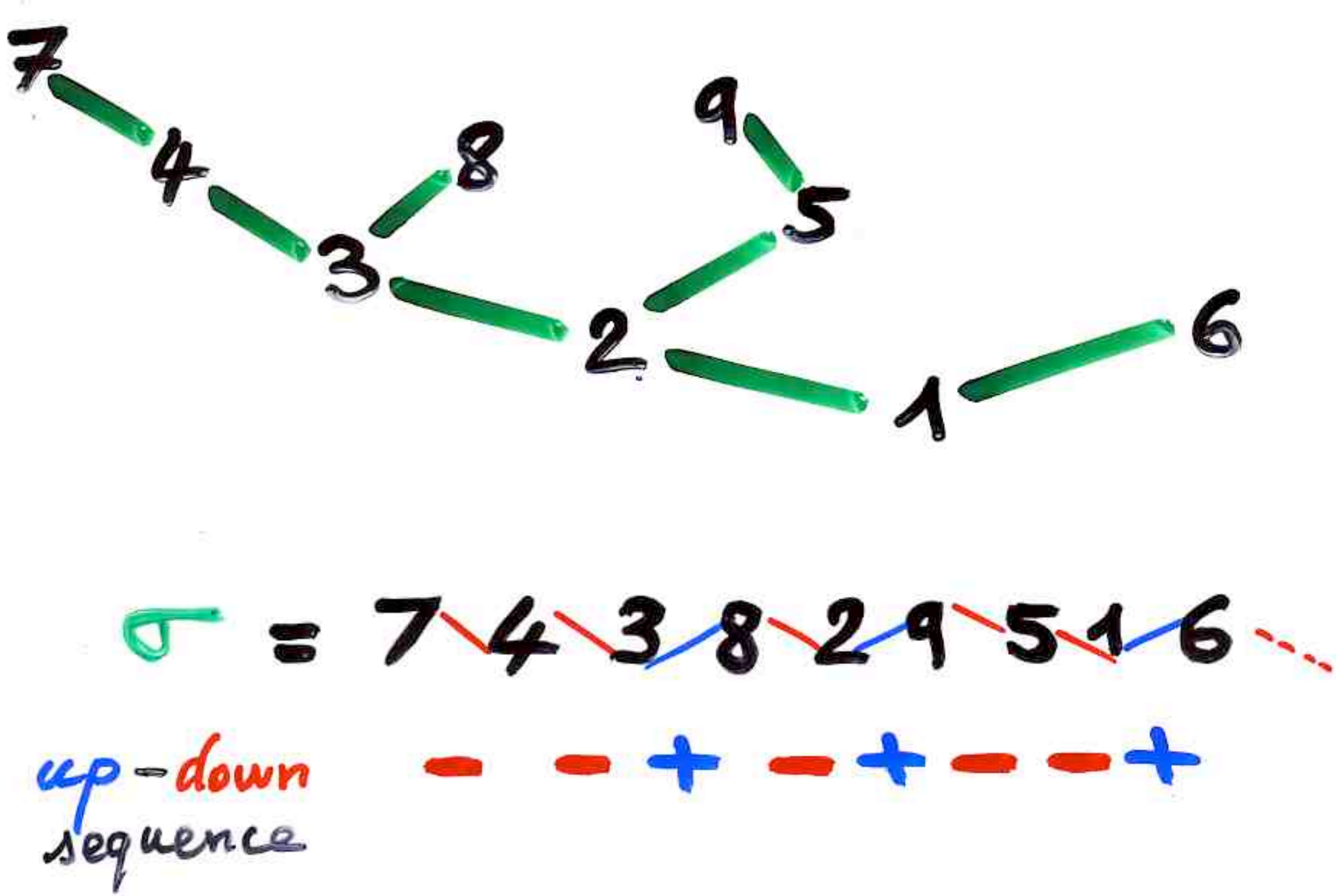


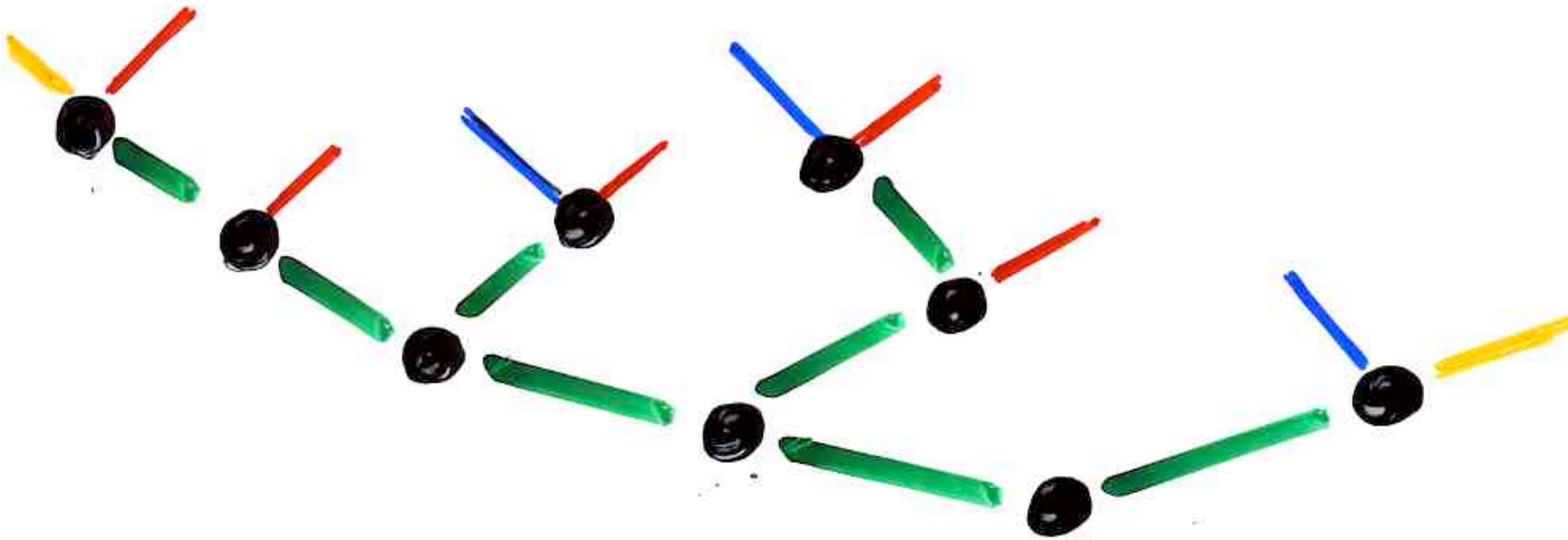


canopy of a binary tree

$C(B) = - - + - + - - +$


σ
permutation
T(σ)
increasing
binary
tree
B
binary tree
UD (σ)
up-down sequence
=
c(B)
canopy


7 4 3 8 9 5 2 1 6
σ = 7 4 3 8 2 9 5 1 6 ...
up-down sequence
− − + − + − − +



$\sigma = 7\ 4\ 3\ 8\ 2\ 9\ 5\ 1\ 6 \ldots$

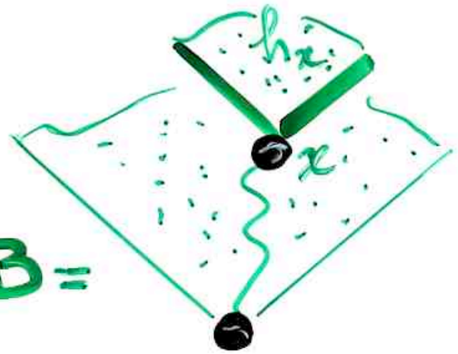
up-down sequence $\quad - \ - \ + \ - \ + \ - \ - \ +$

"hook-length formula"

$$\frac{n!}{\prod_x h_x}$$

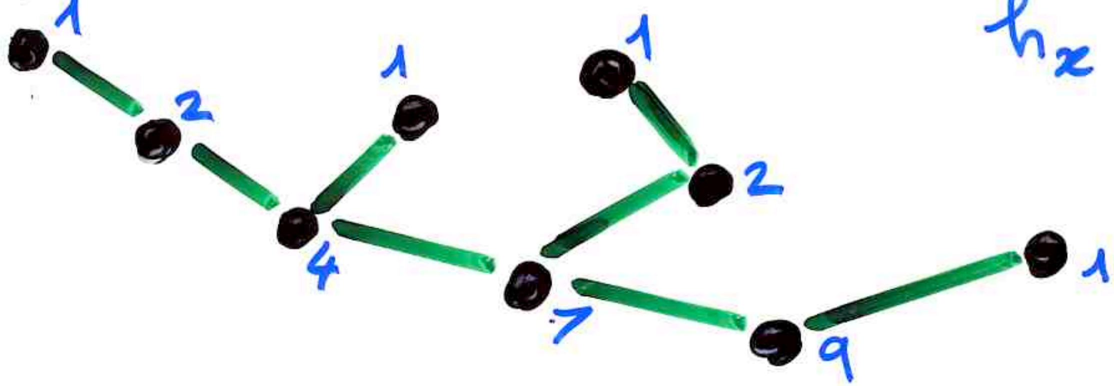
n nb of vertices

product of size of sub-trees

$B =$ 

nb of increasing binary tree for a binary tree B

ex:

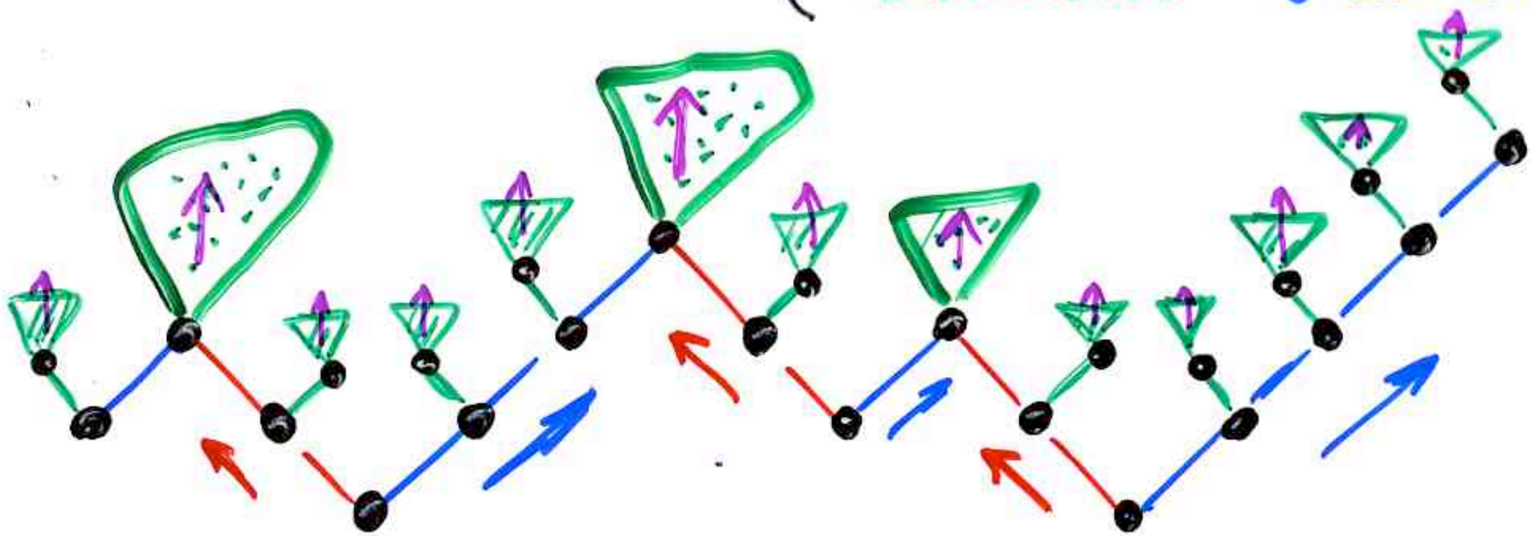


"hook-length" h_x

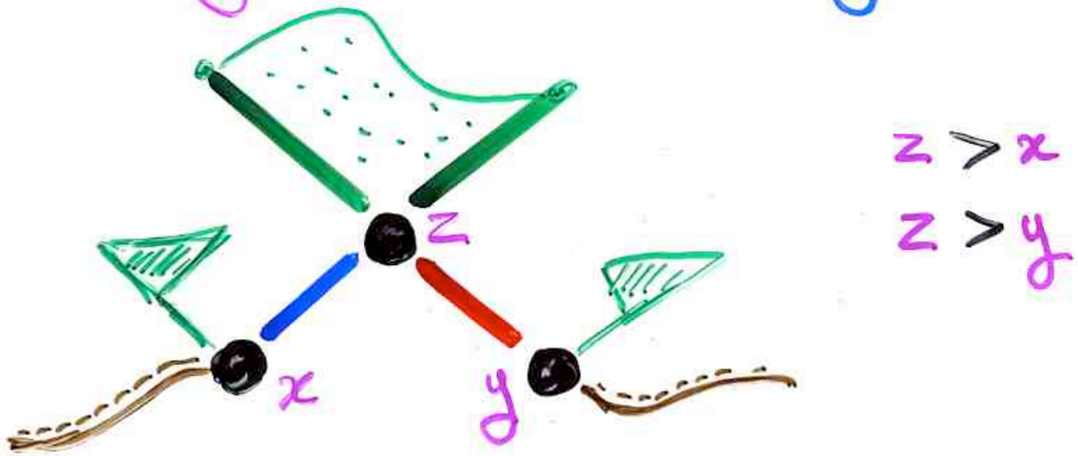
$$\frac{9!}{2^2 \cdot 4 \cdot 7 \cdot 9} = 360$$

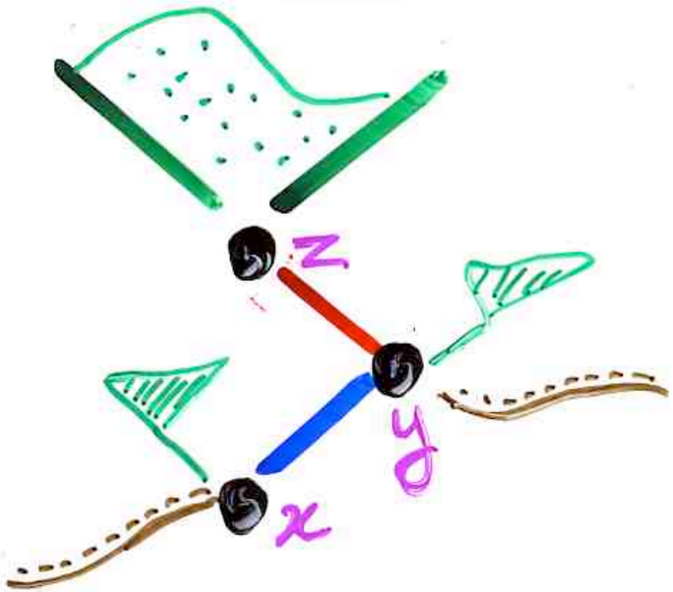
"jeu de taquin"
for
increasing binary trees

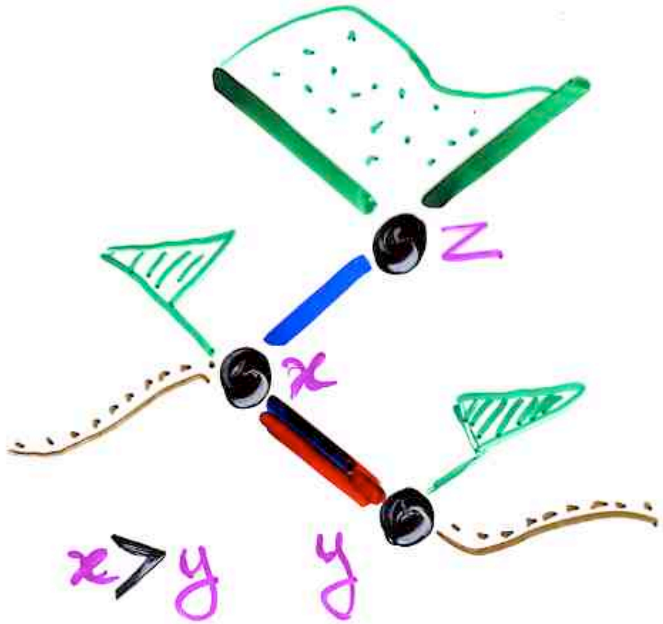
Def- Increasing woods ("buissons" croissants)



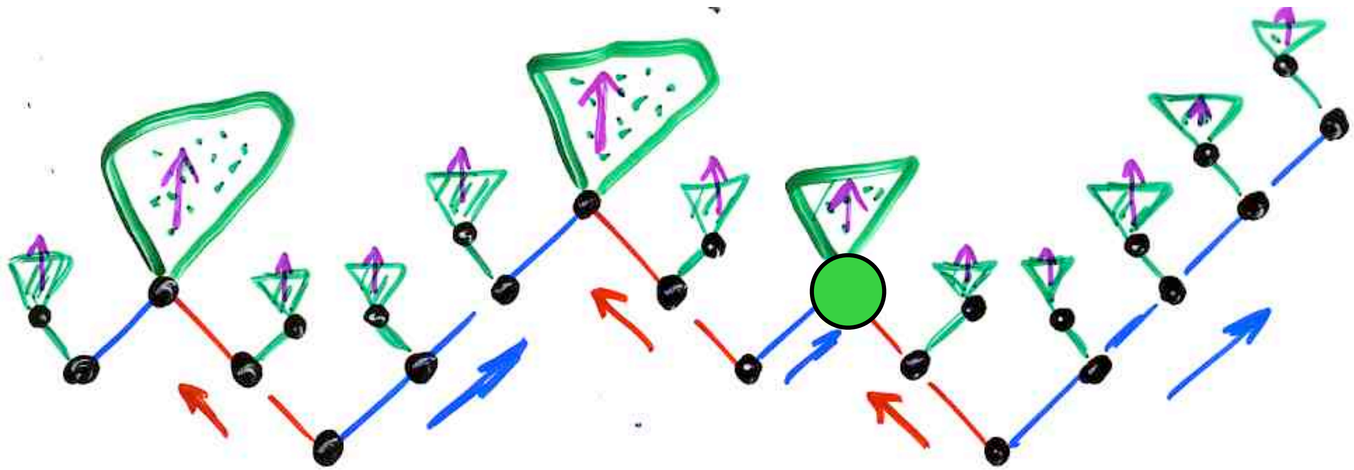
Def - Sliding in an increasing woods

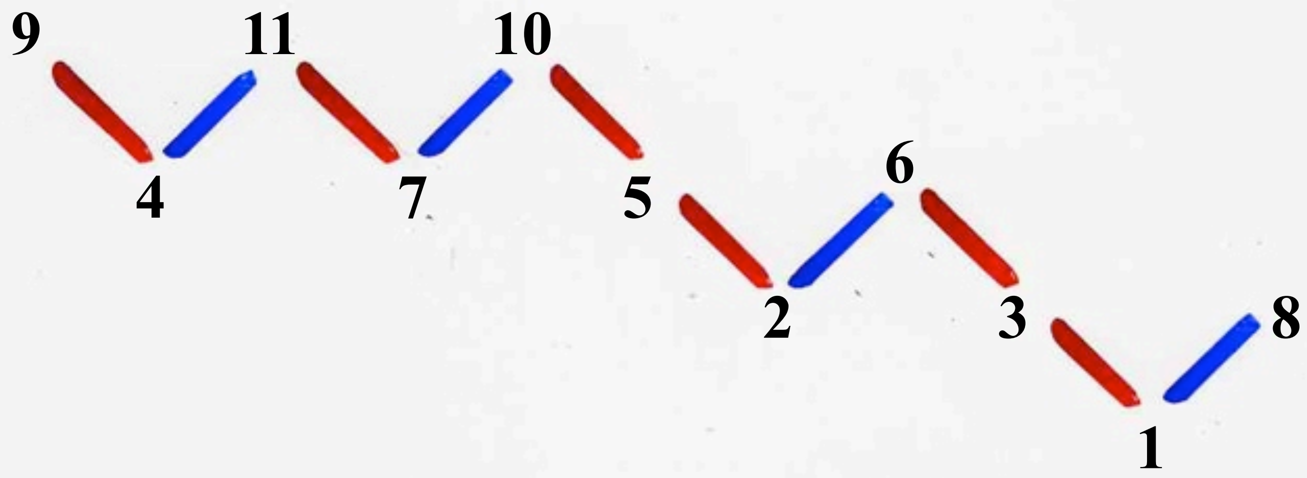






"jeu de taquin" for increasing woods

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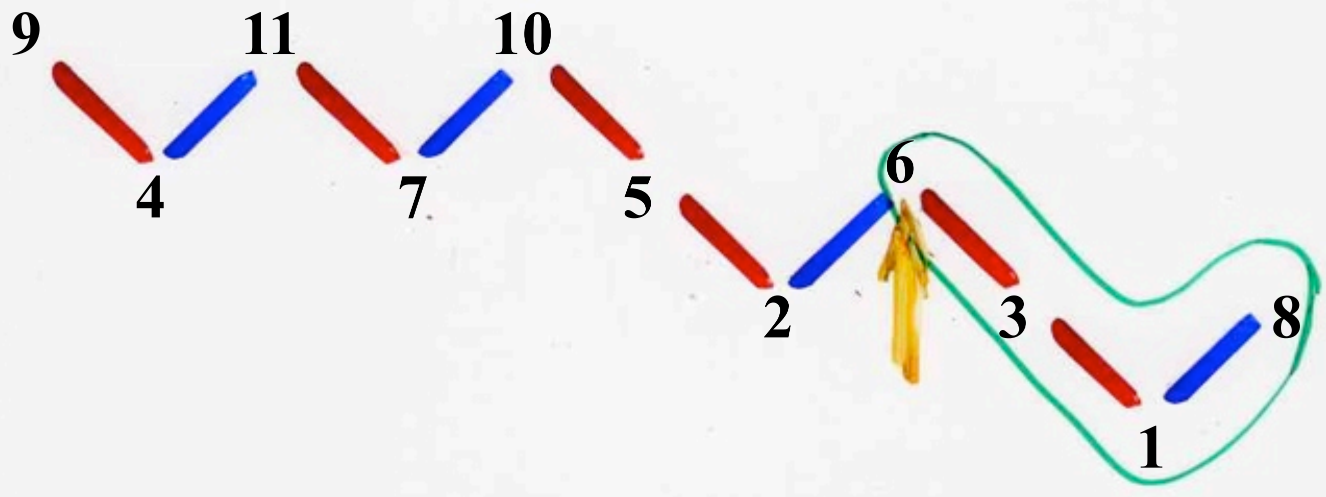
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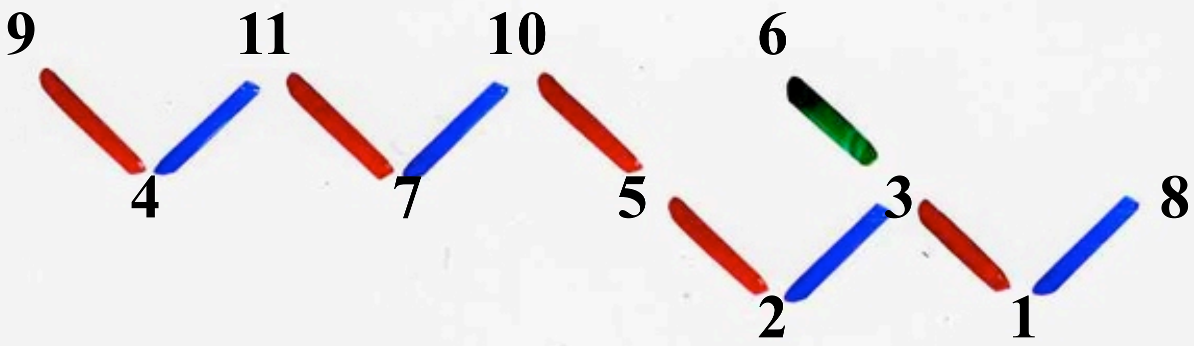
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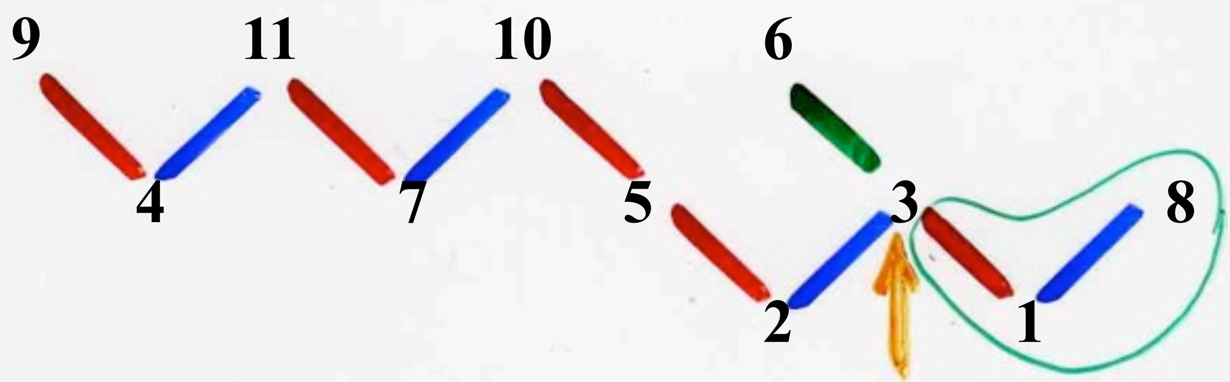
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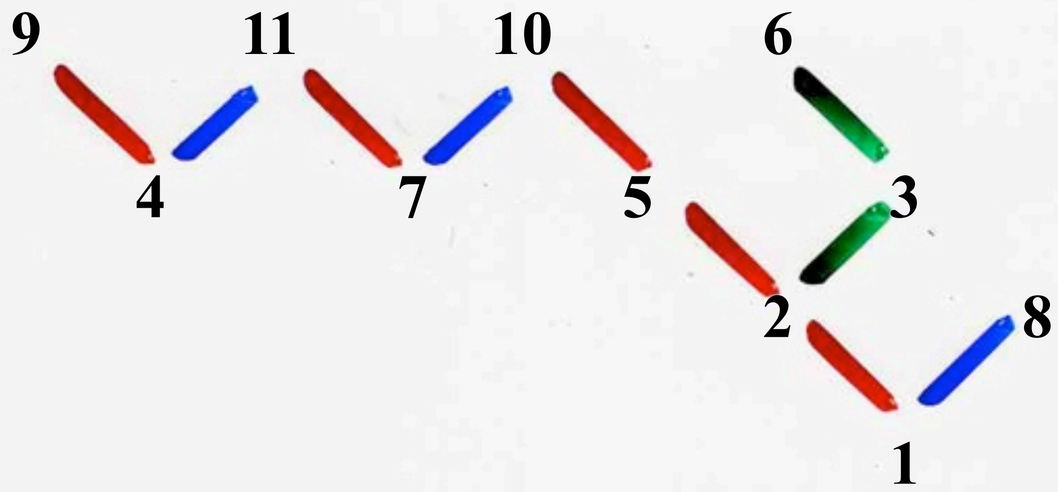
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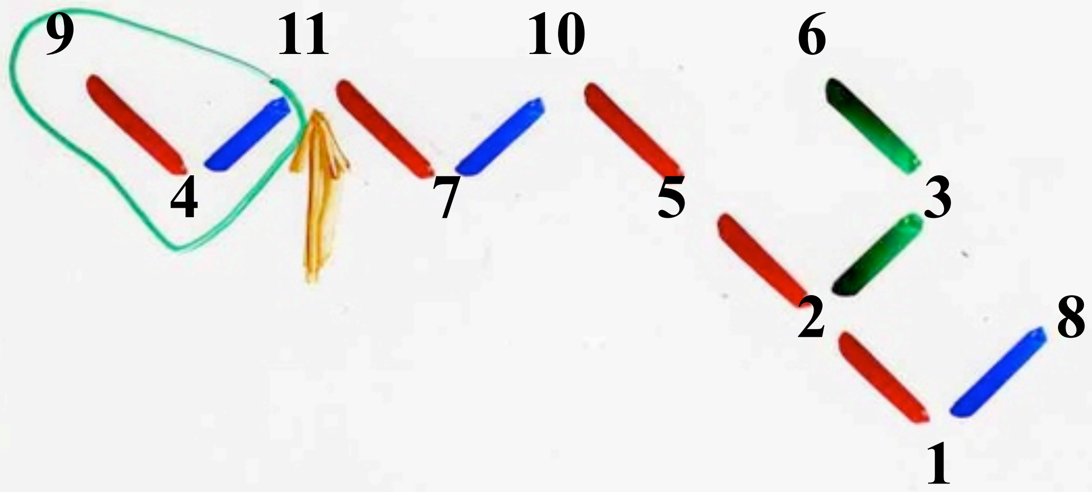
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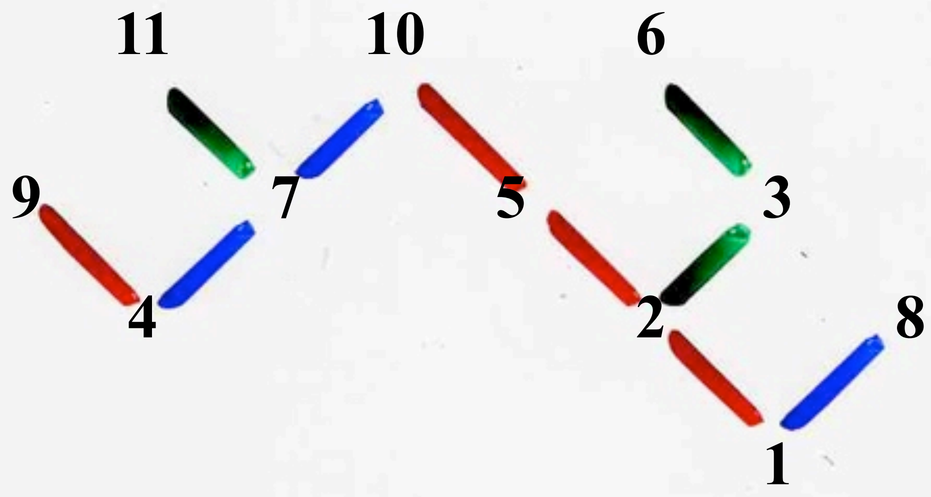
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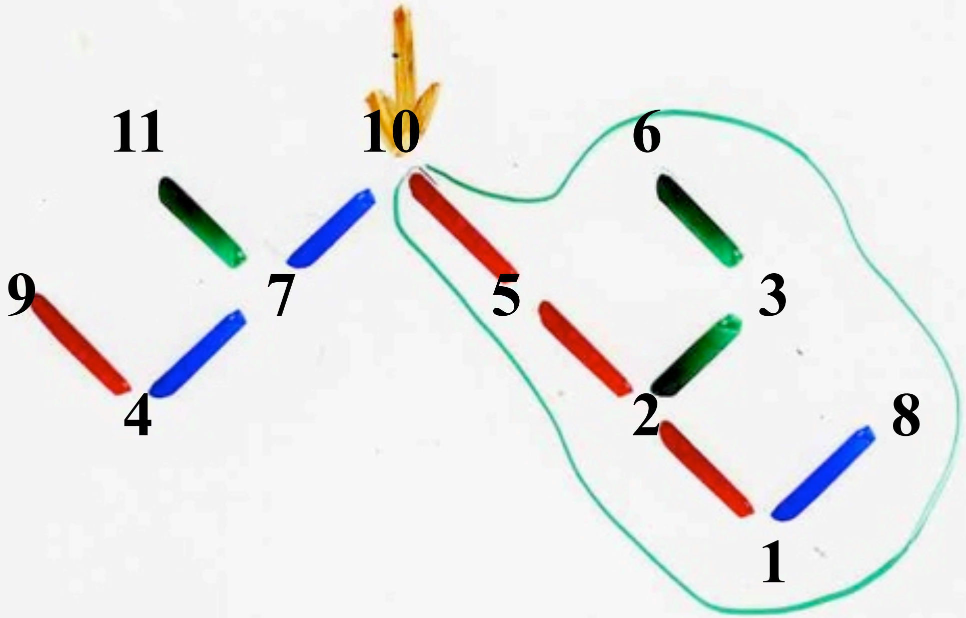
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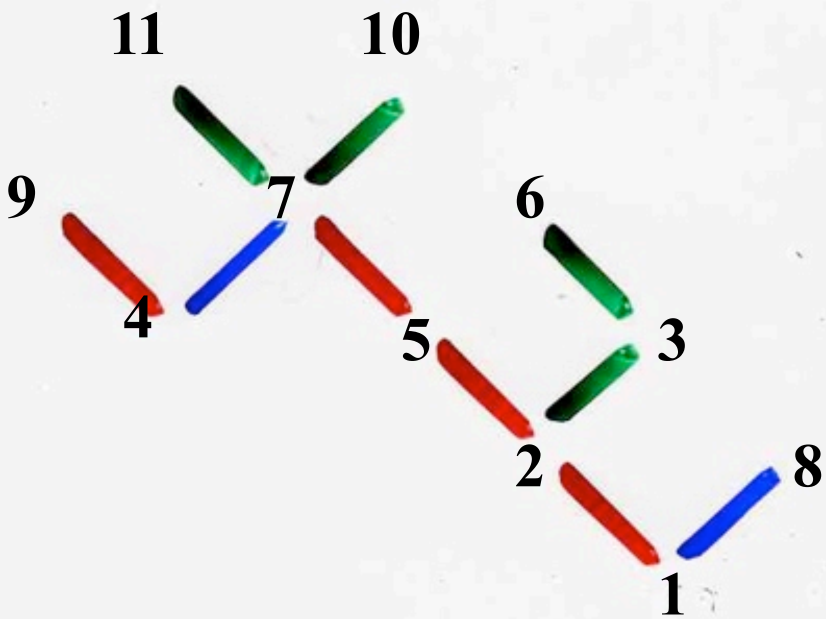
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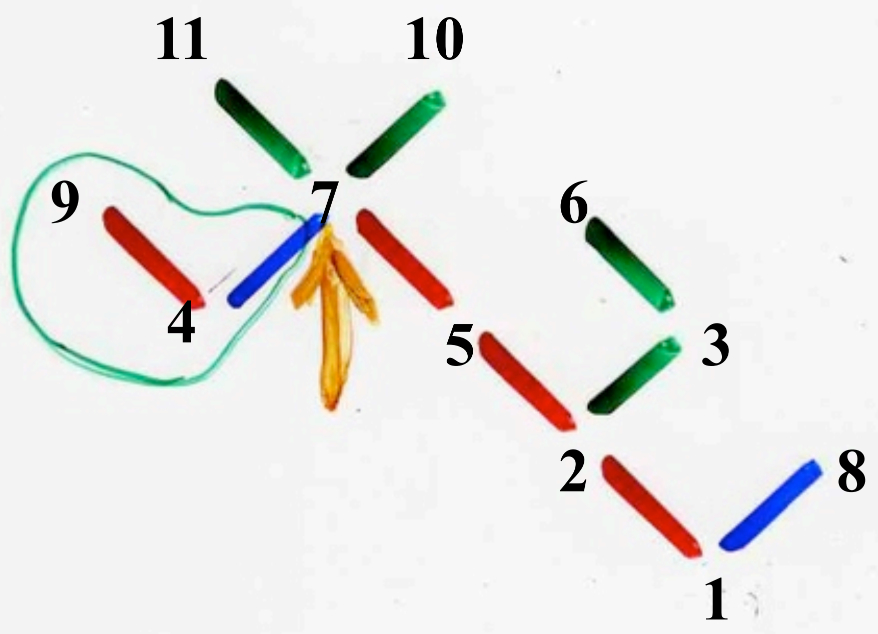
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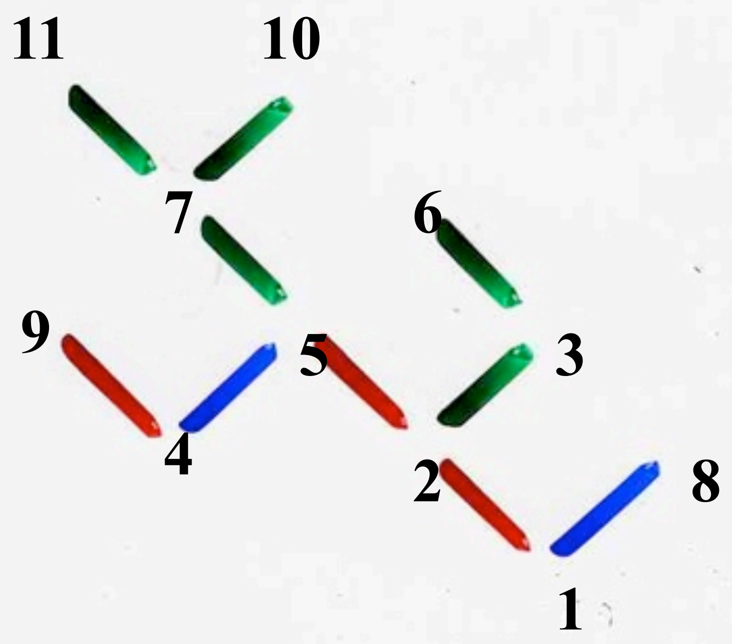
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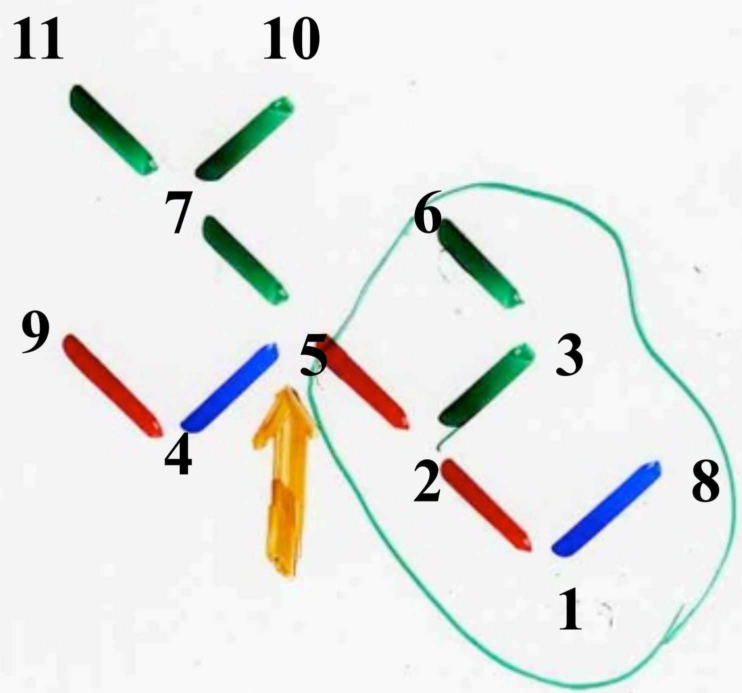
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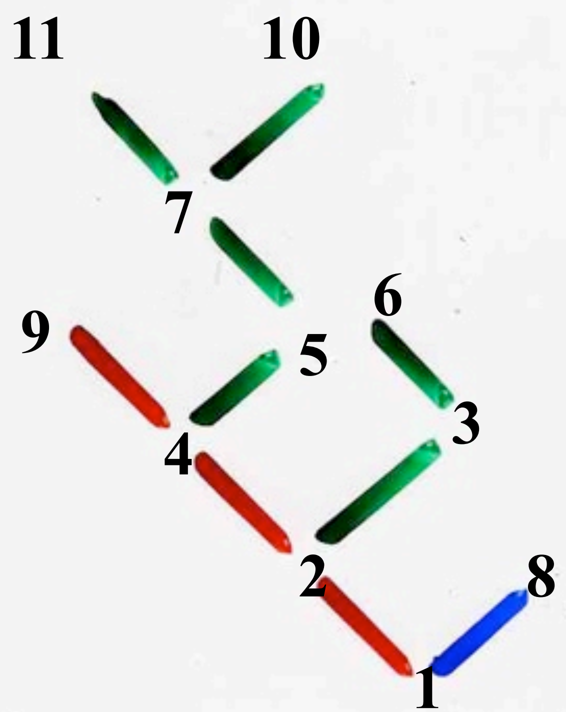
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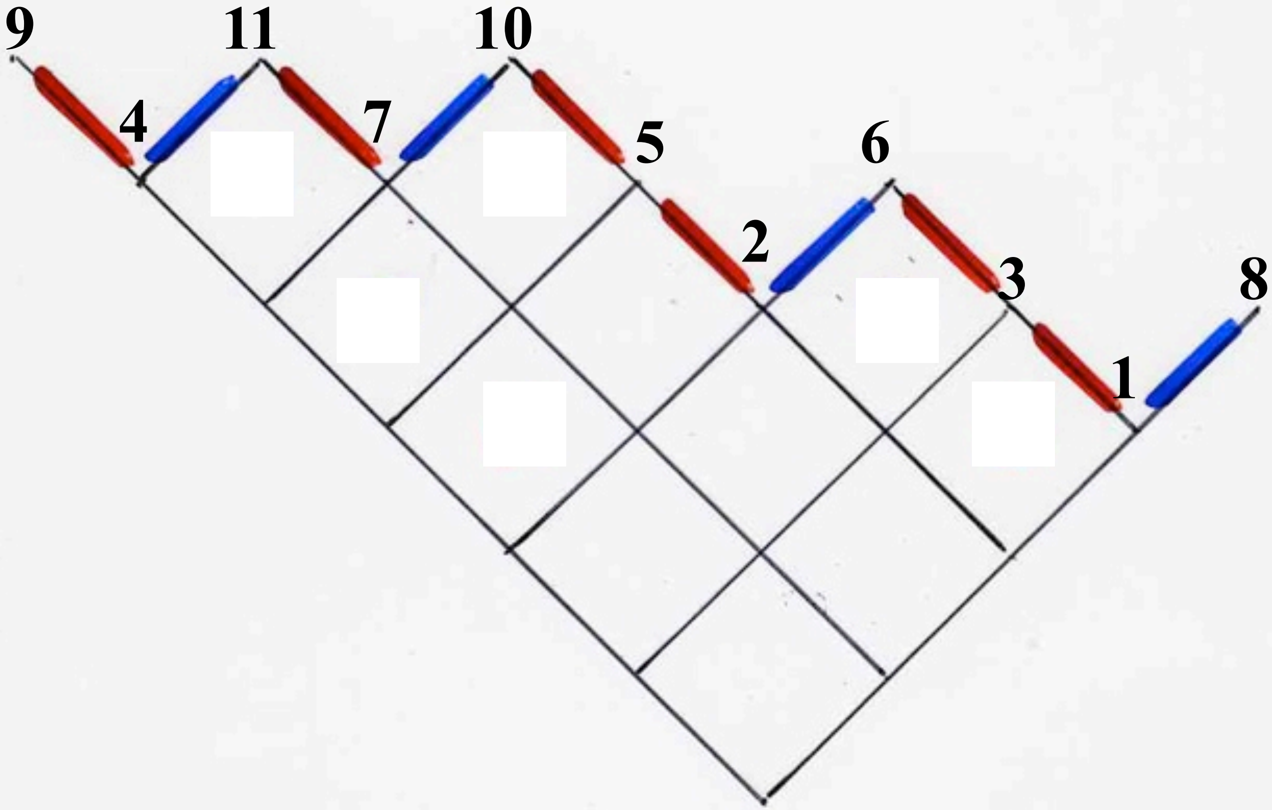
"jeu de taquin"
from increasing binary tree

$\longrightarrow$

to Catalan alternative tableau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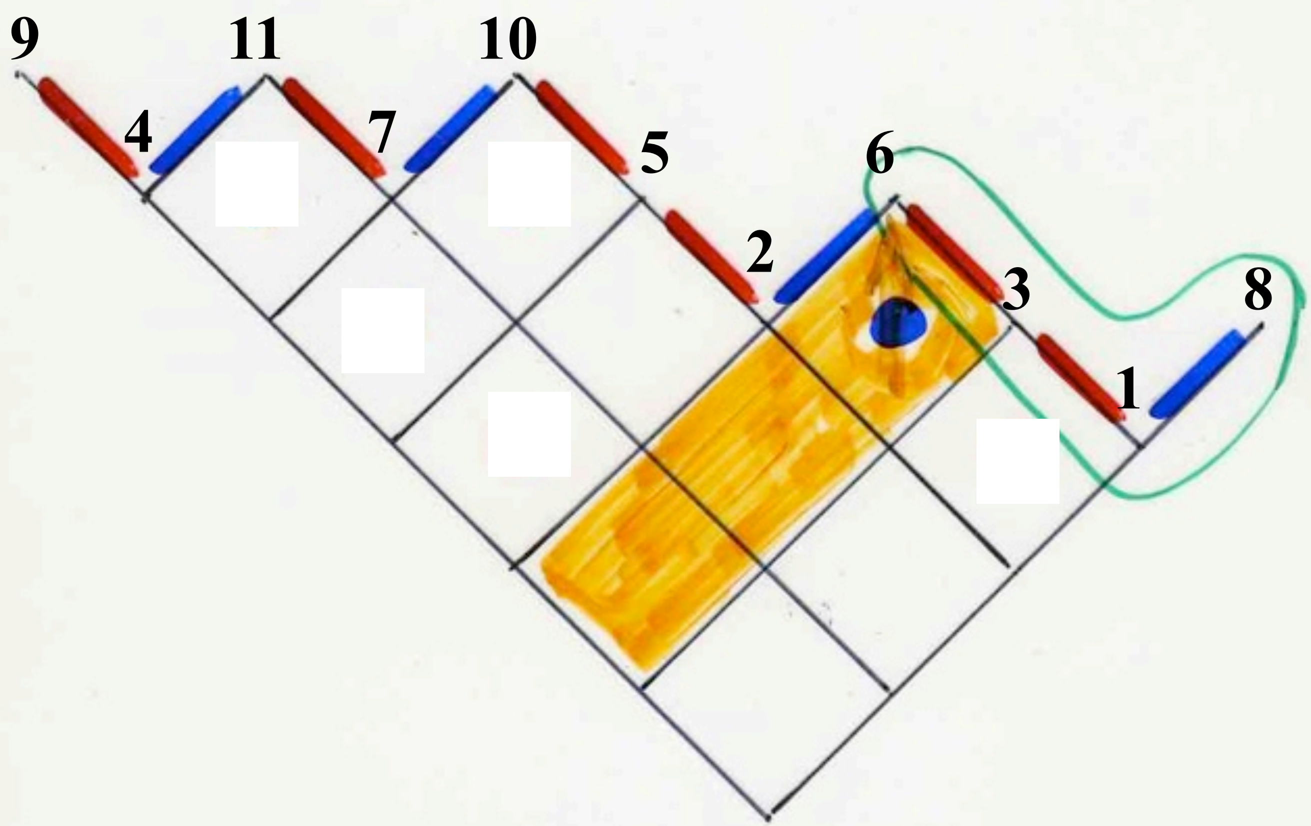
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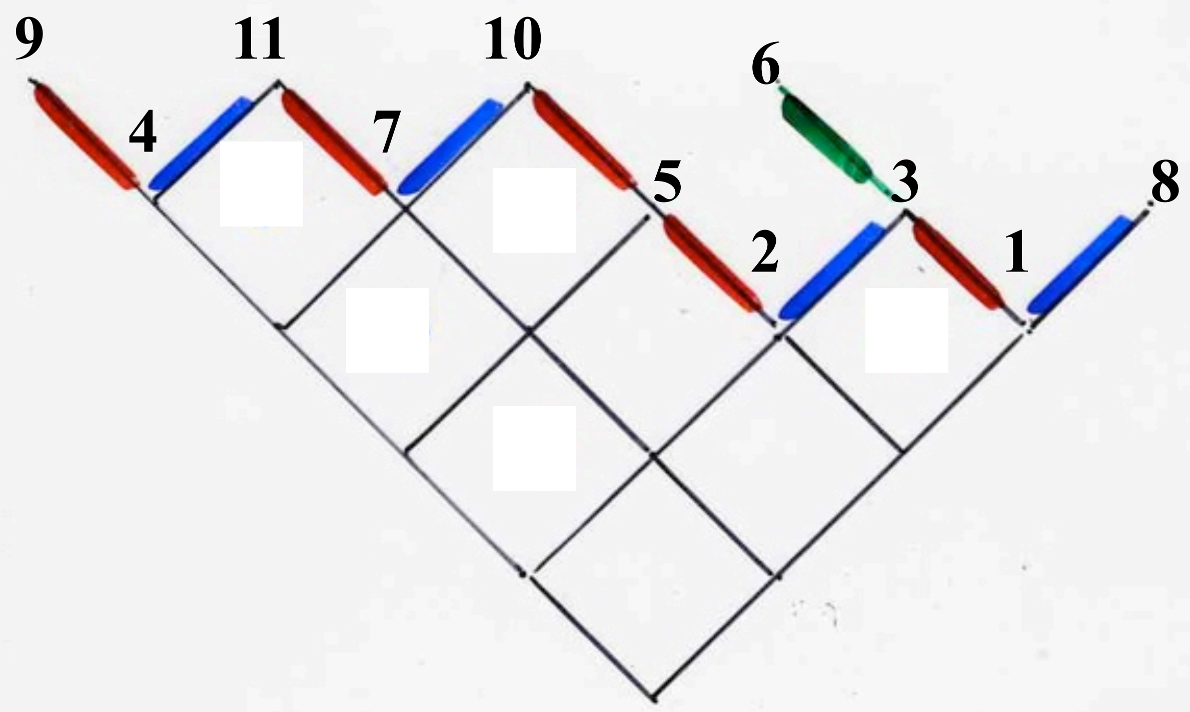
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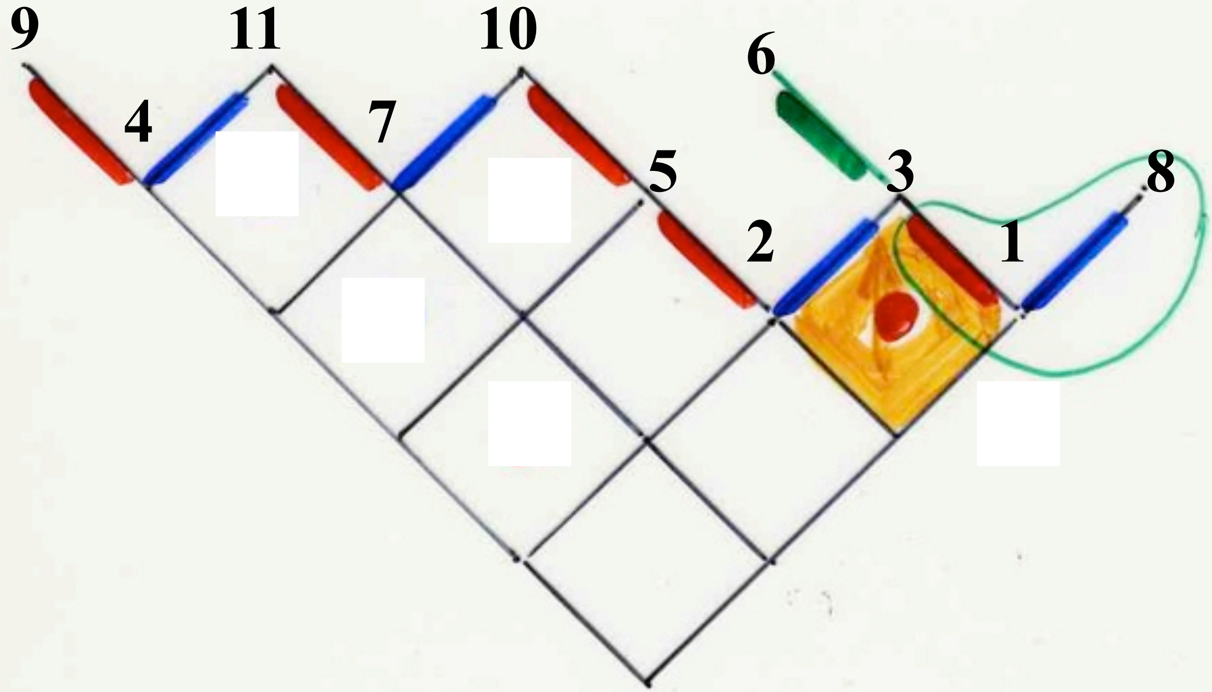
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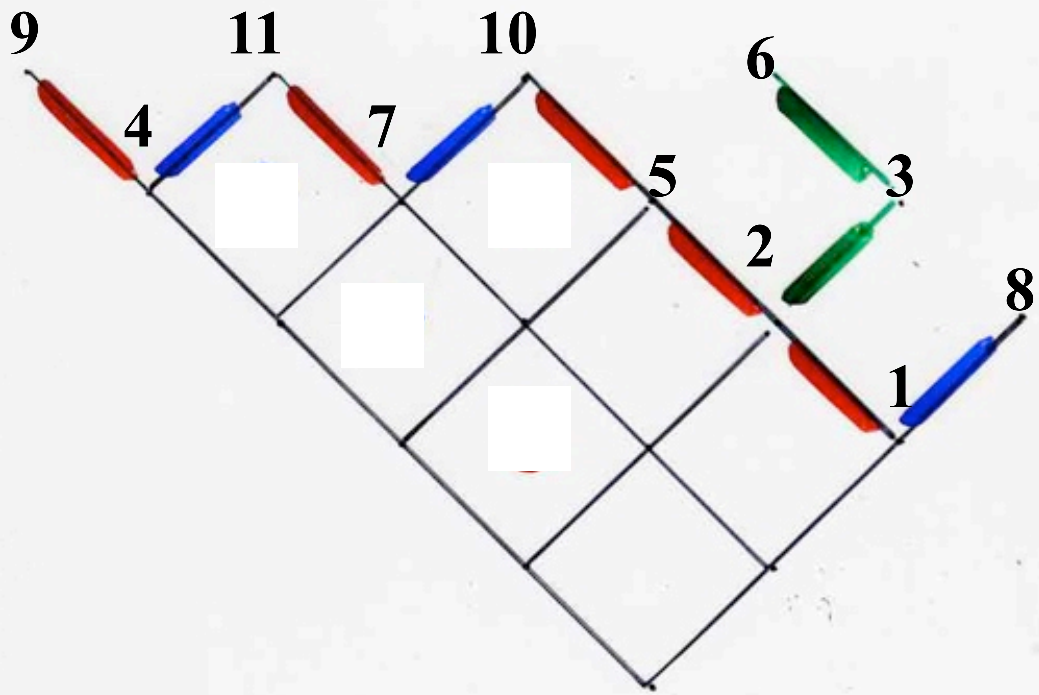
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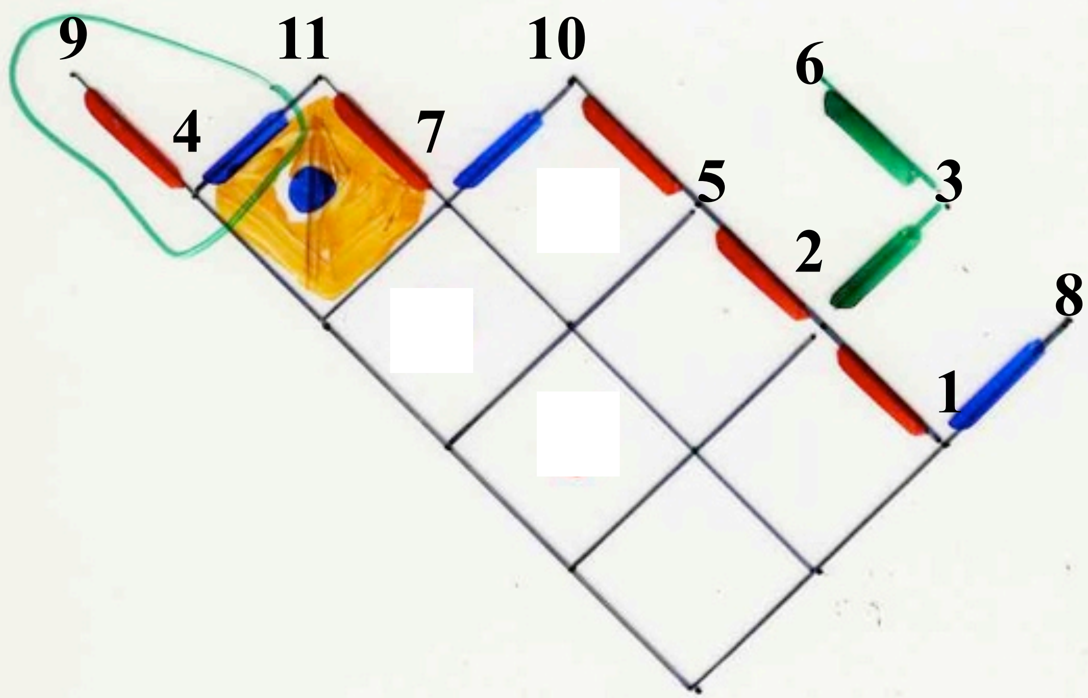
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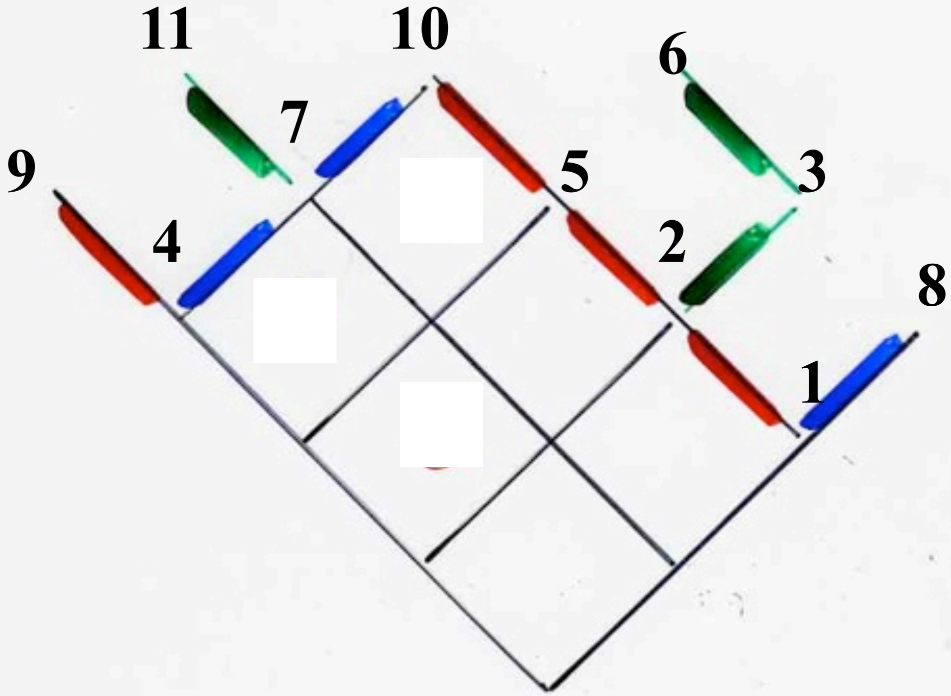

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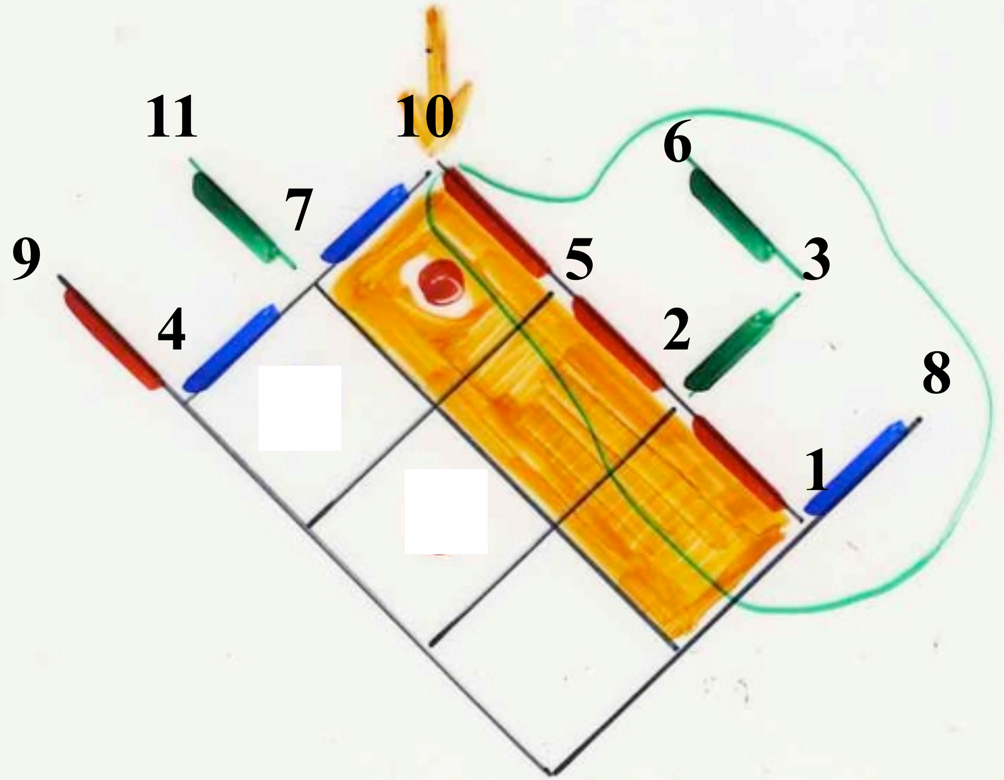
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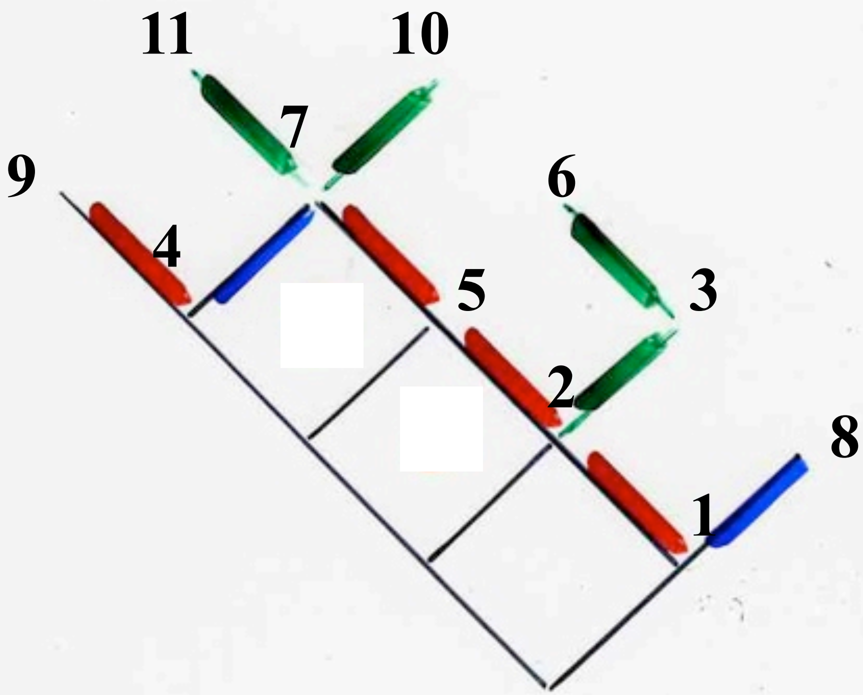
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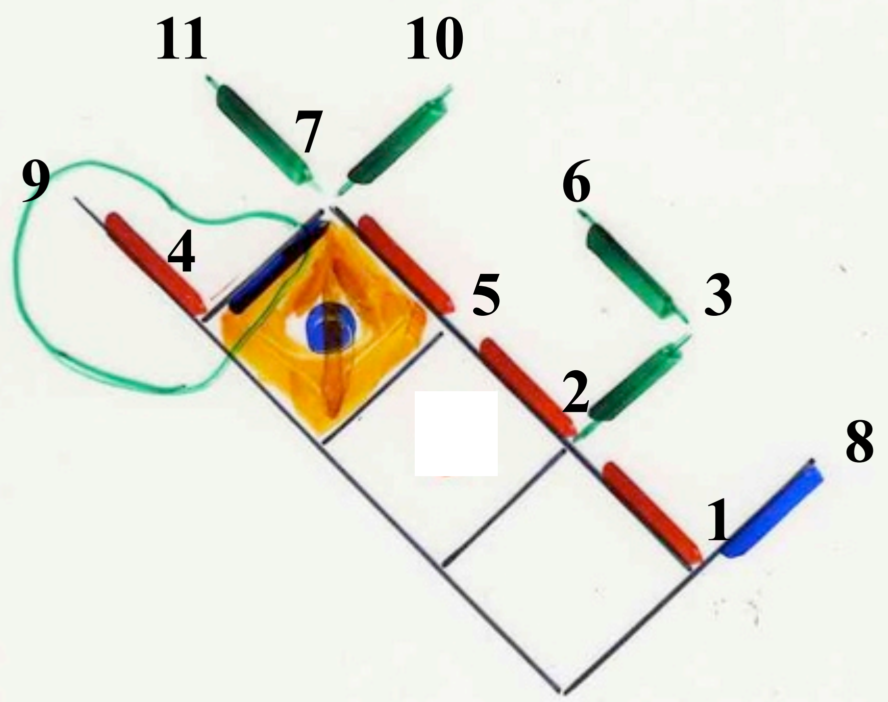
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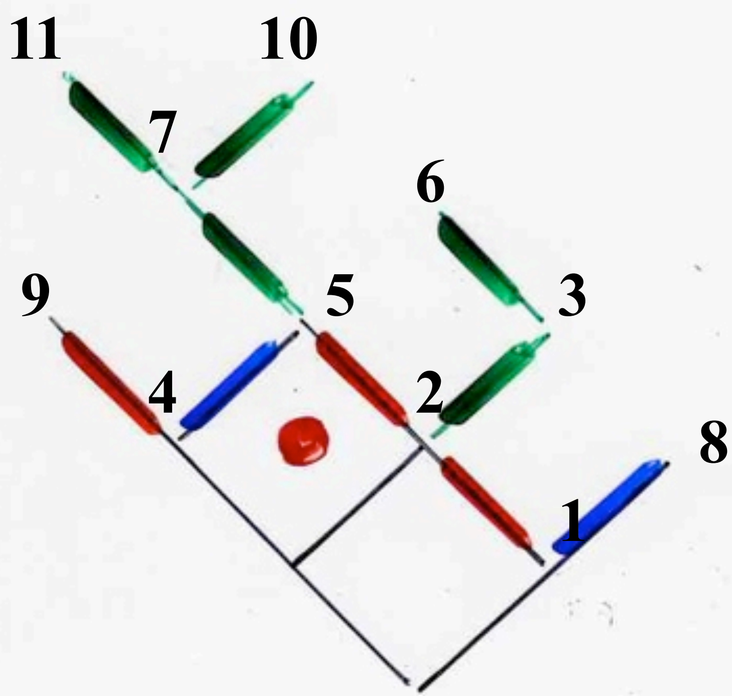

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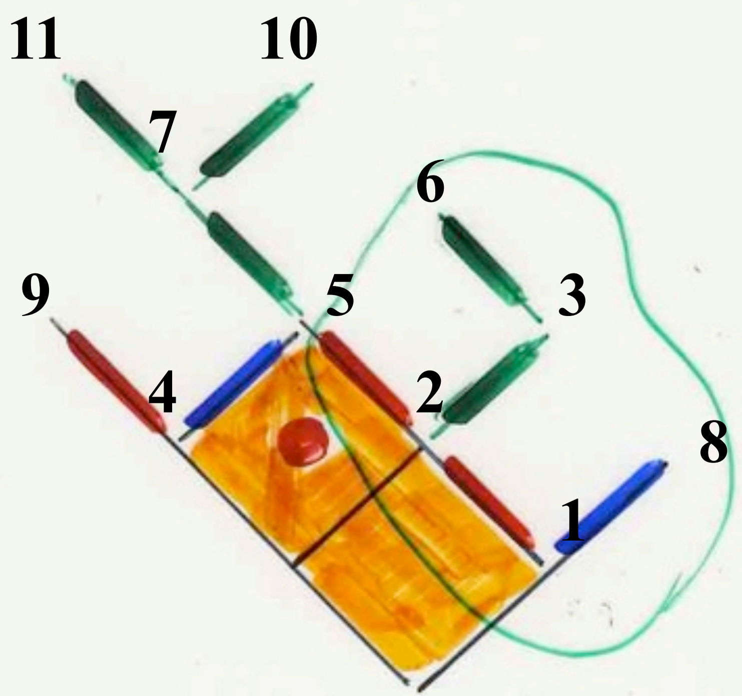
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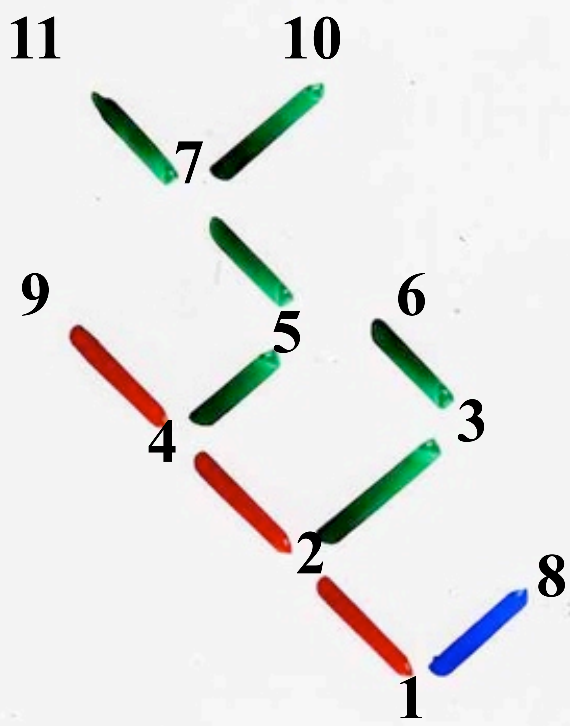
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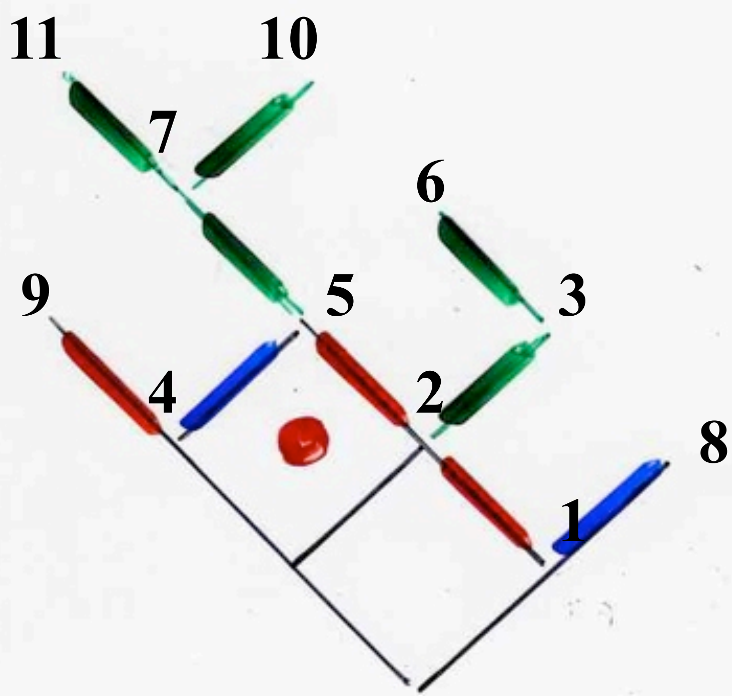
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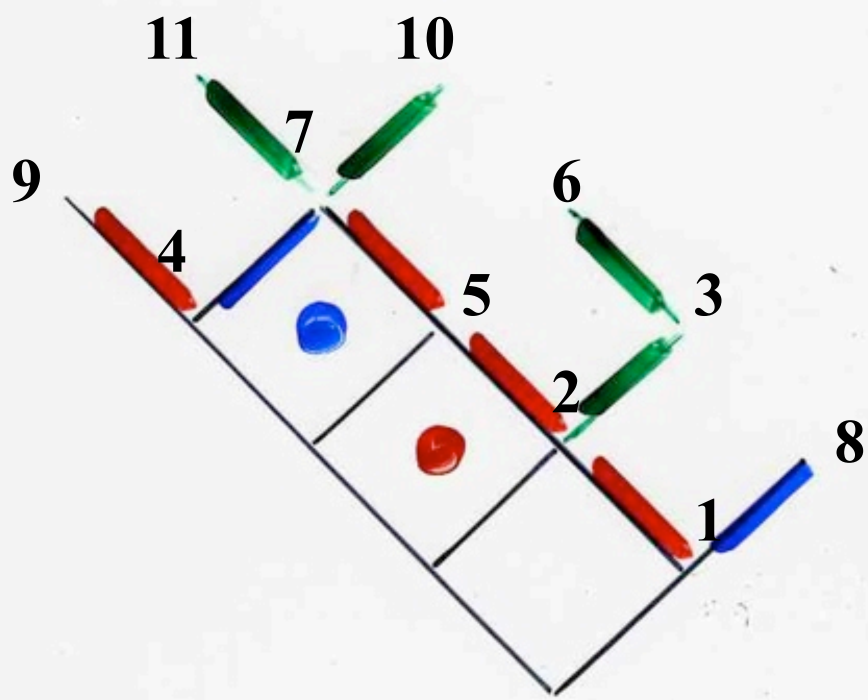
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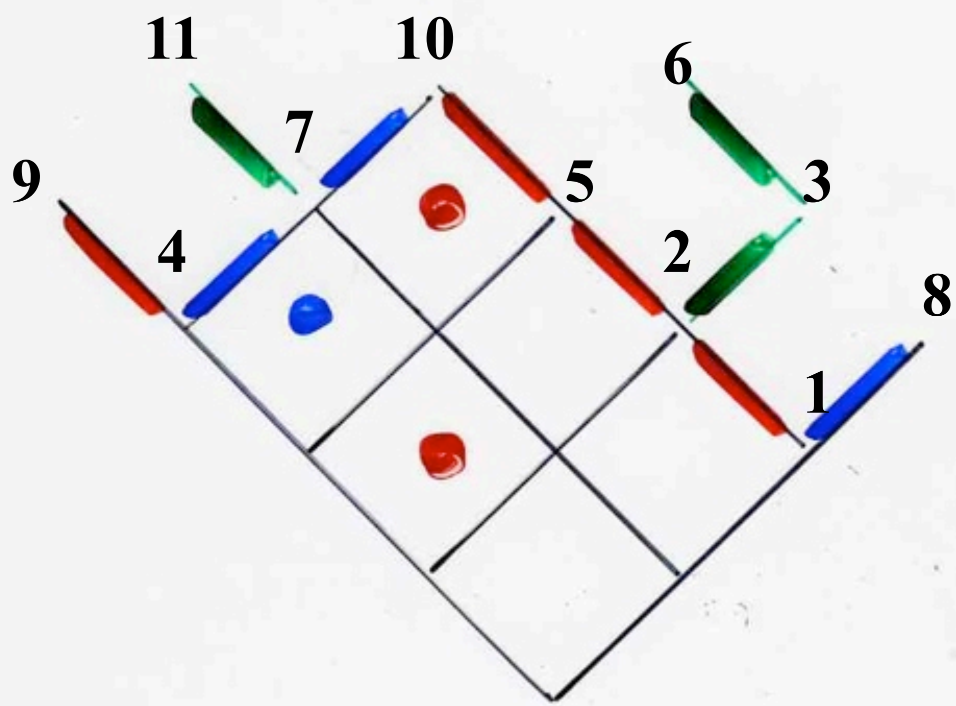

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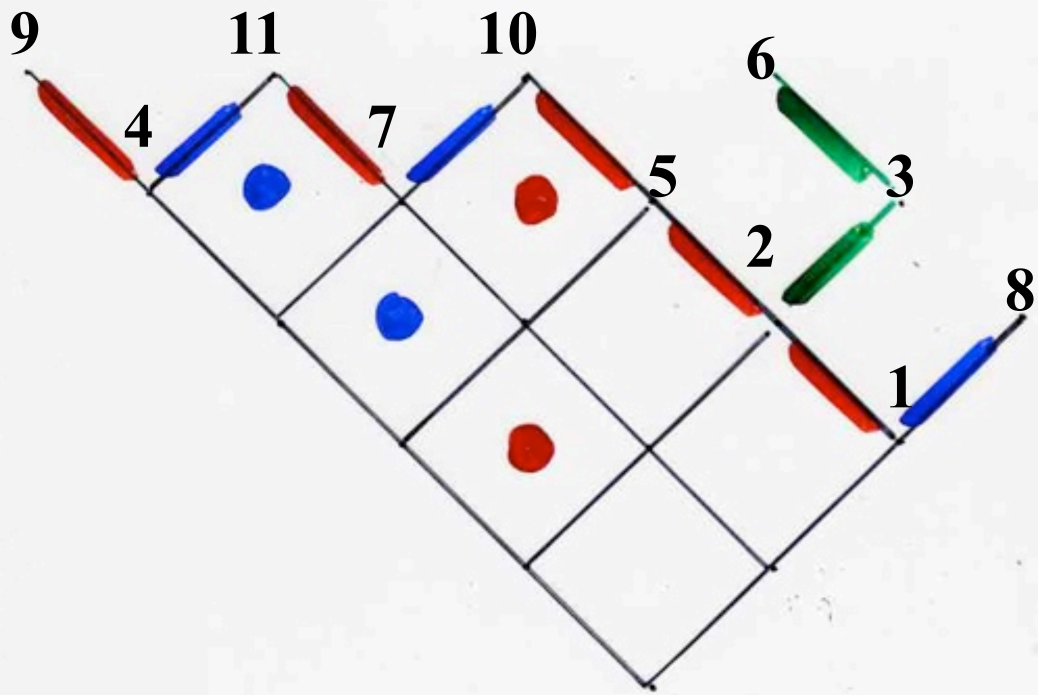

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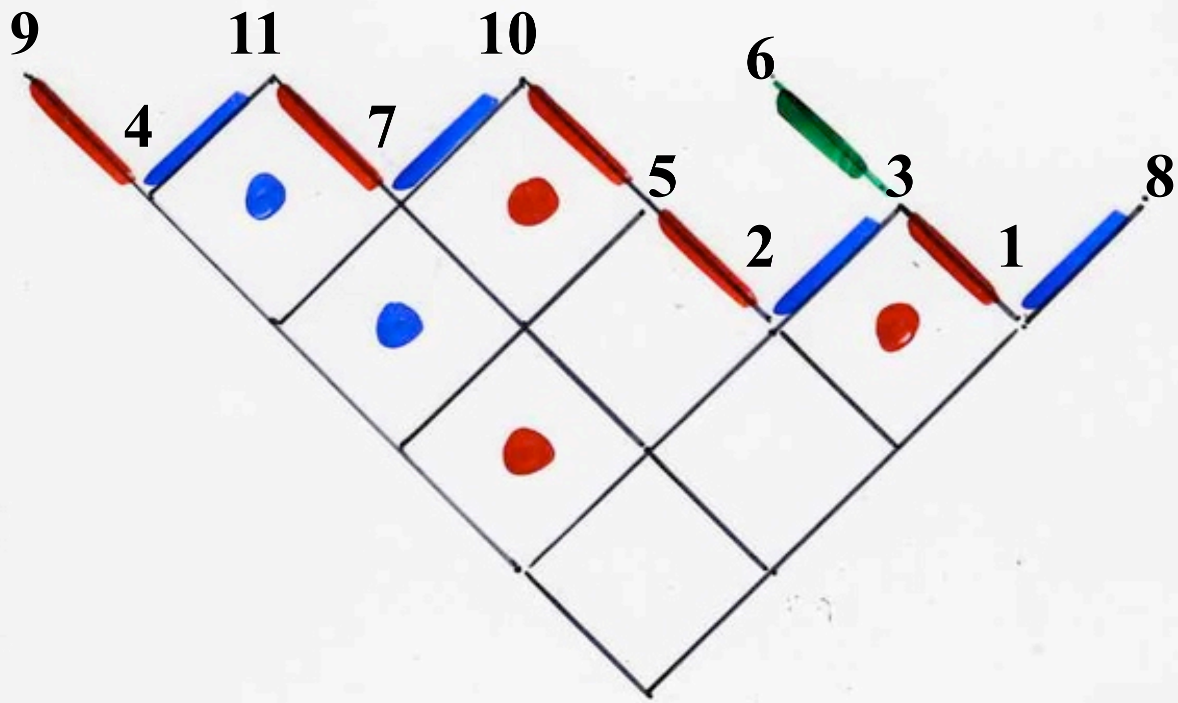
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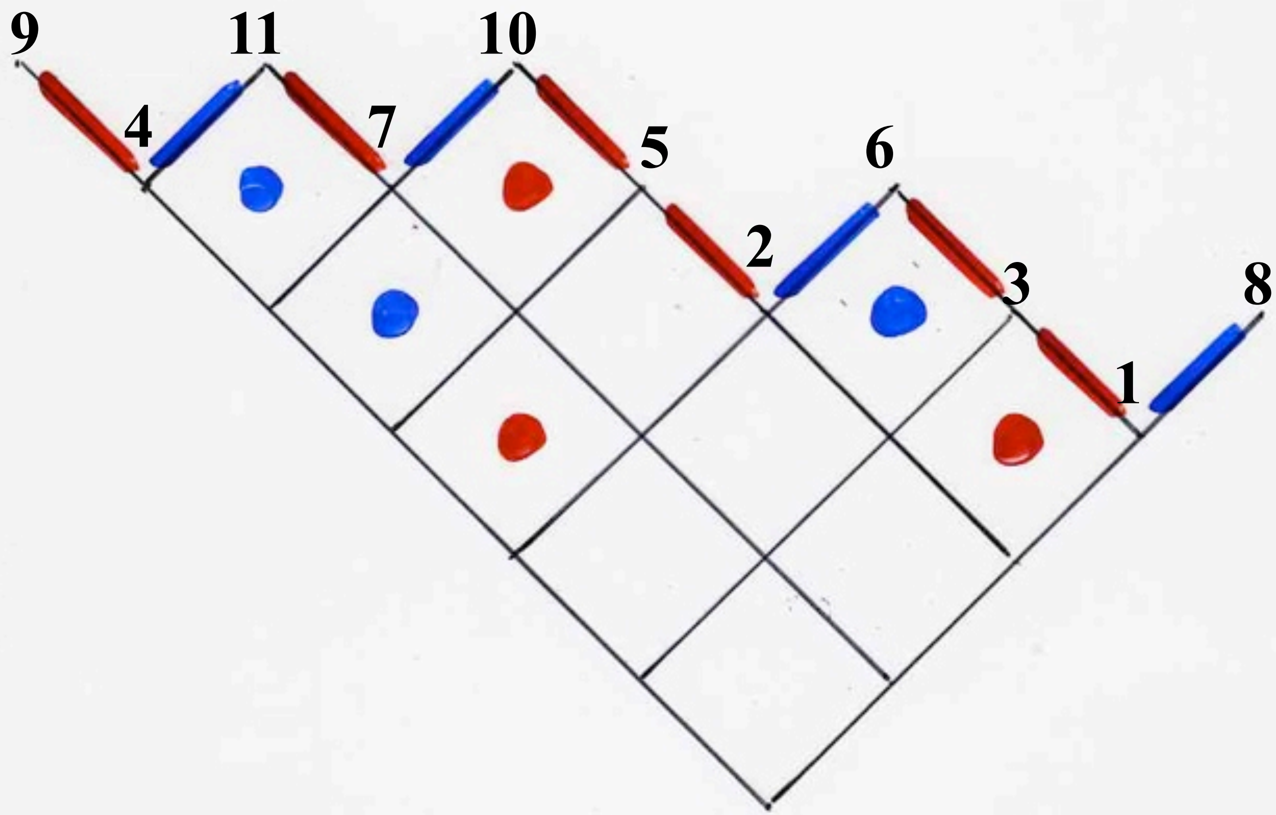
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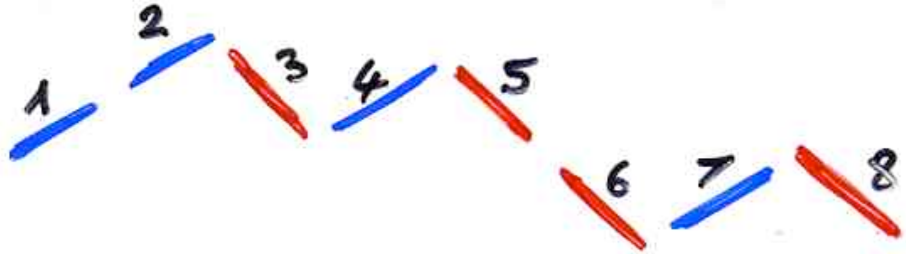
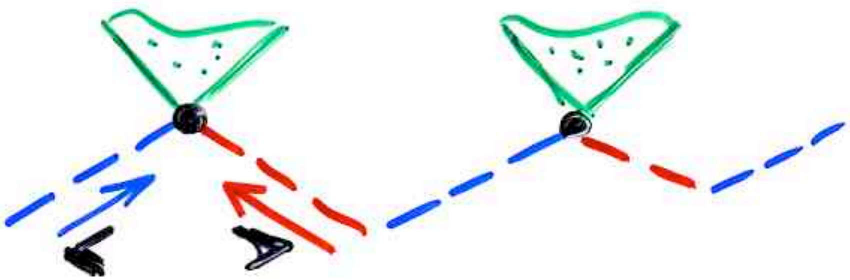
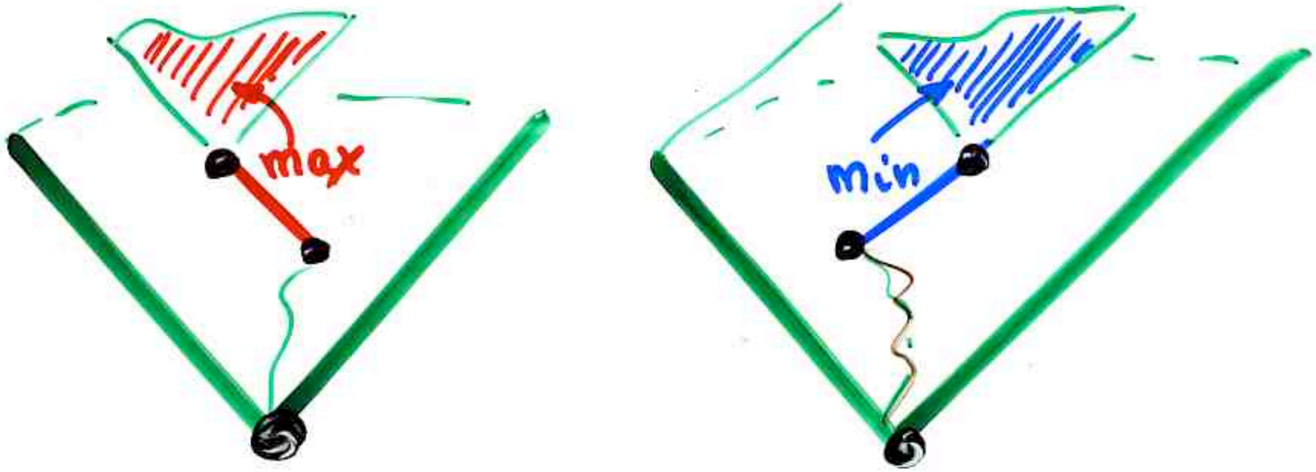
alternative binary trees

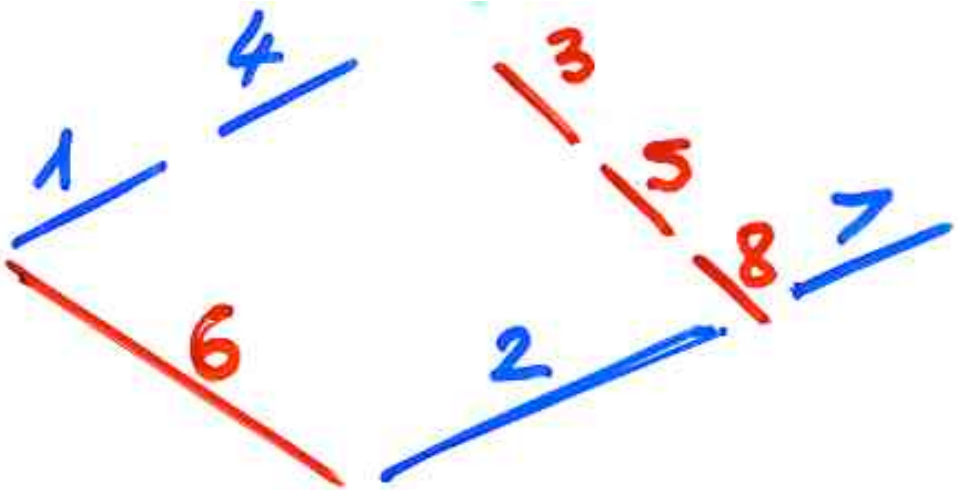


Def. edge-alternative woods

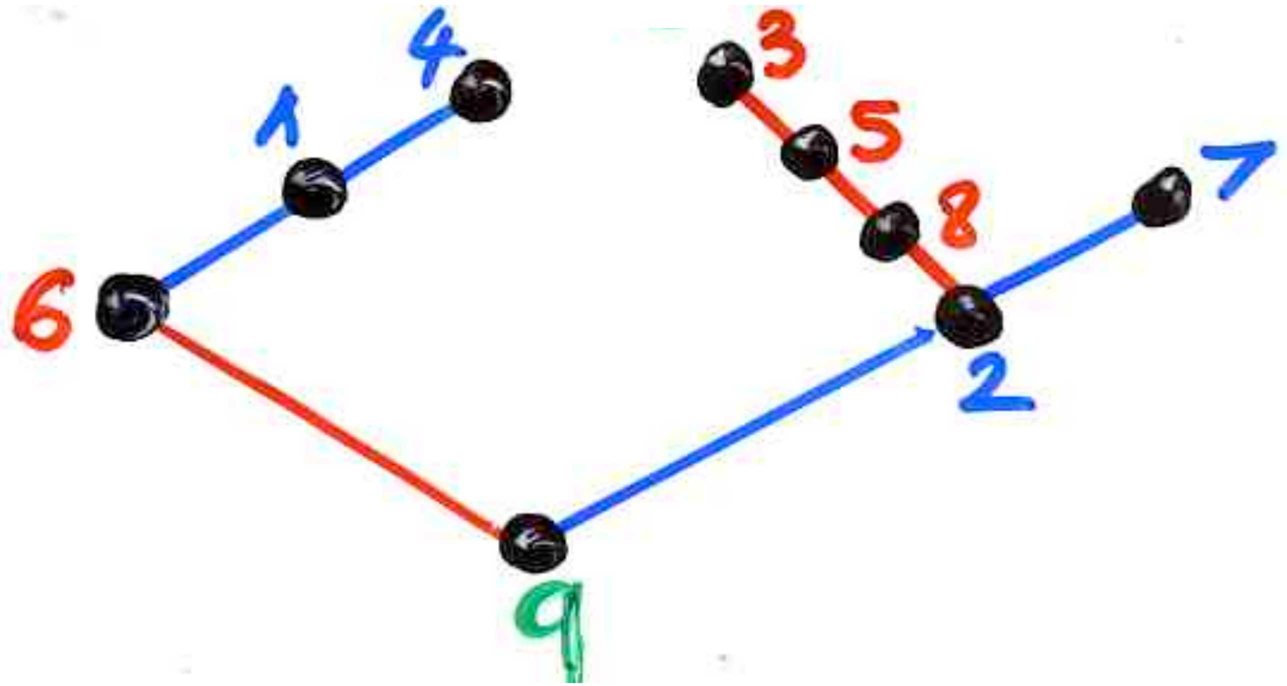


Def edge-alternative binary trees



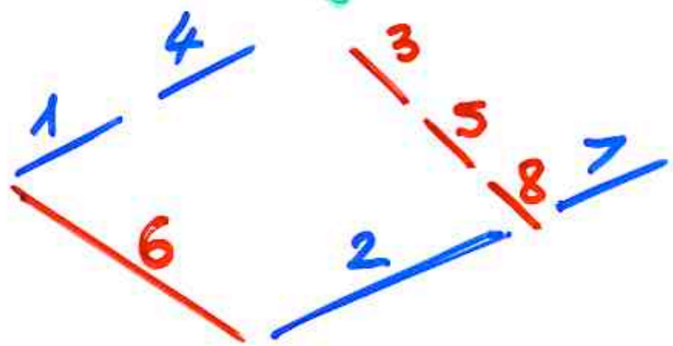


Def alternative binary tree

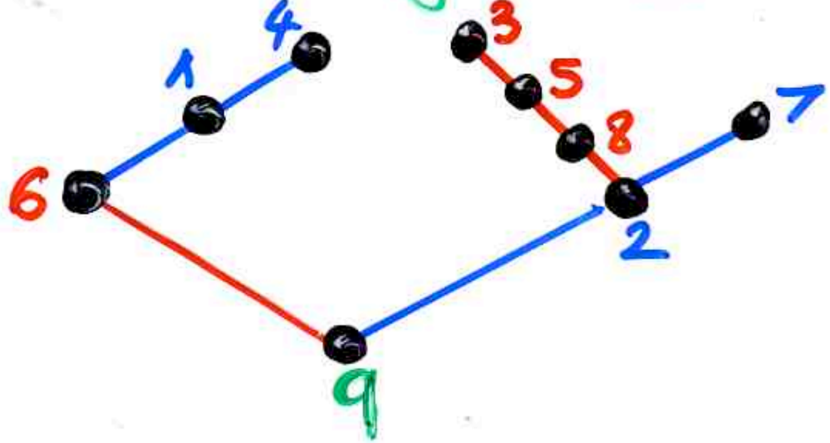


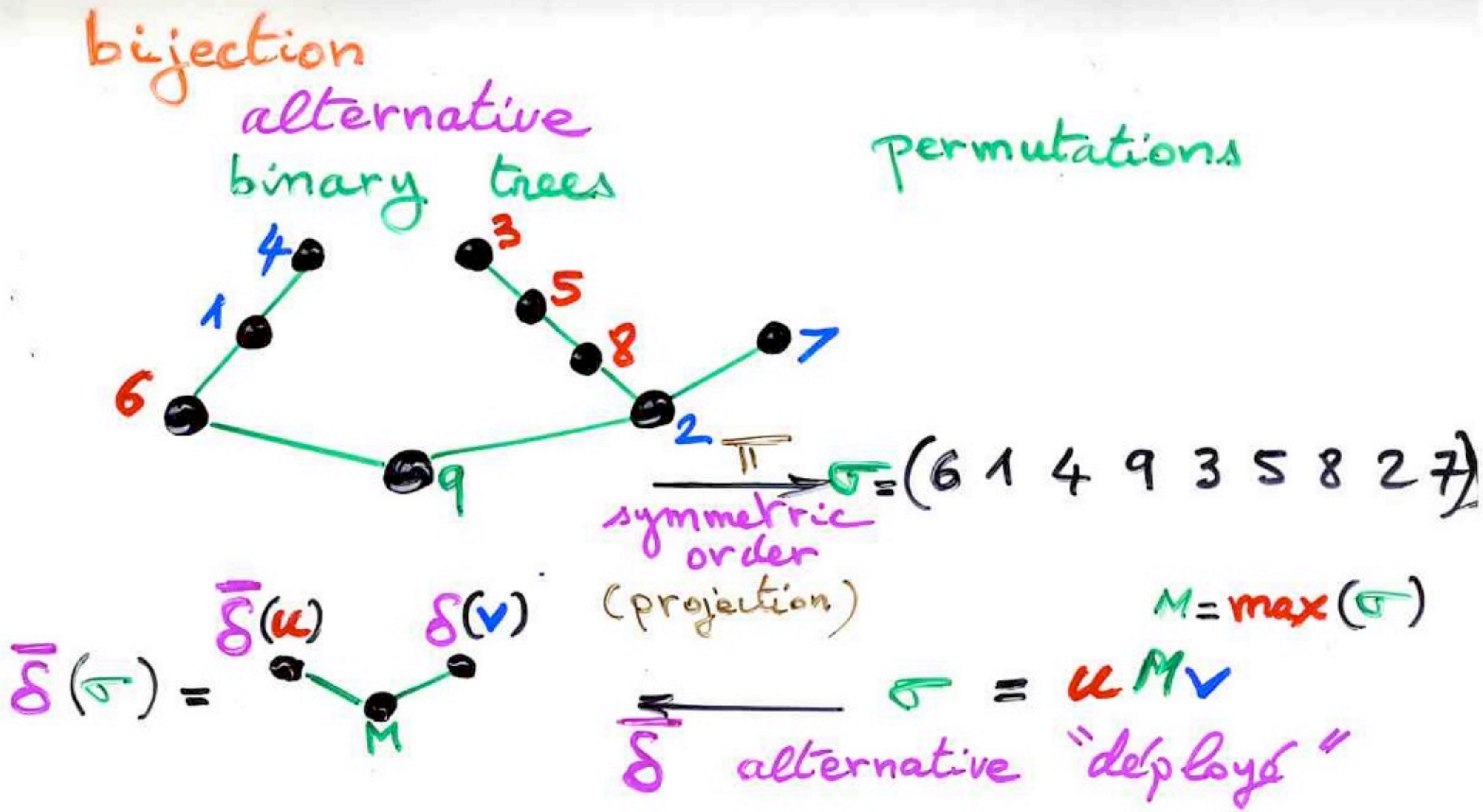
bijection

edge-alternative binary tree



alternative binary tree

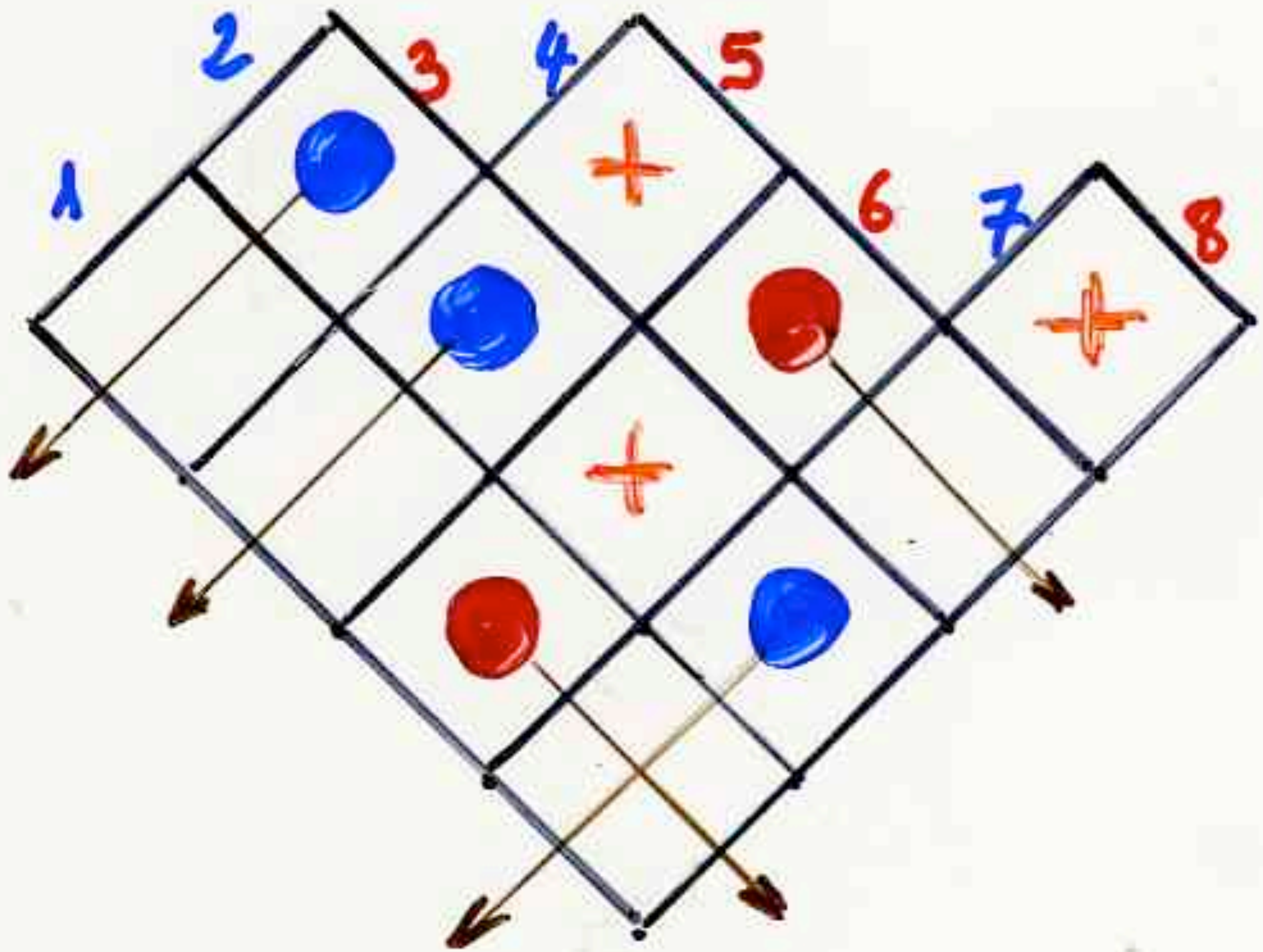



bijection
alternative binary trees
permutations
4 1 6 3 5 8 2 7 9
π
σ = (6 1 4 9 3 5 8 2 7)
symmetric order (projection)
M = max(σ)
δ̄(u) δ(v)
δ̄(σ) =
M
σ = u M v
δ̄ alternative "déployé"

"hook length"
formula
same as for
increasing
binary tree

$$\frac{n!}{\prod_{x} h_x}$$

jeu de taquin
for
alternative
binary
trees



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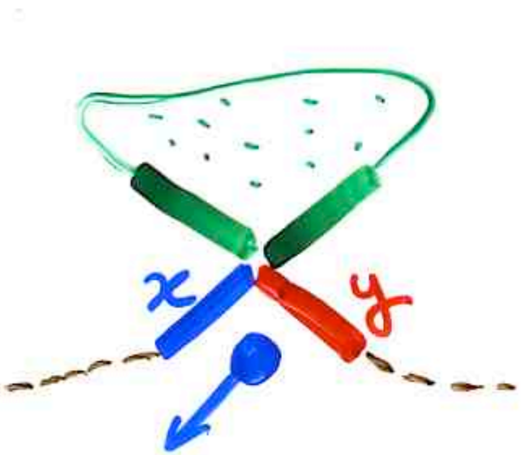
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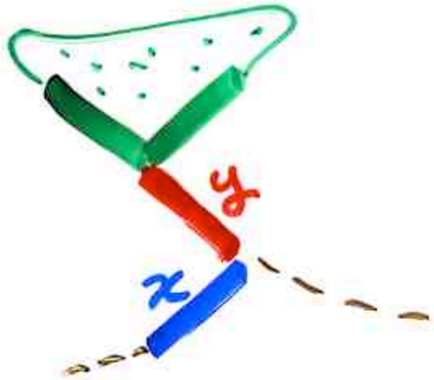
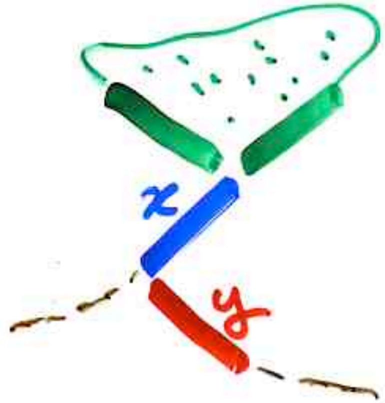
y


y
x


x
y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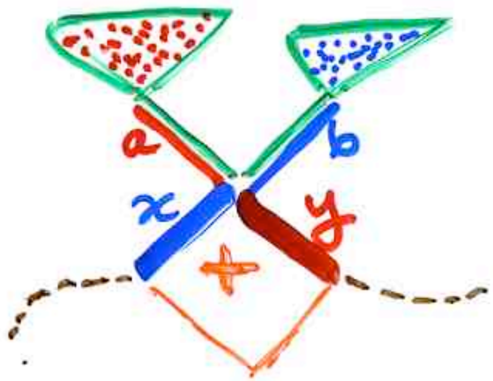
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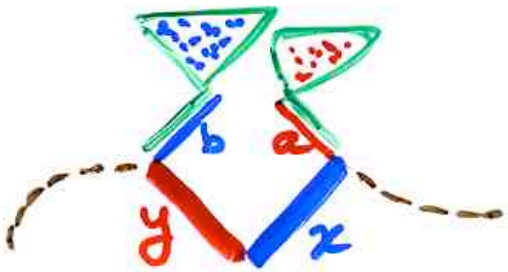
b
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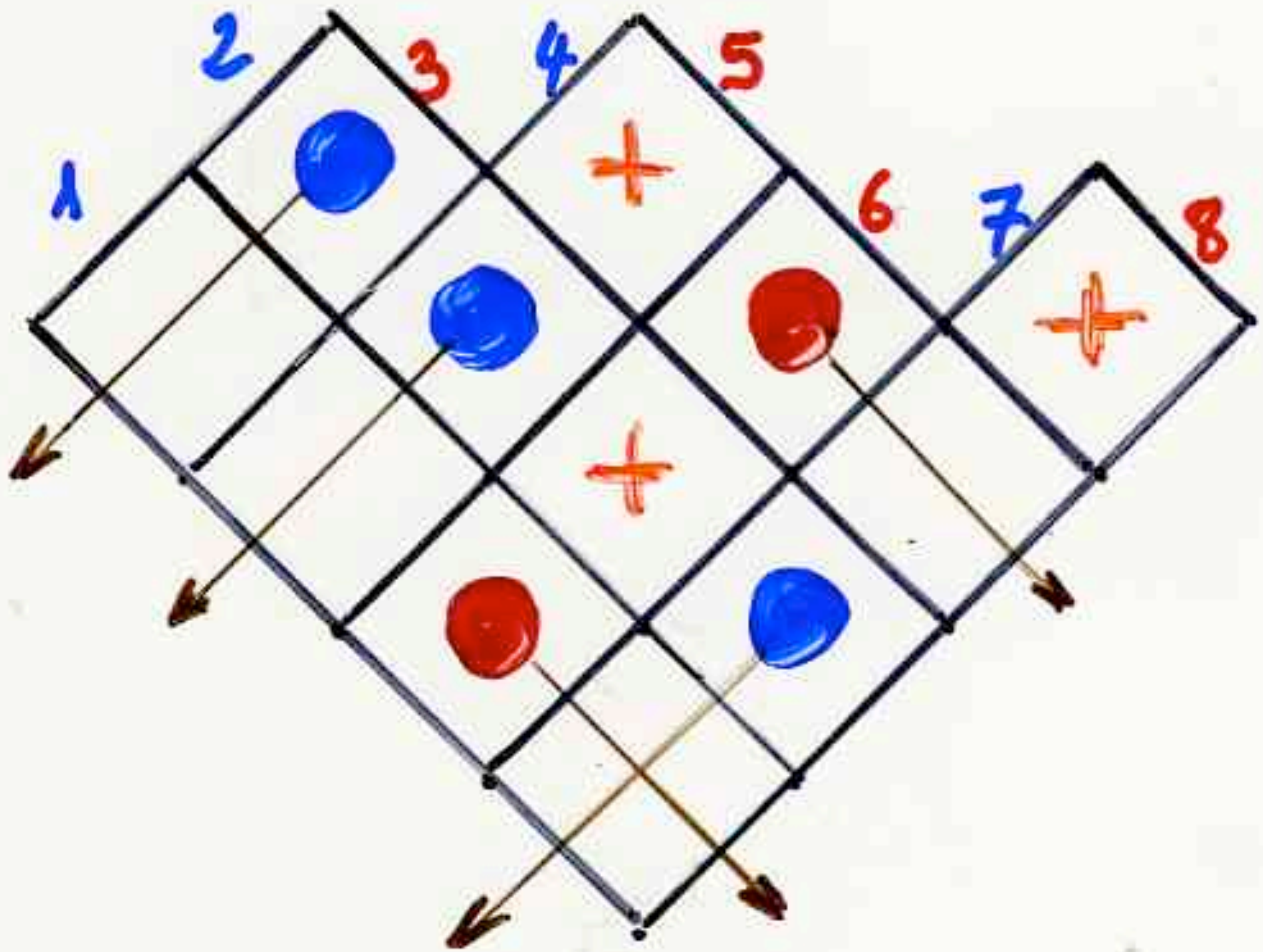
y
x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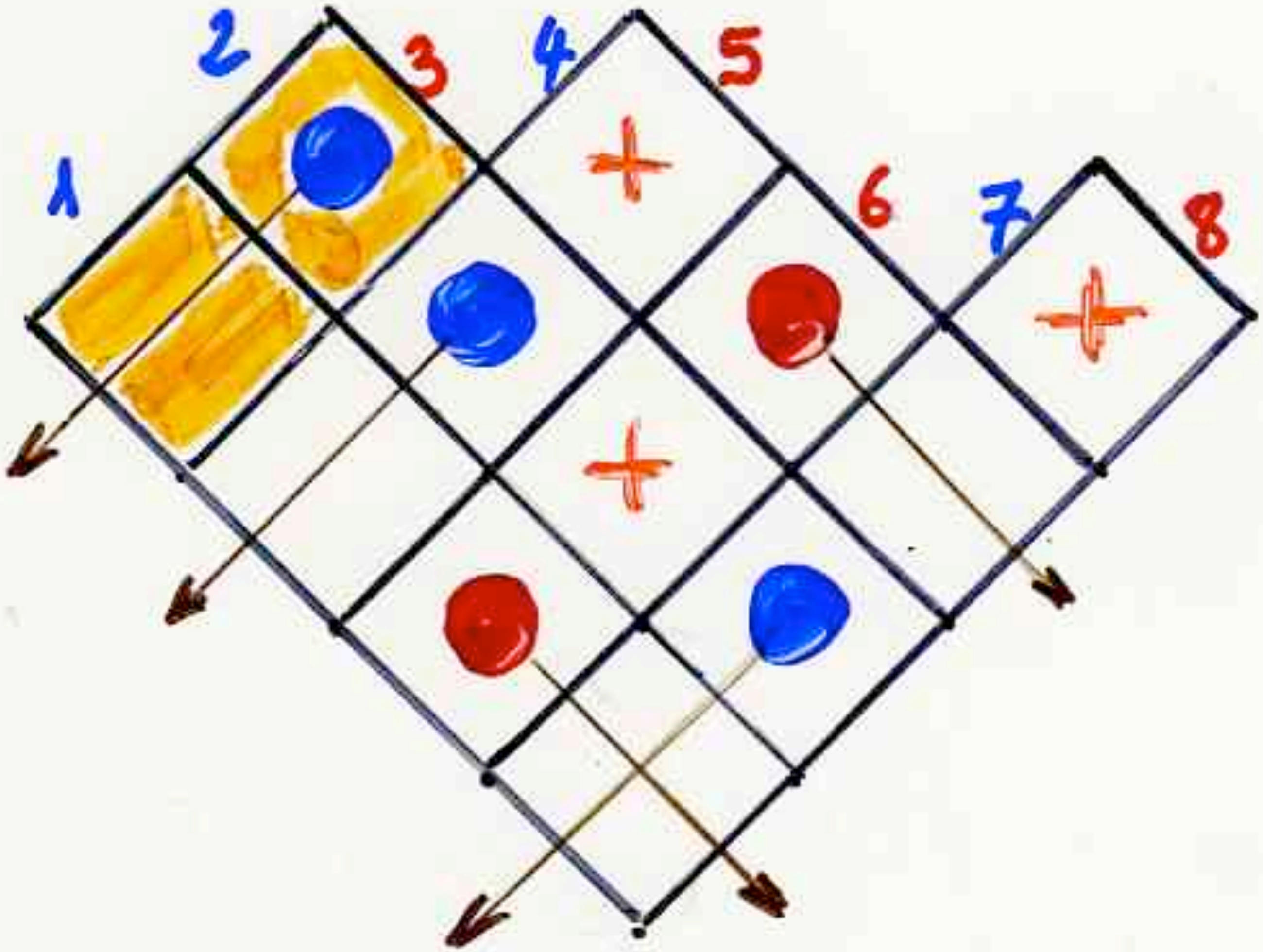
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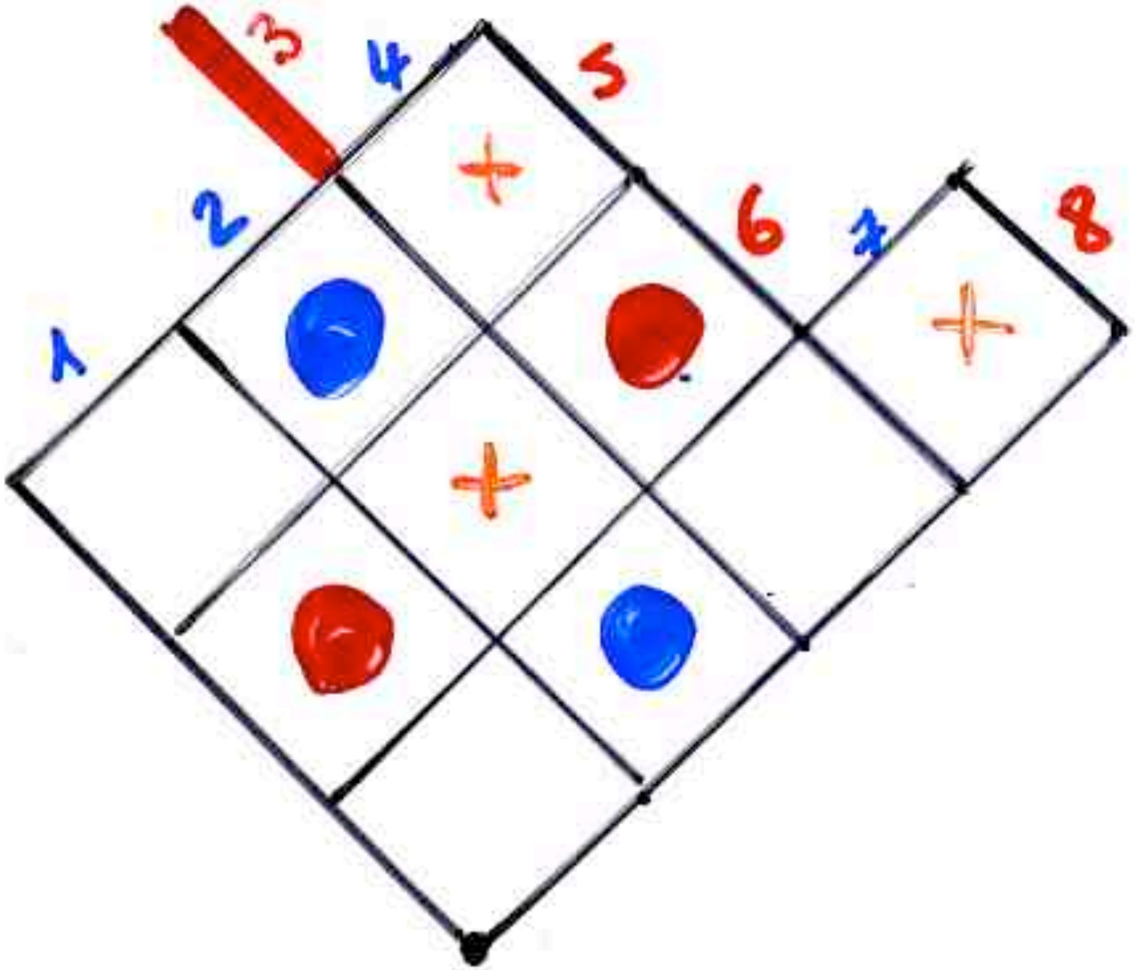
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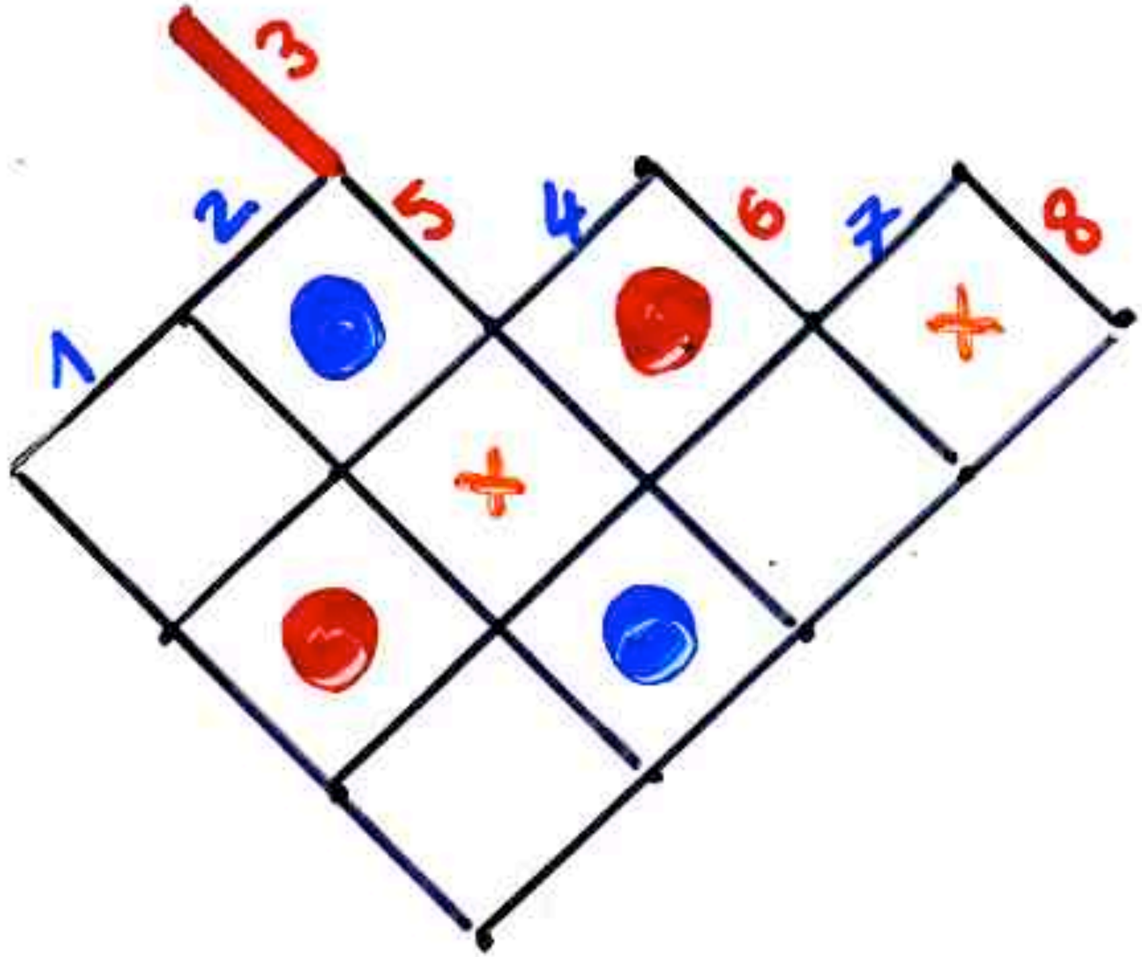
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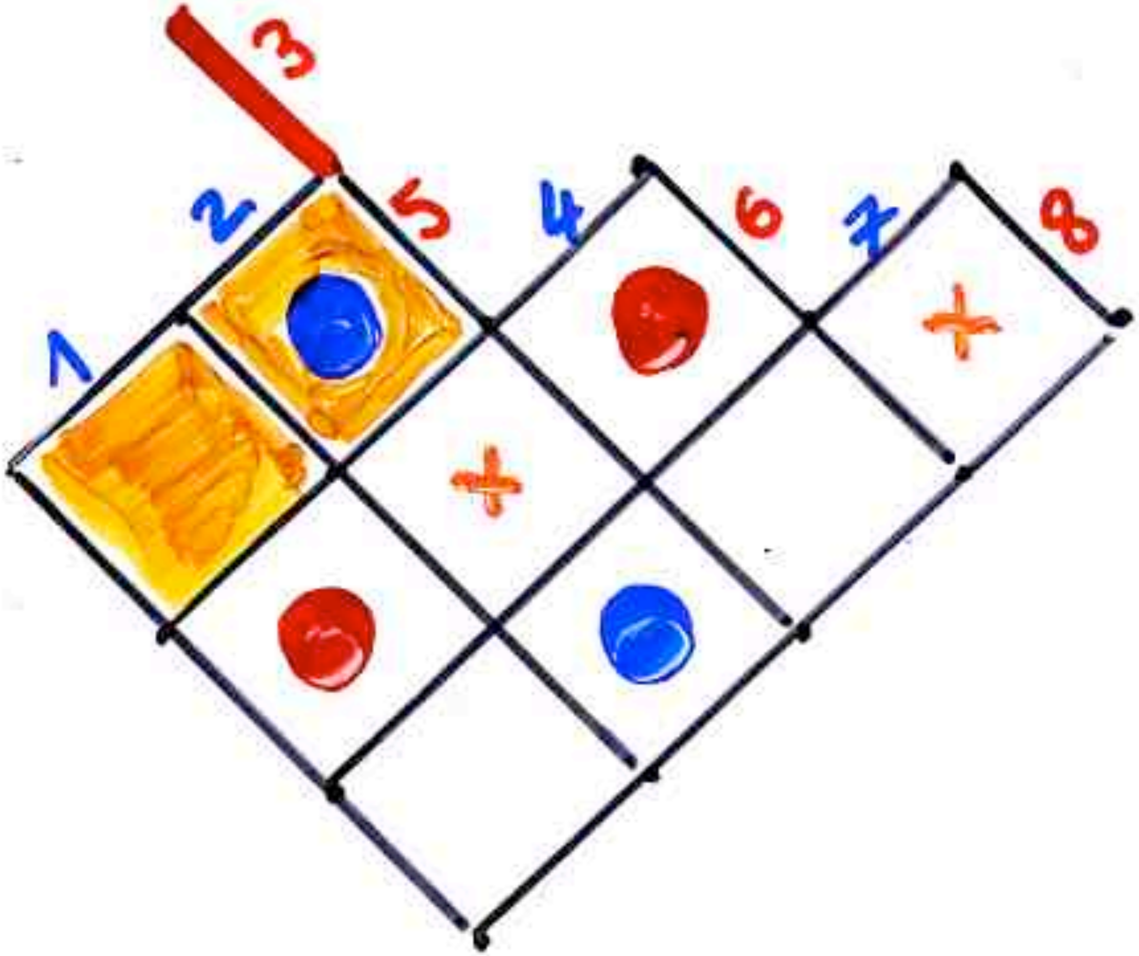

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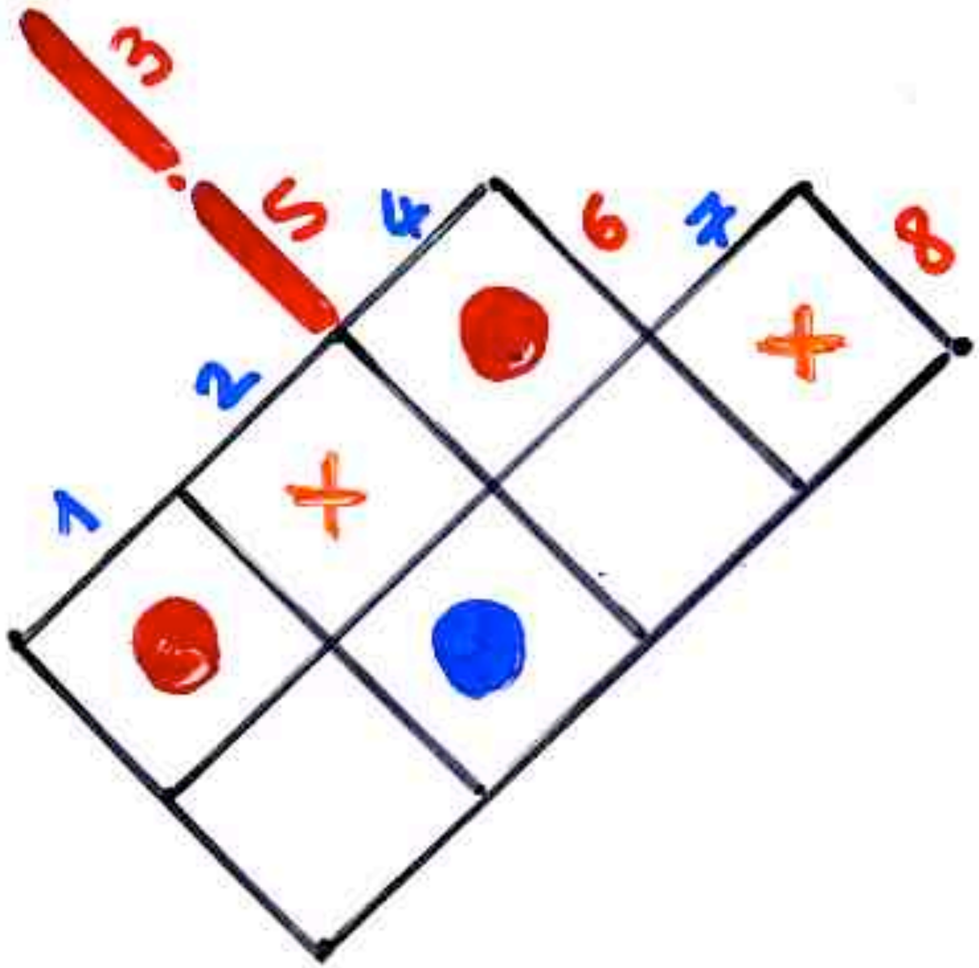

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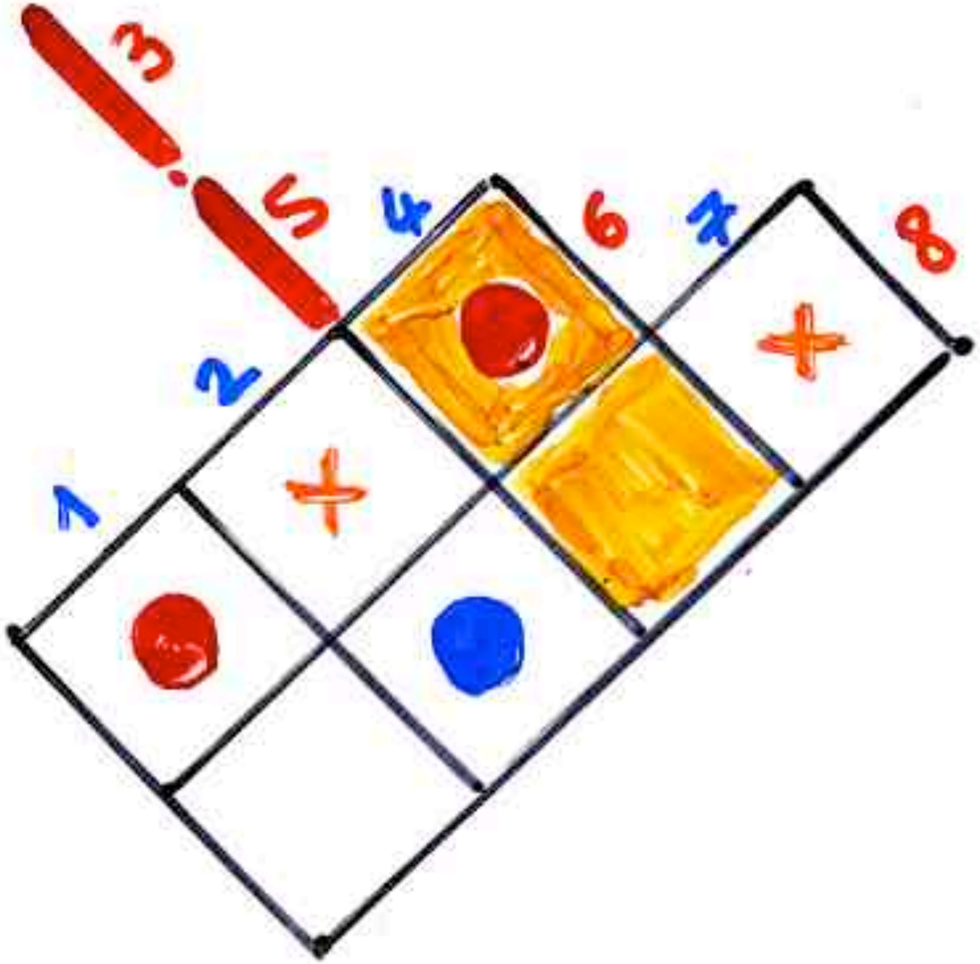
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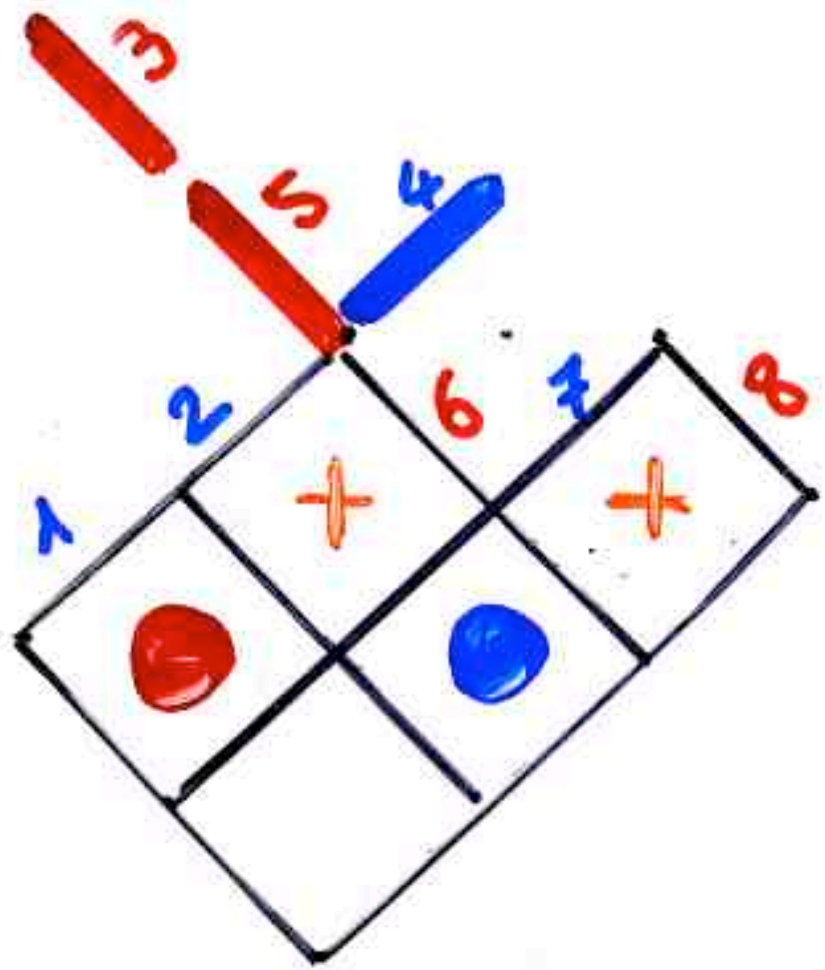
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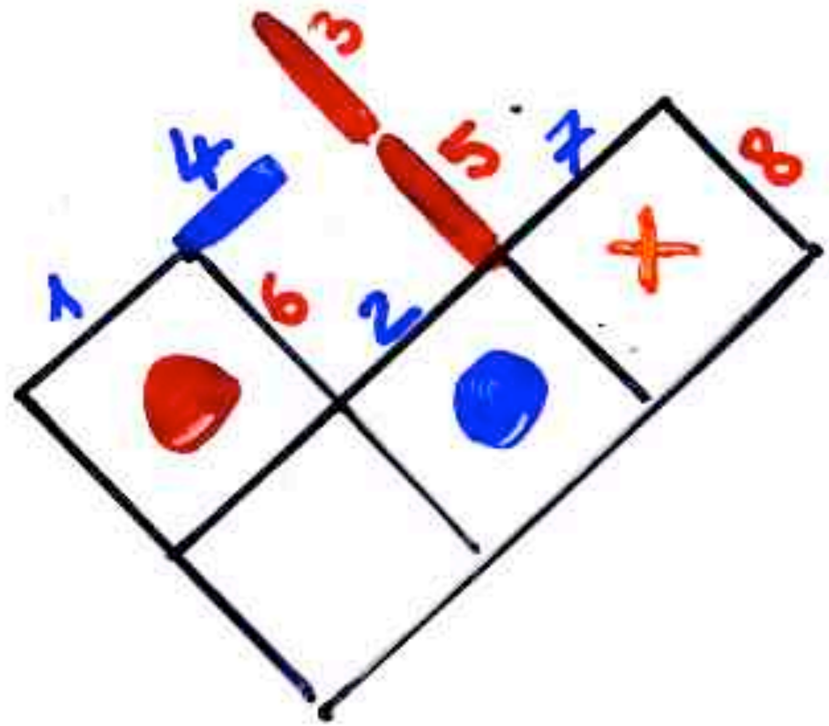

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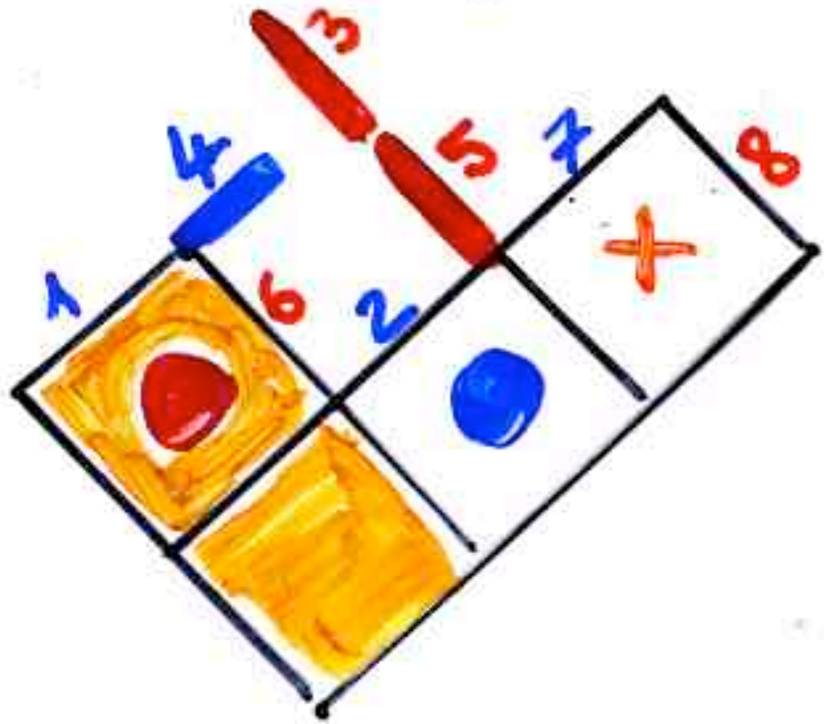

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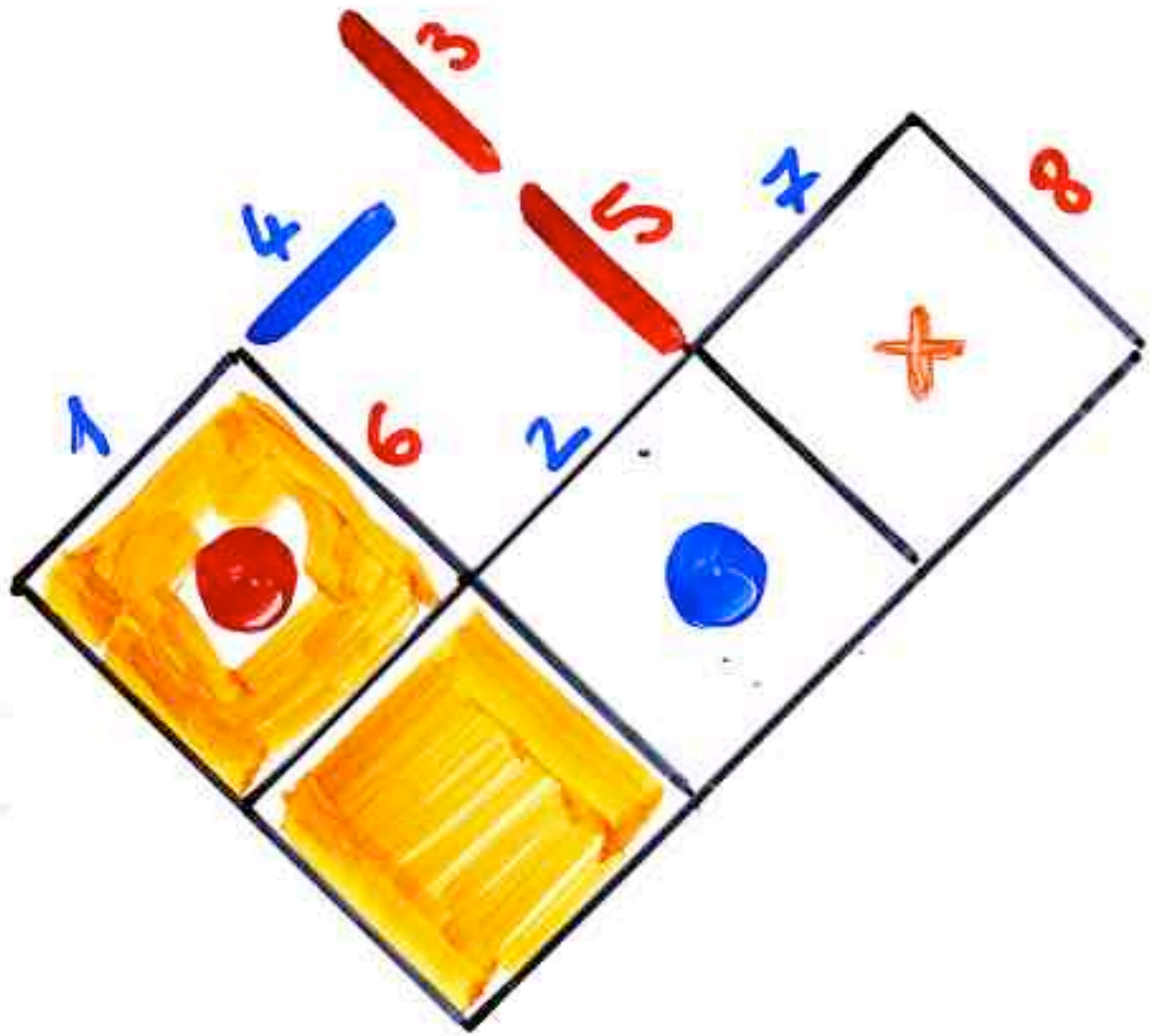

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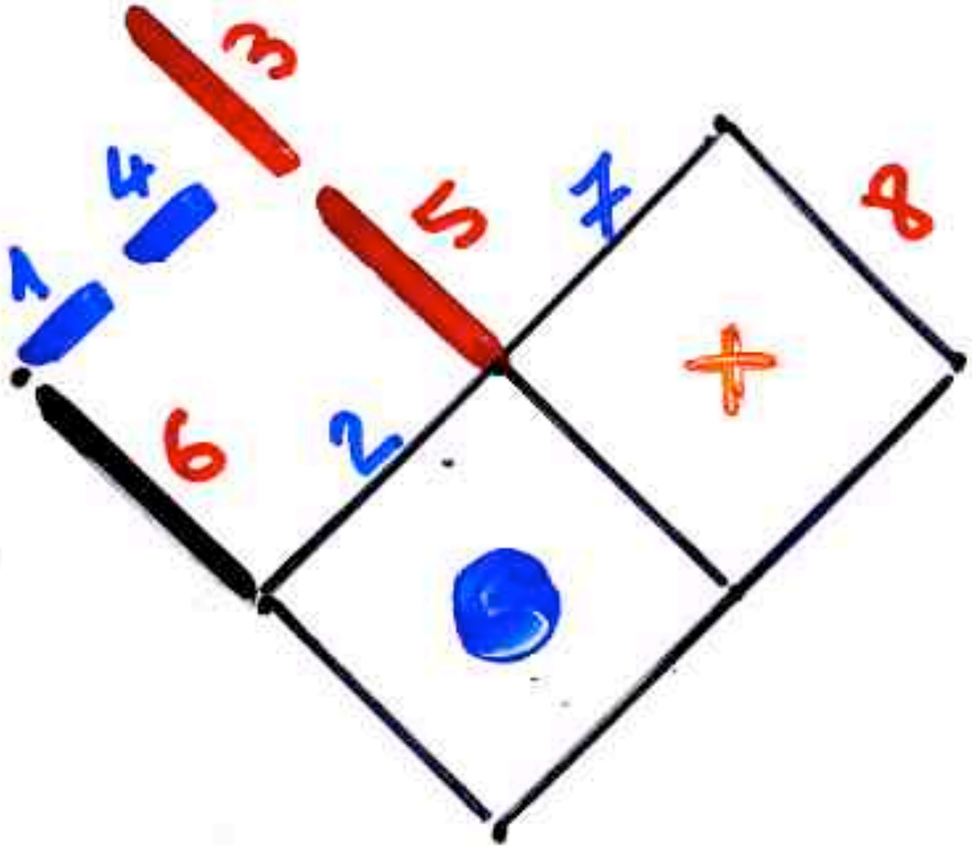
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6
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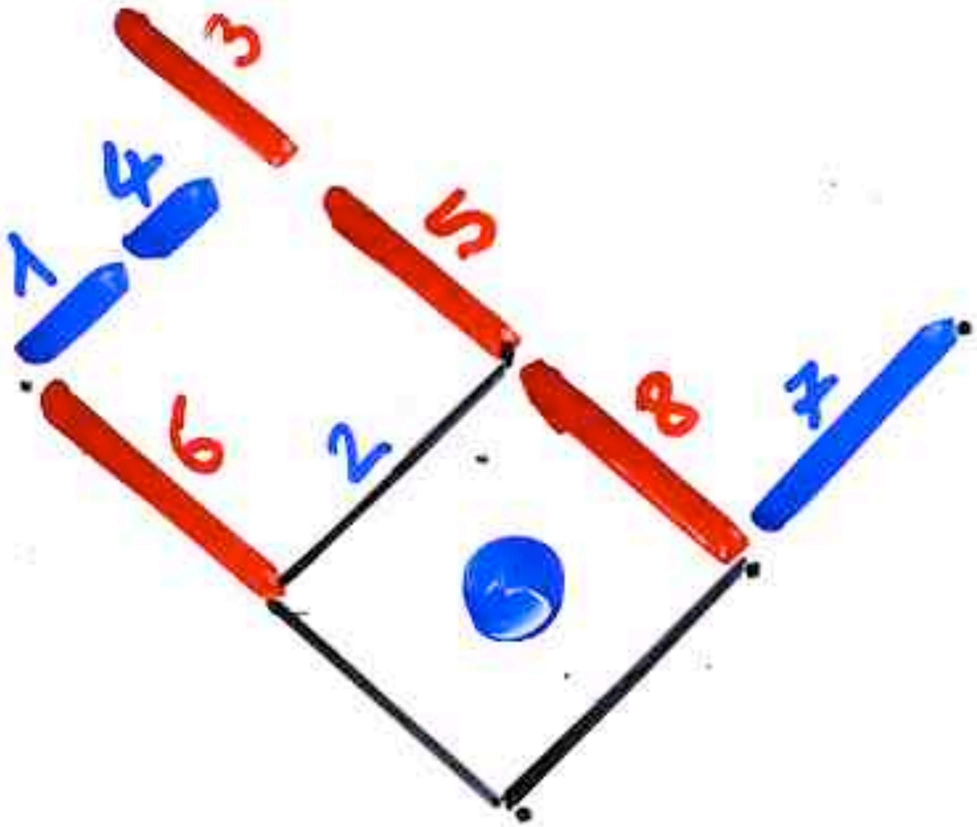
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8
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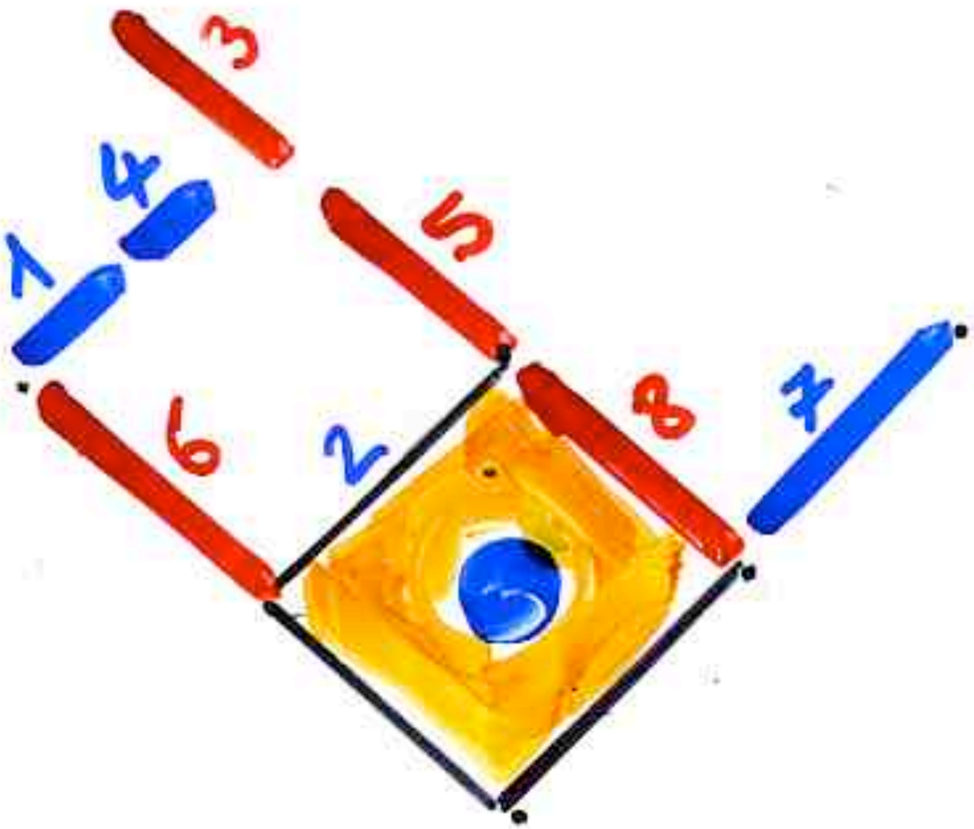

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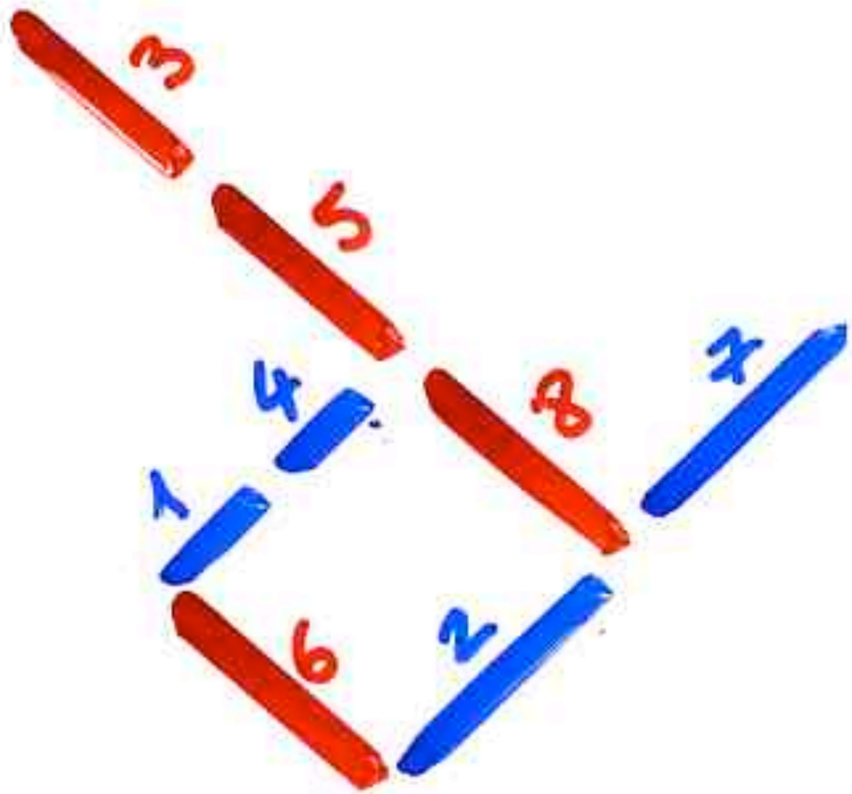
8
7



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6
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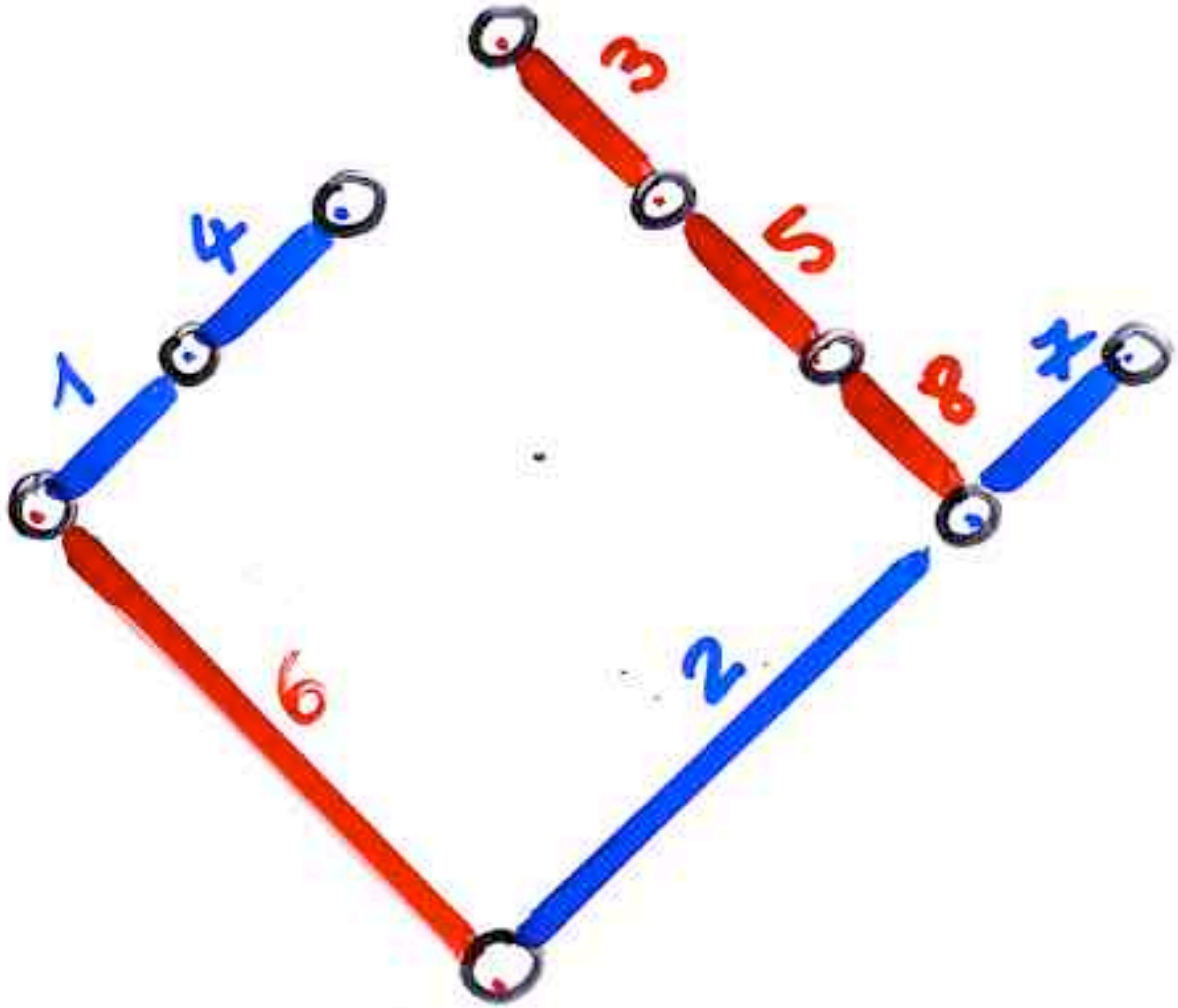

3
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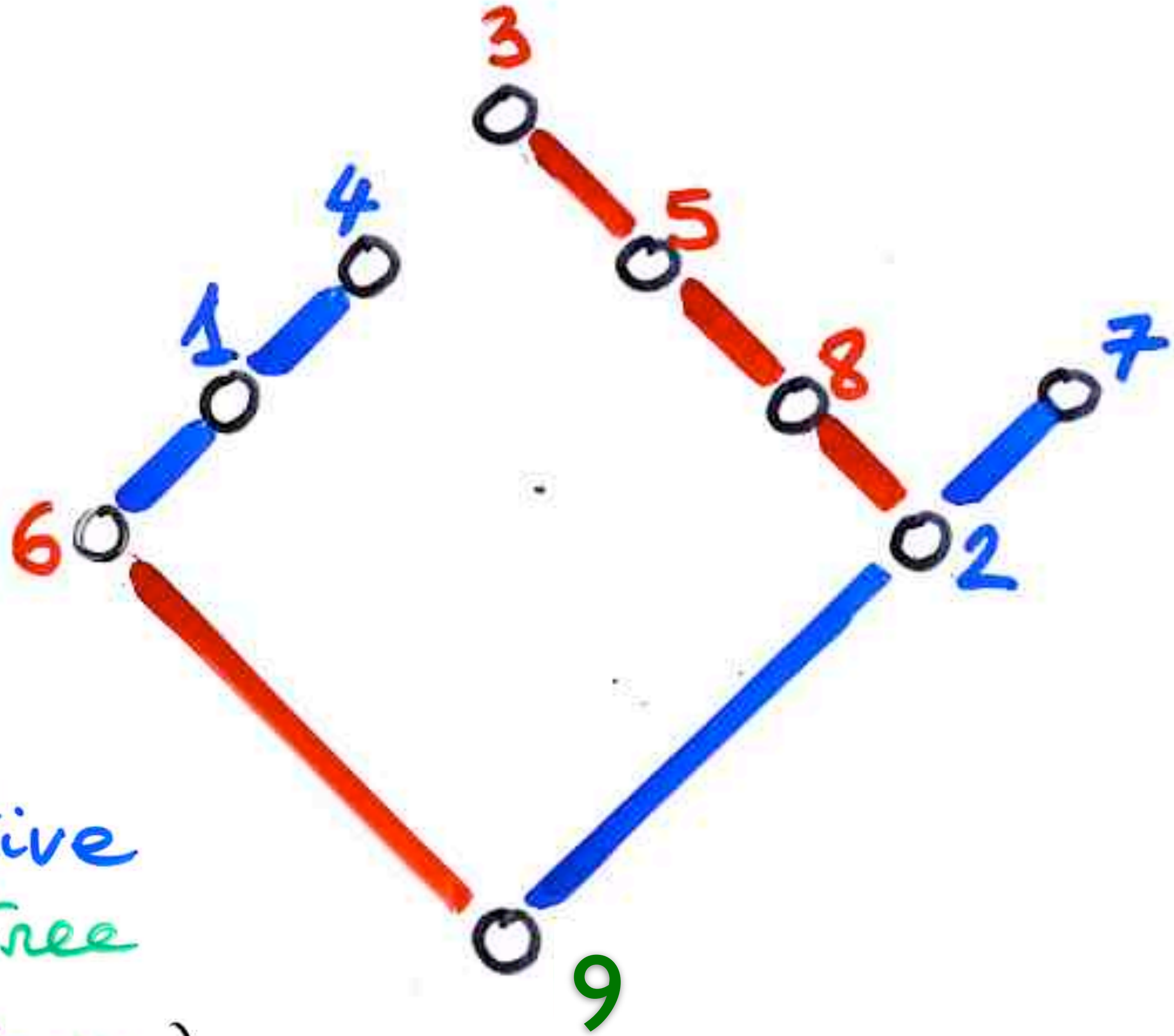
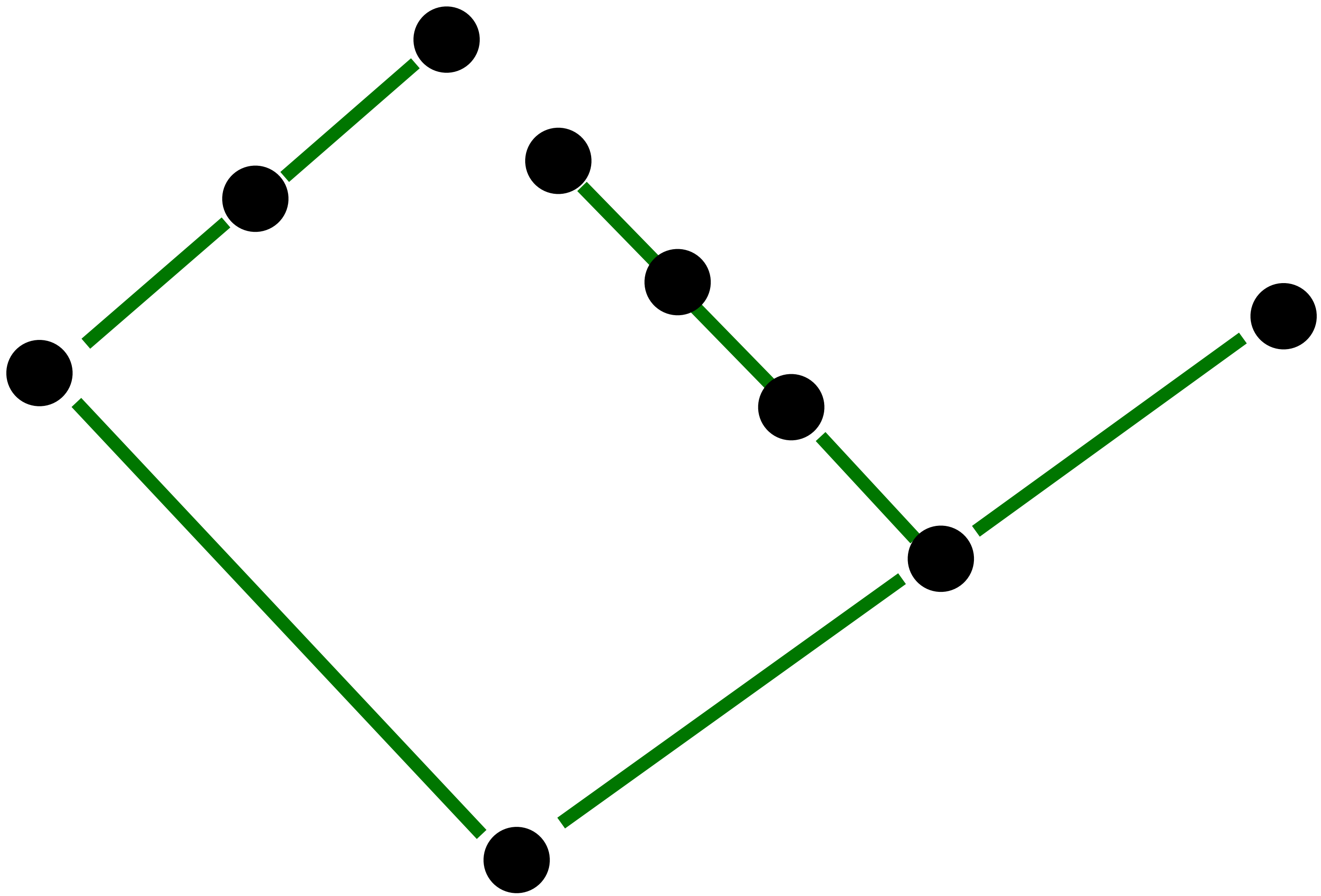
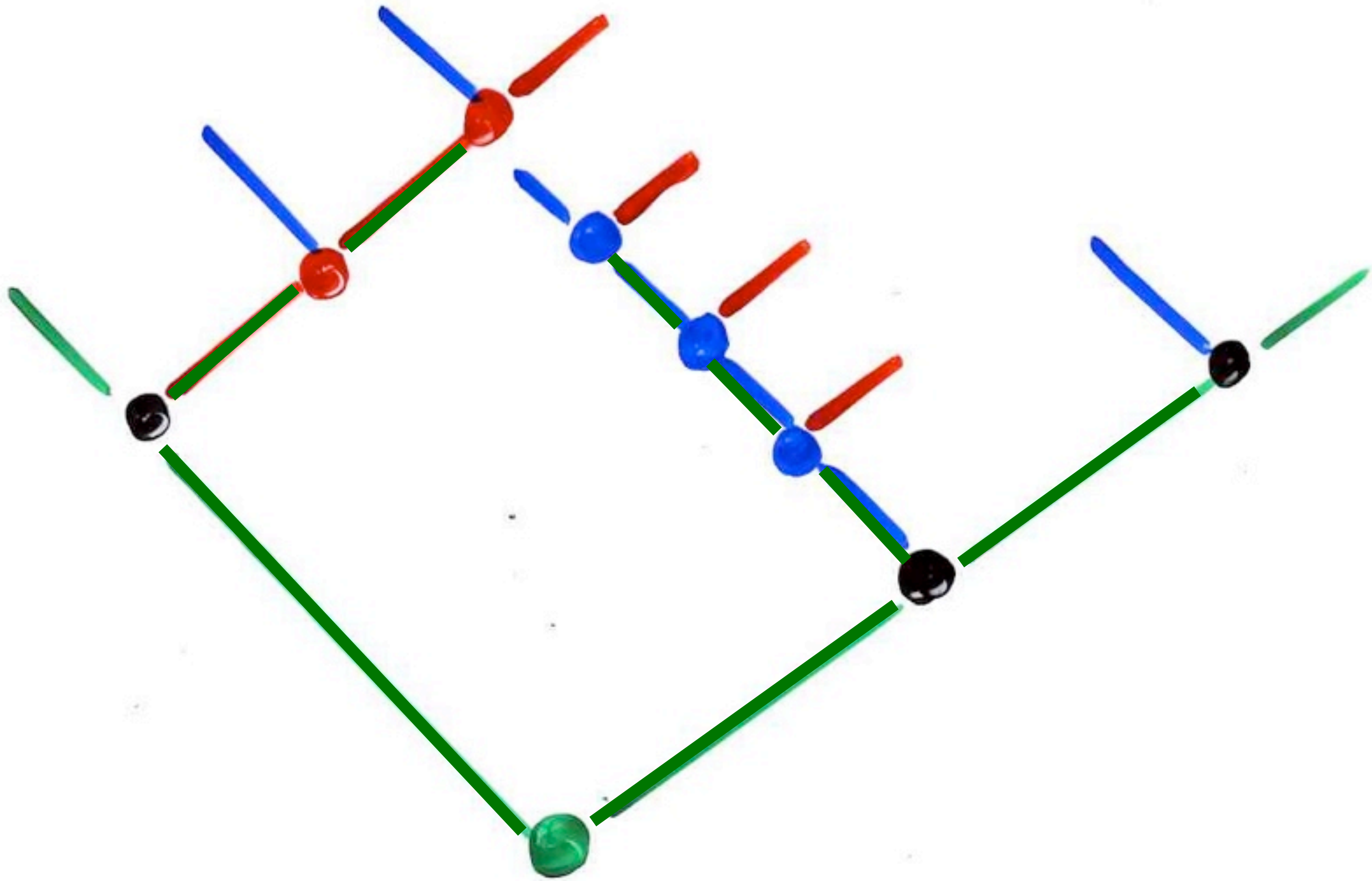
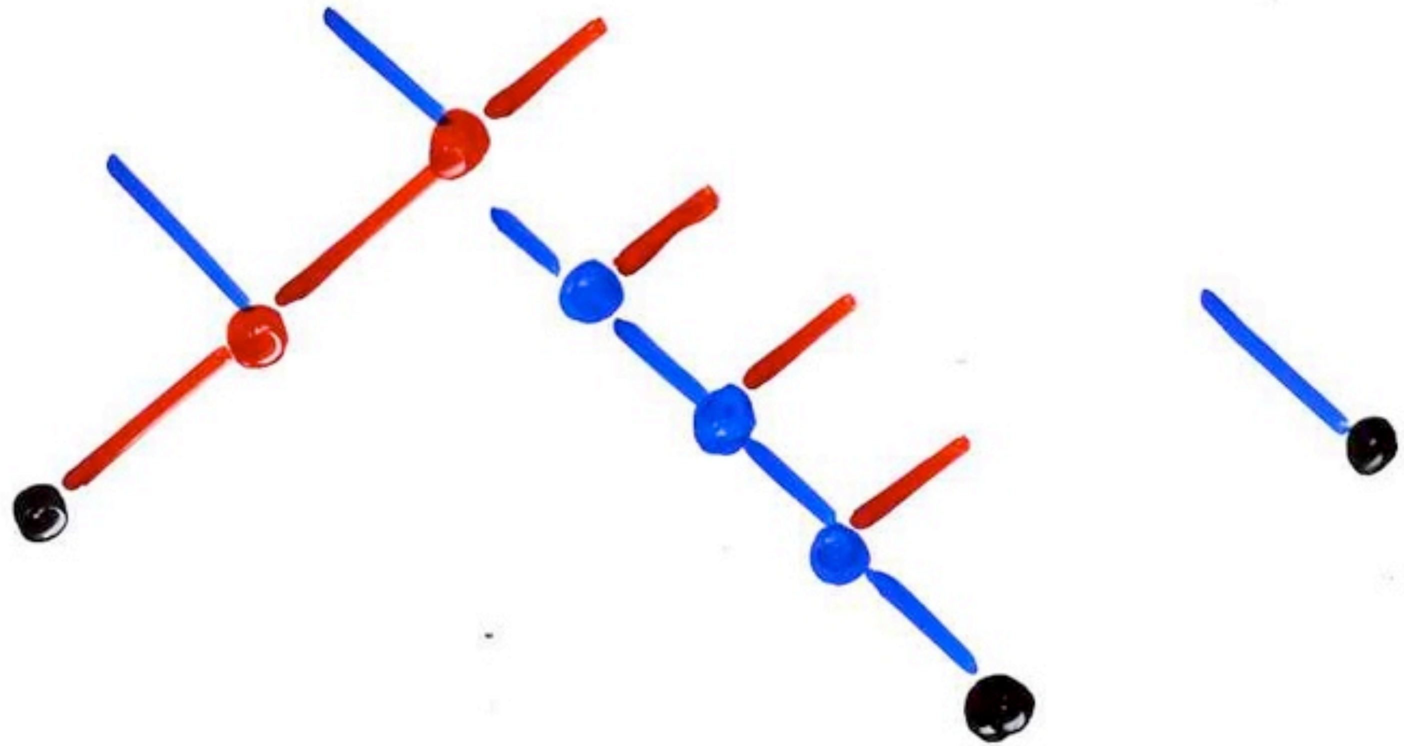
8
7
6
2

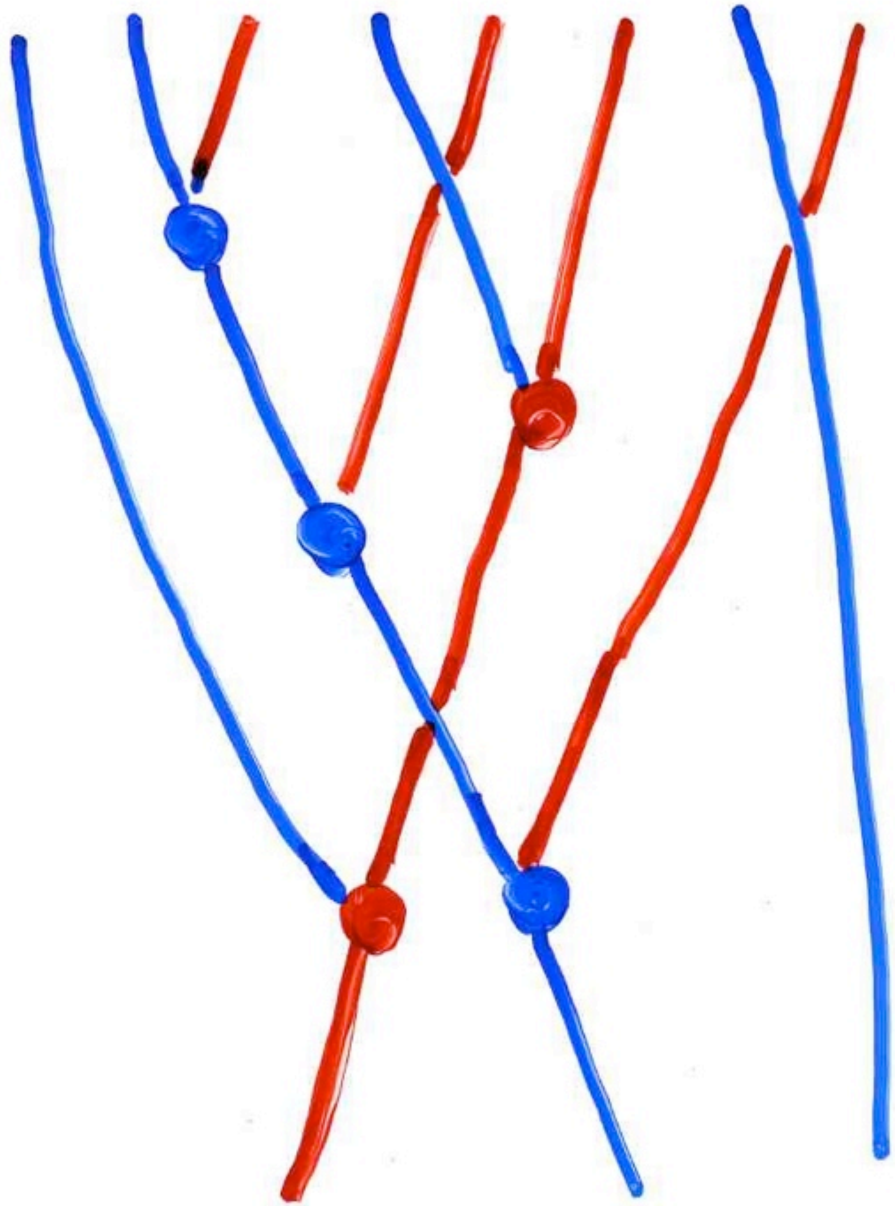


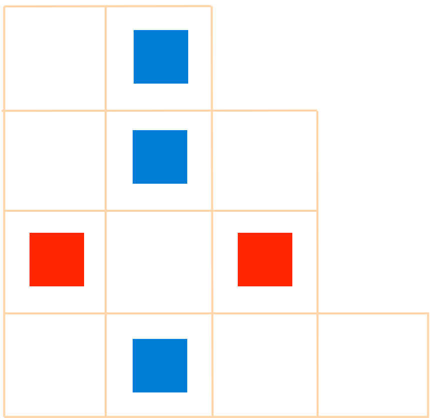
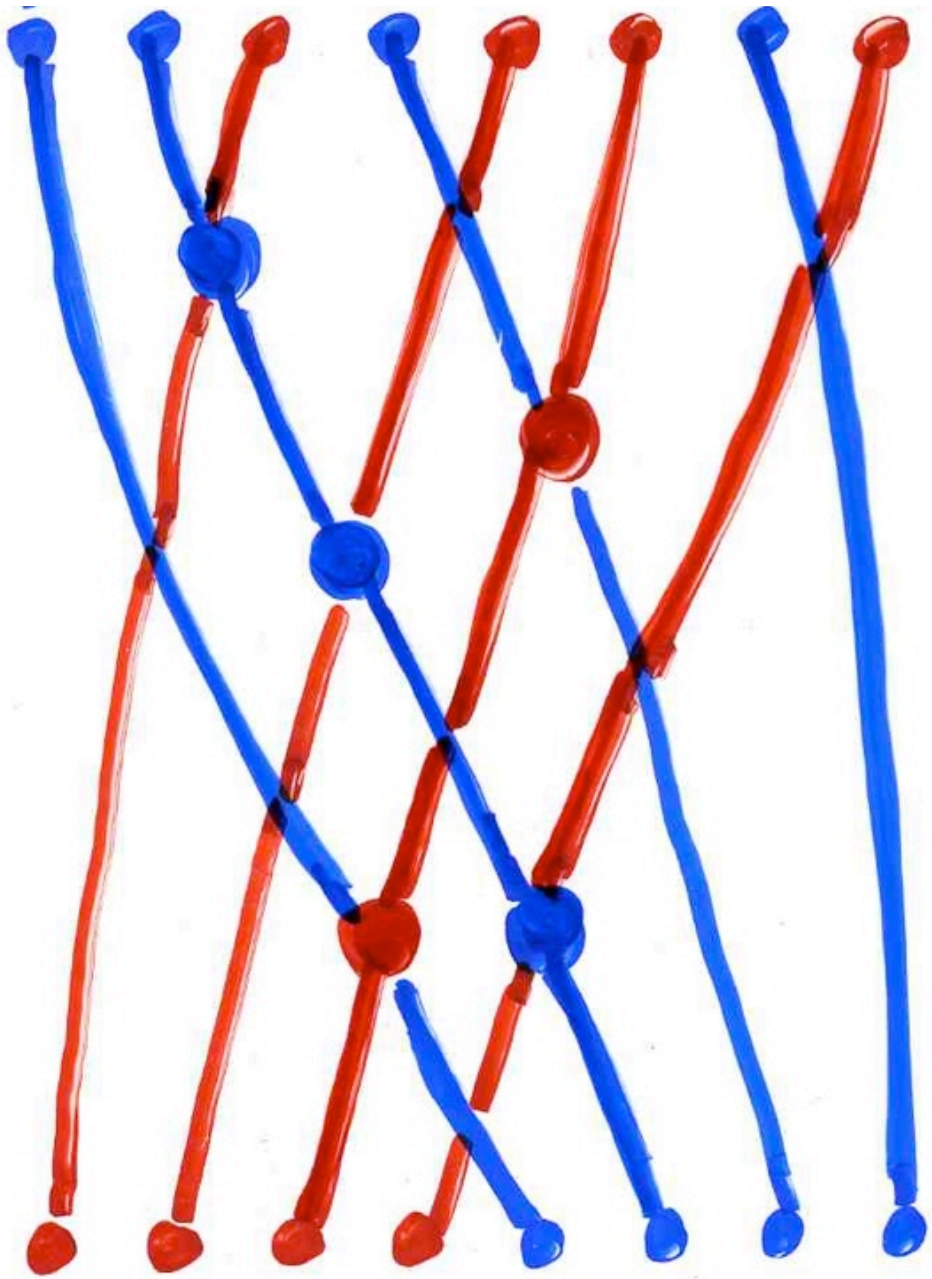
alternative
binary tree
(P. Nadeau)



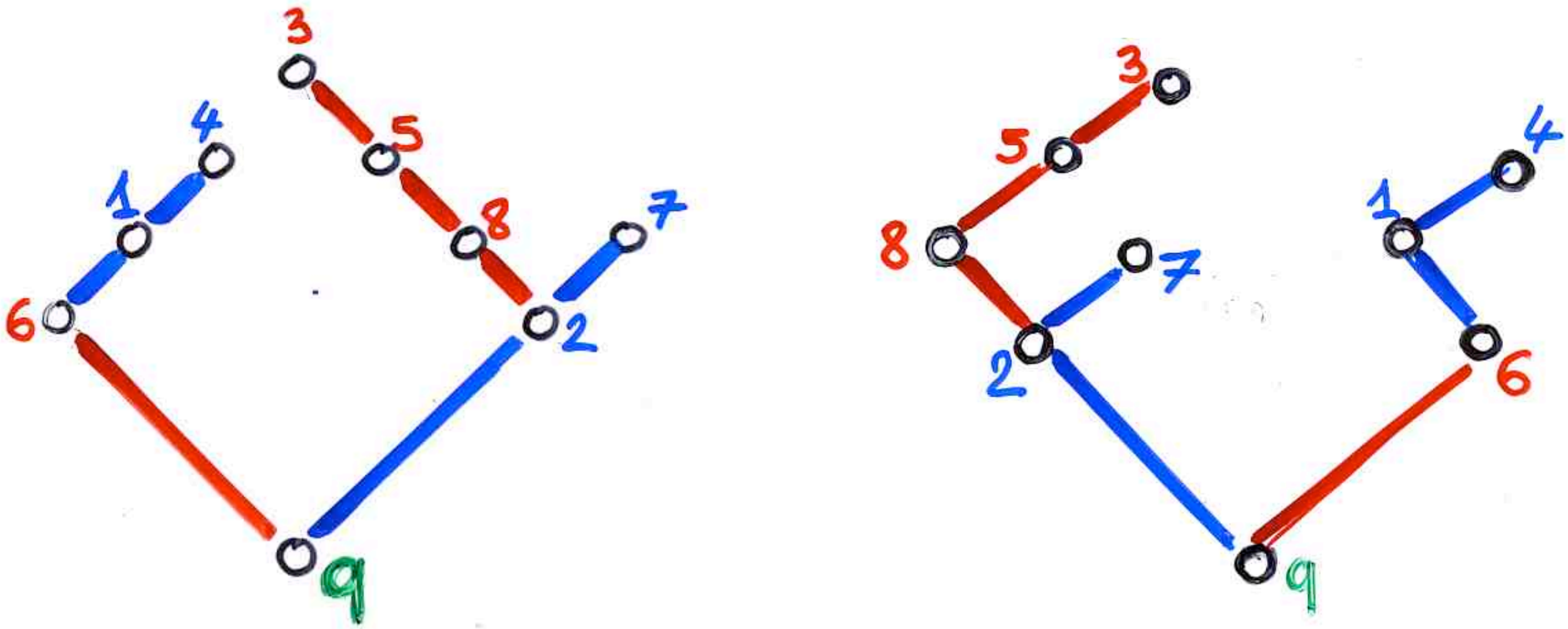




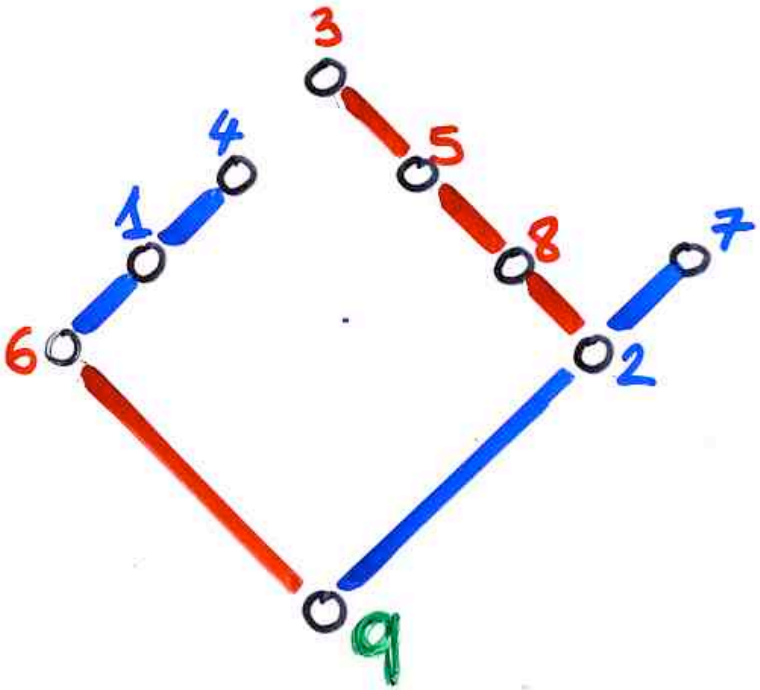


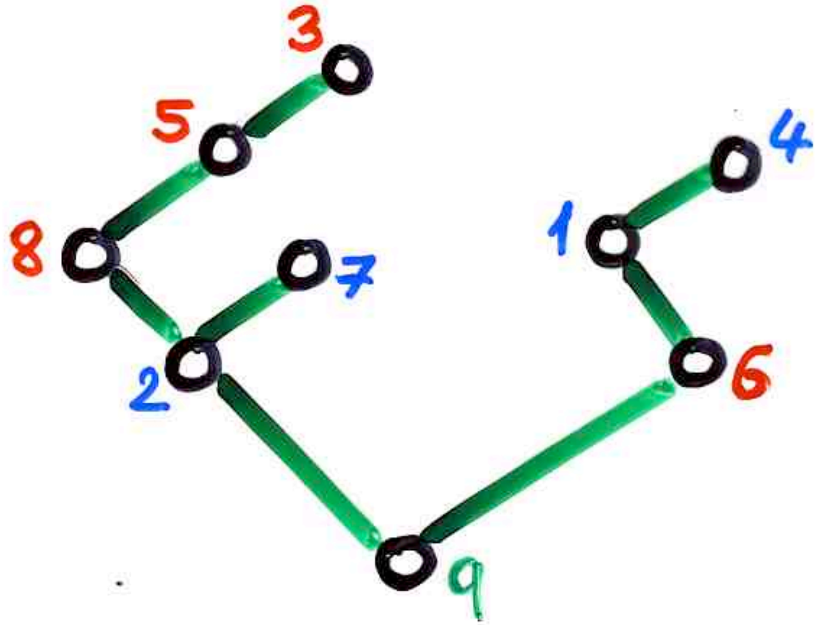



The twisted symmetric order



"twisted"
symmetric
order





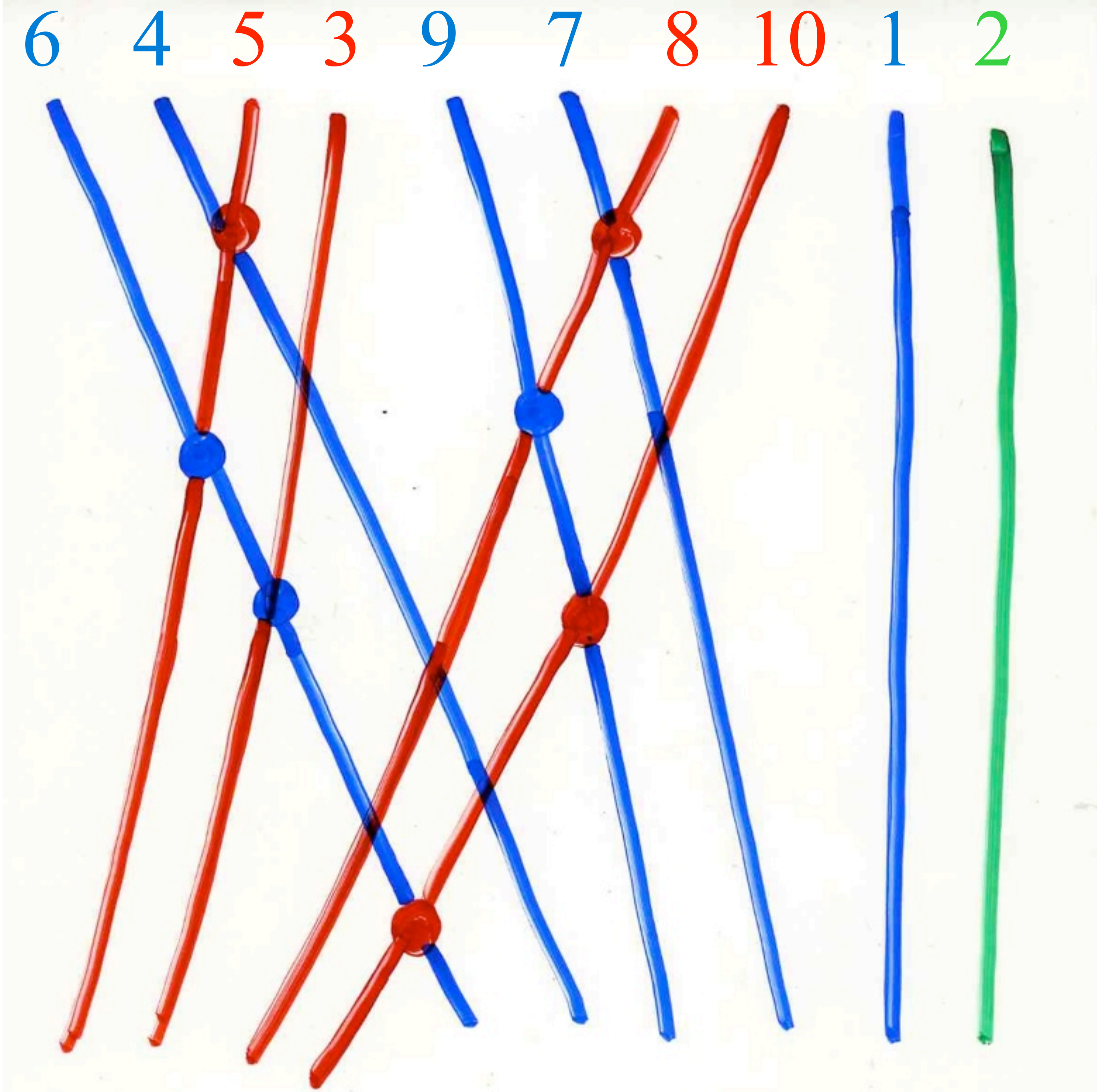
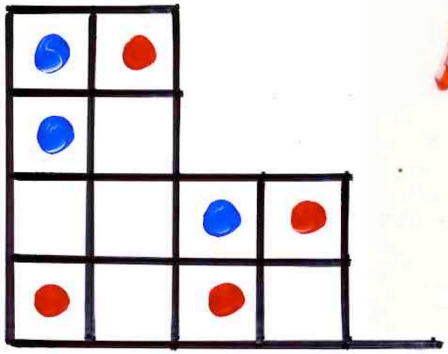
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 3 & 2 & 7 & 9 & 1 & 4 & 6 \end{pmatrix}$$

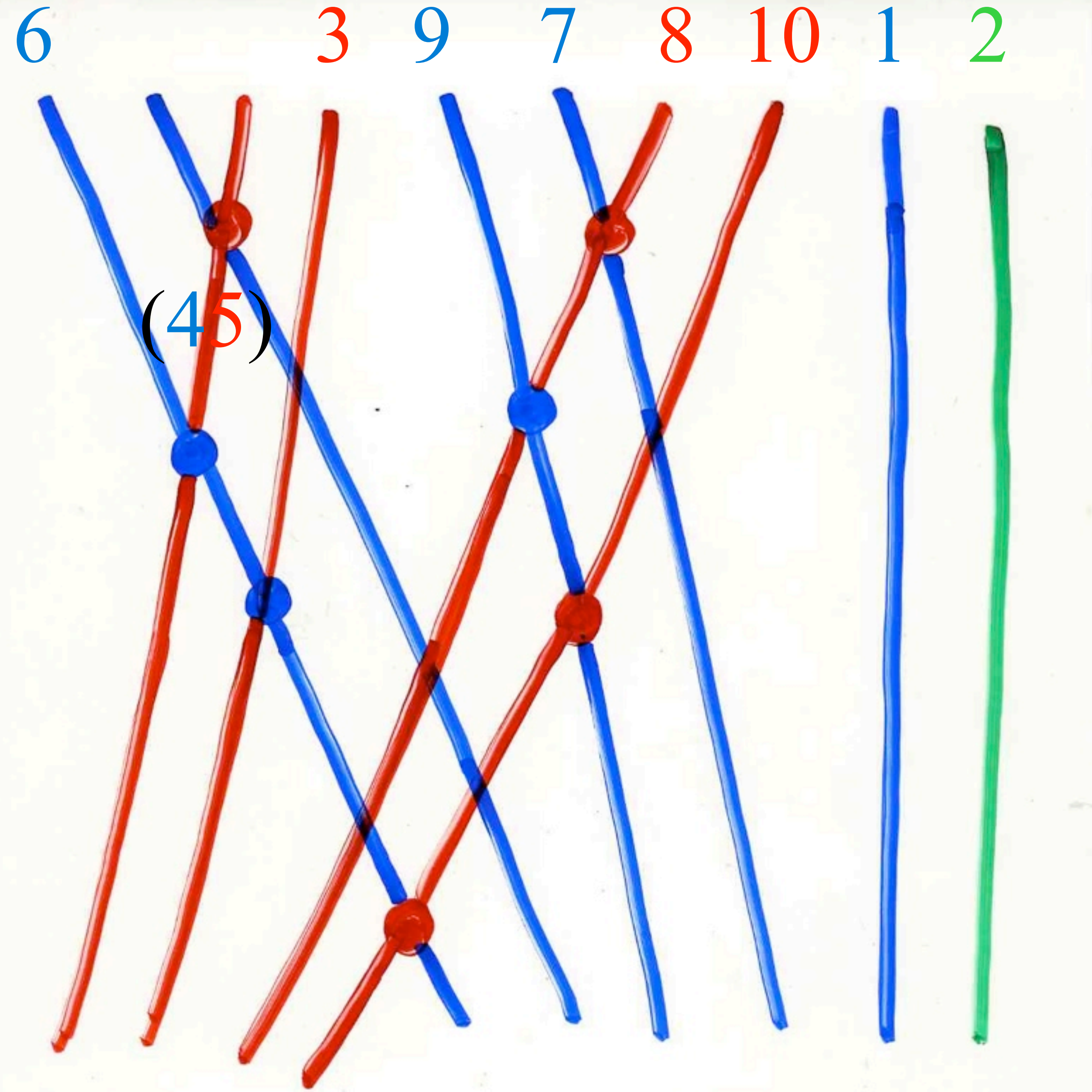
"twisted"
symmetric
order

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 3 & 8 & 2 & 9 & 5 & 1 & 6 \end{pmatrix}$$

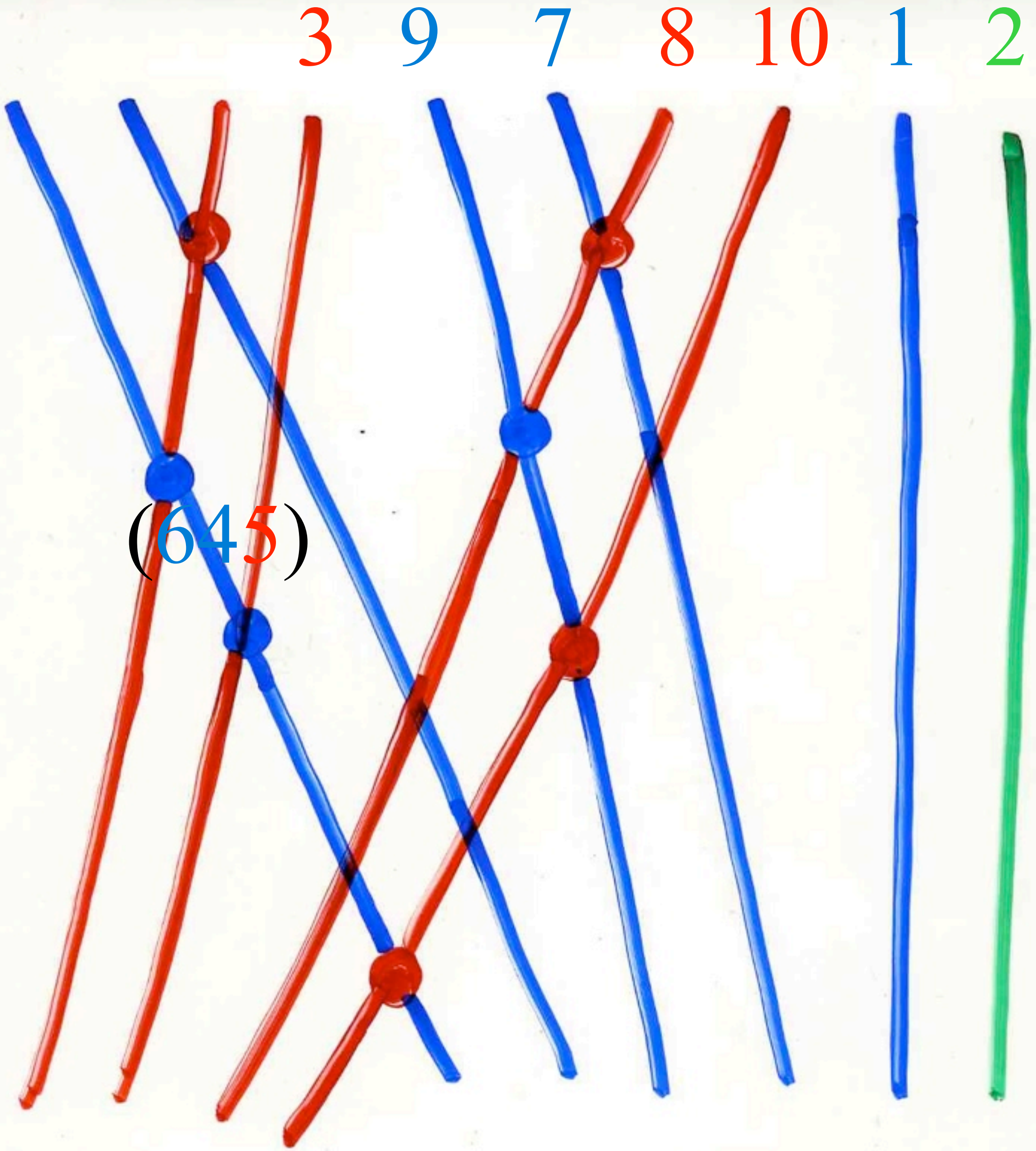
complements

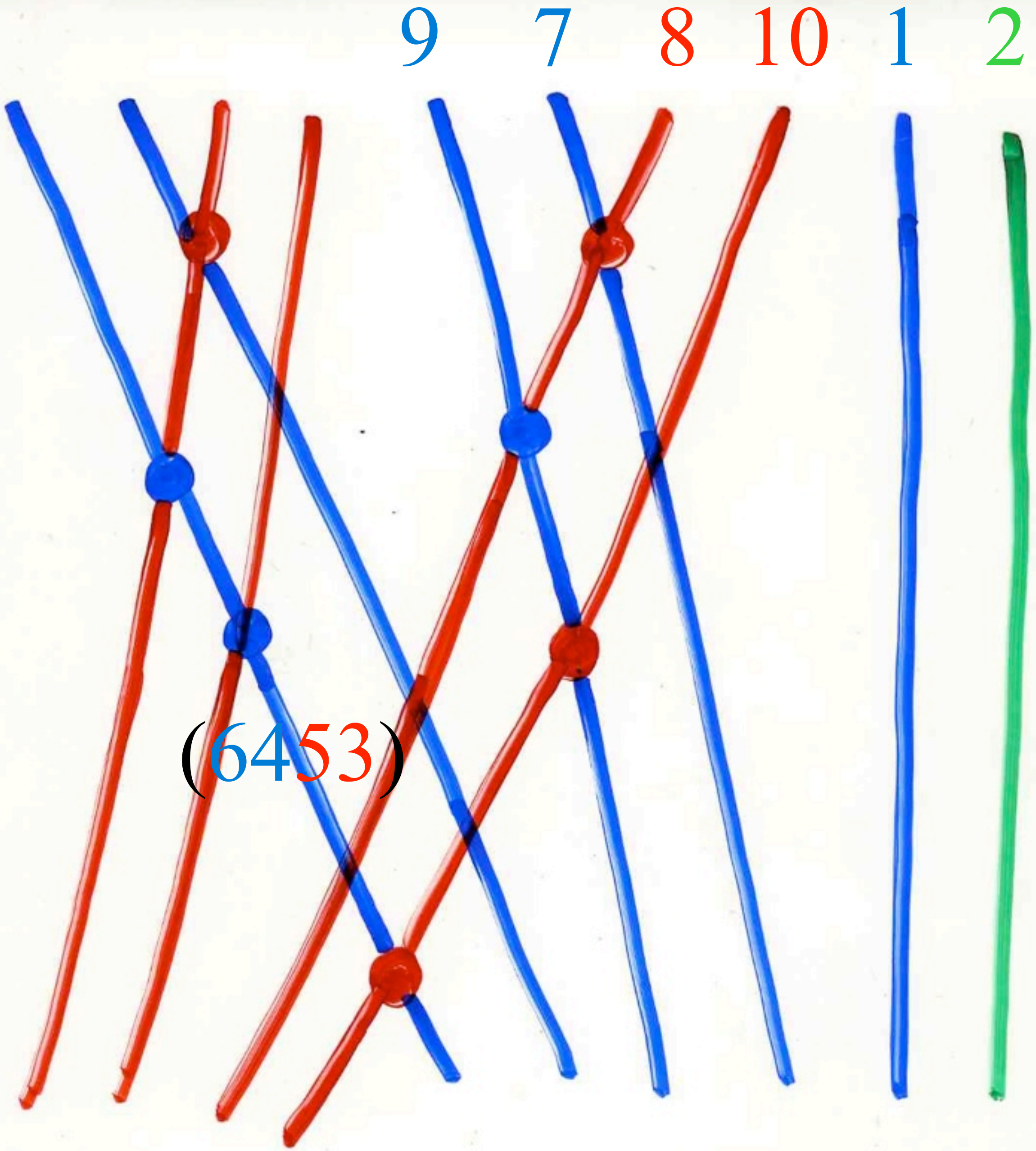
The “exchange-fusion” algorithm in the Catalan case

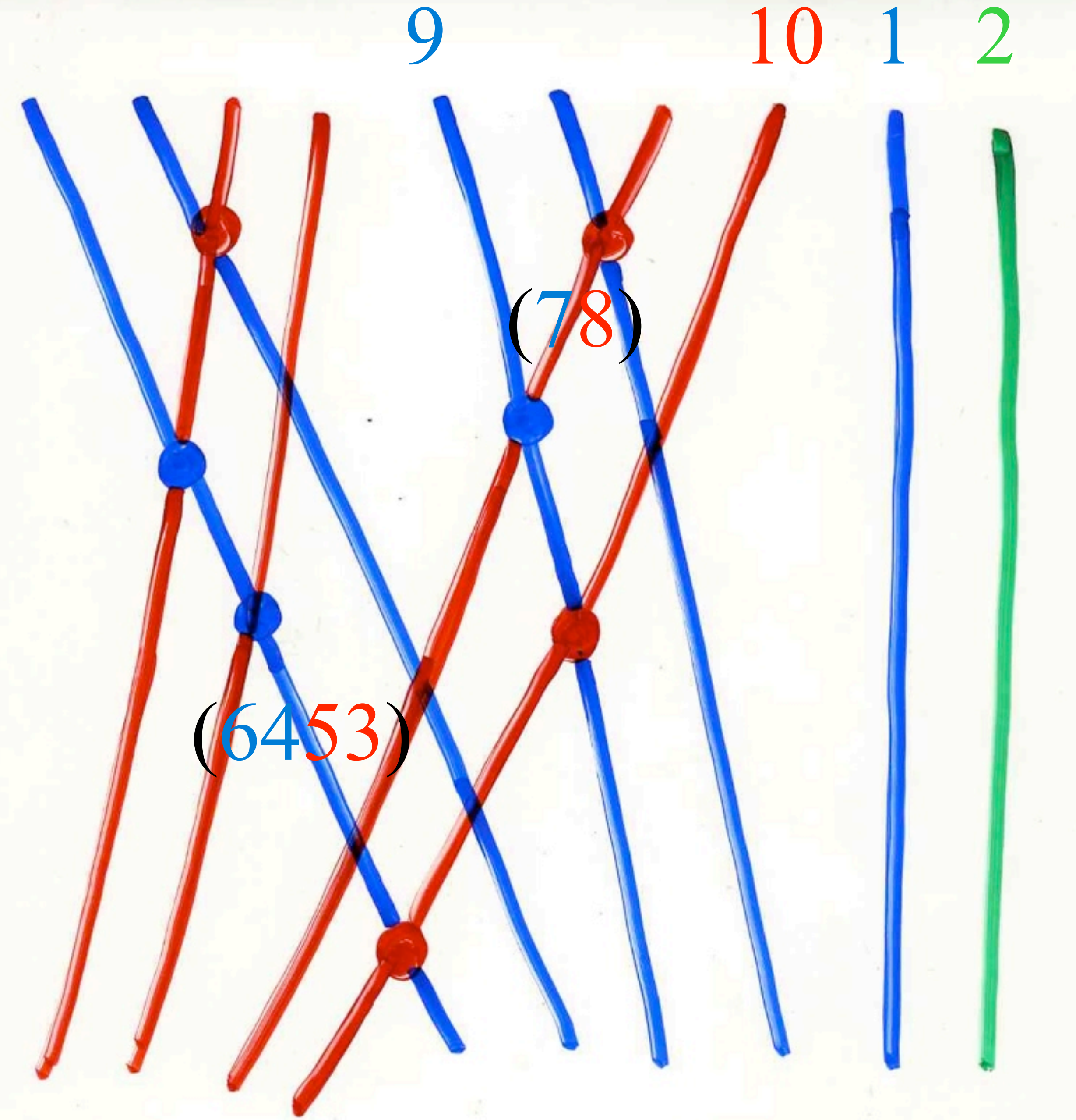

6 4 5 3 9 7 8 10 1 2


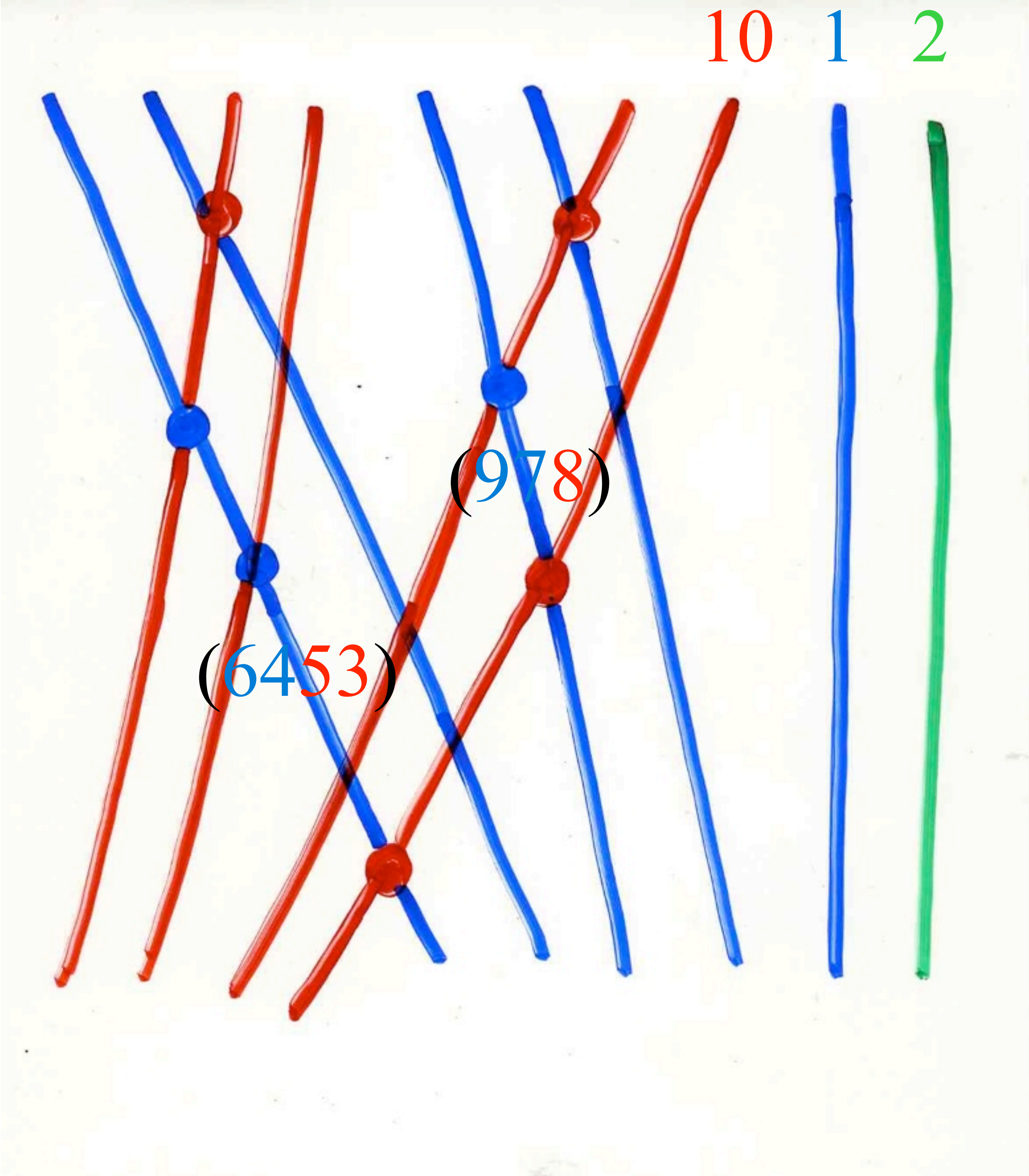


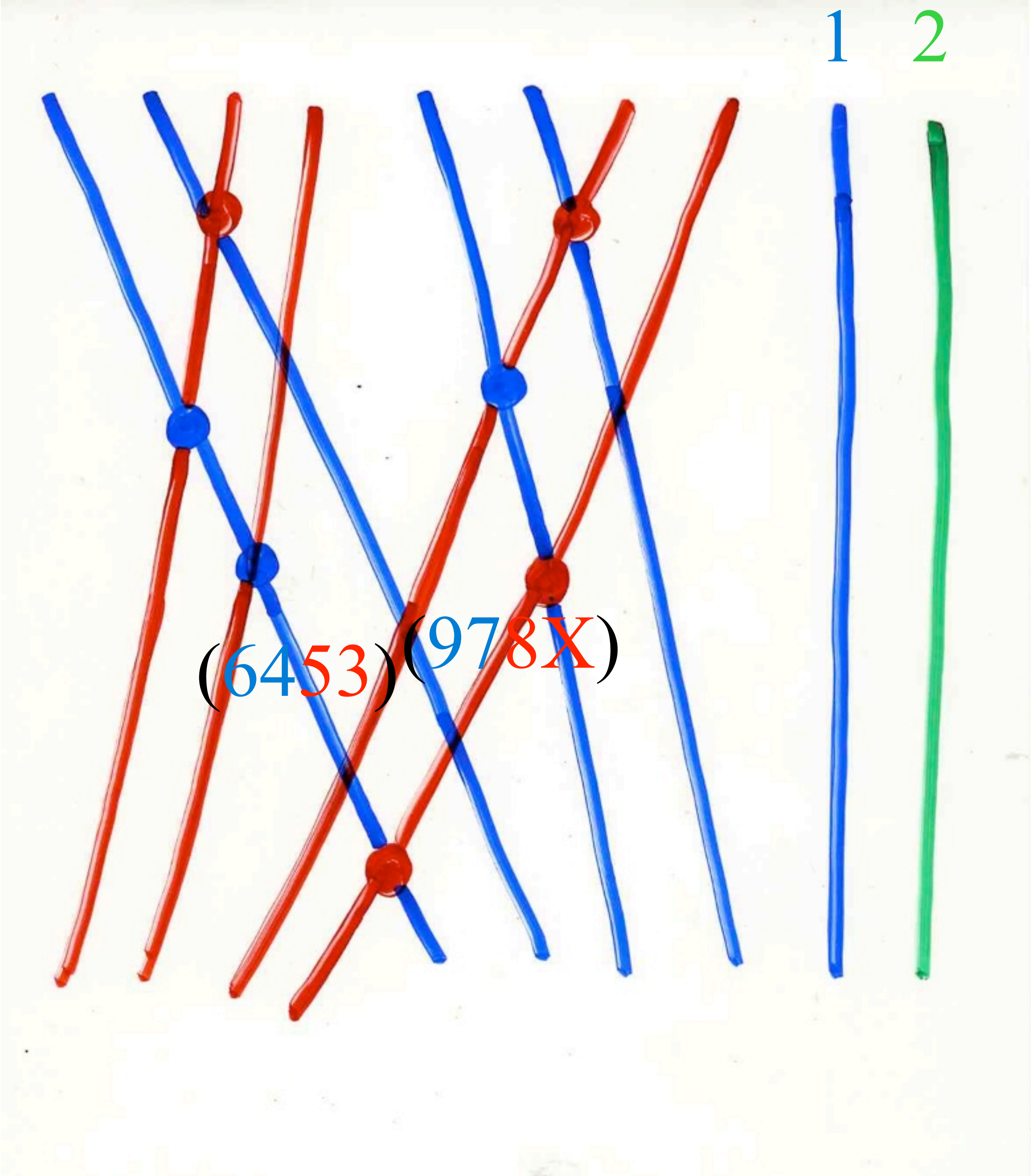
6
3
9
7
8
10
1
2
(45)

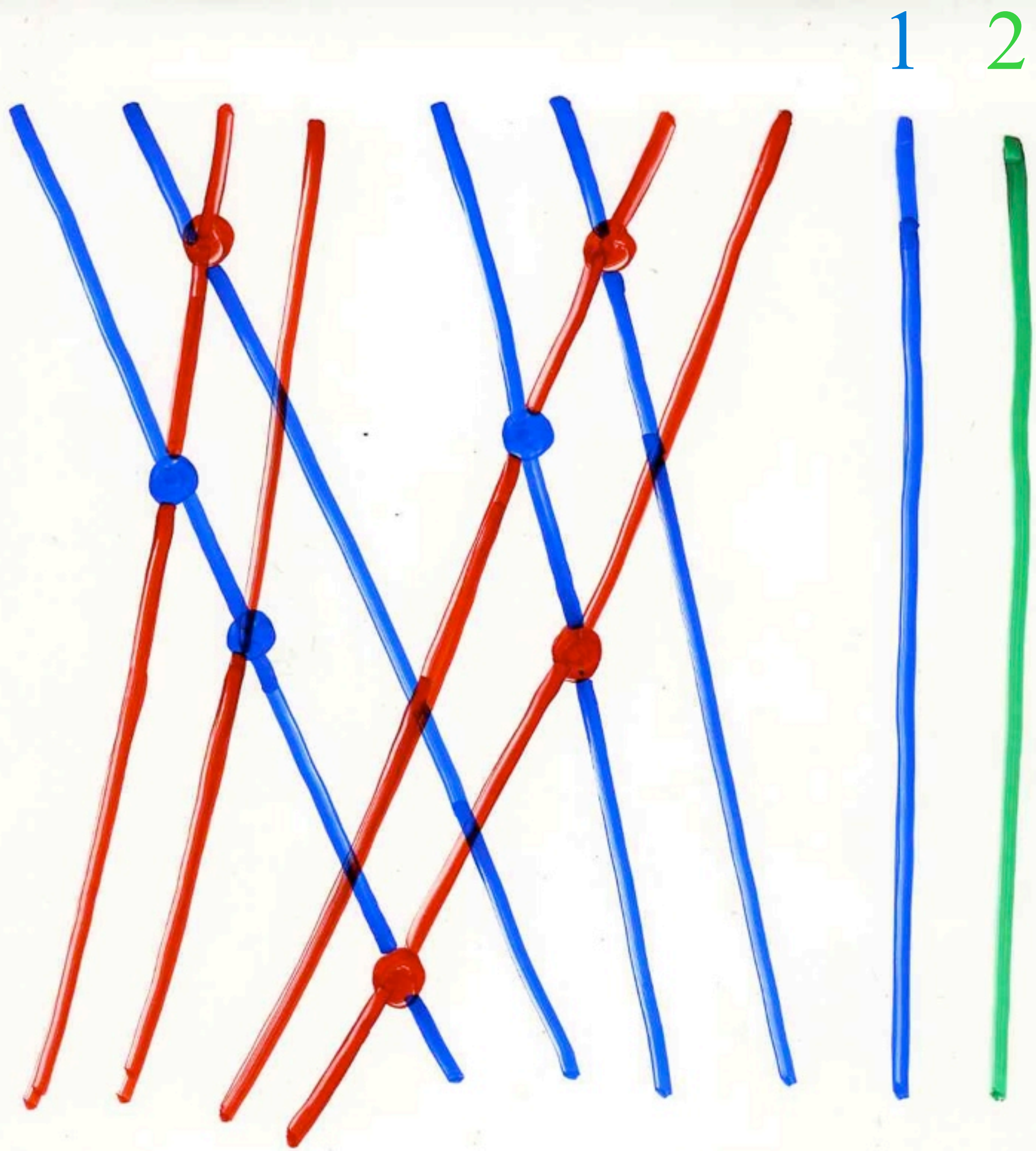

3 9 7 8 10 1 2
(645)


9 7 8 10 1 2
(6453)

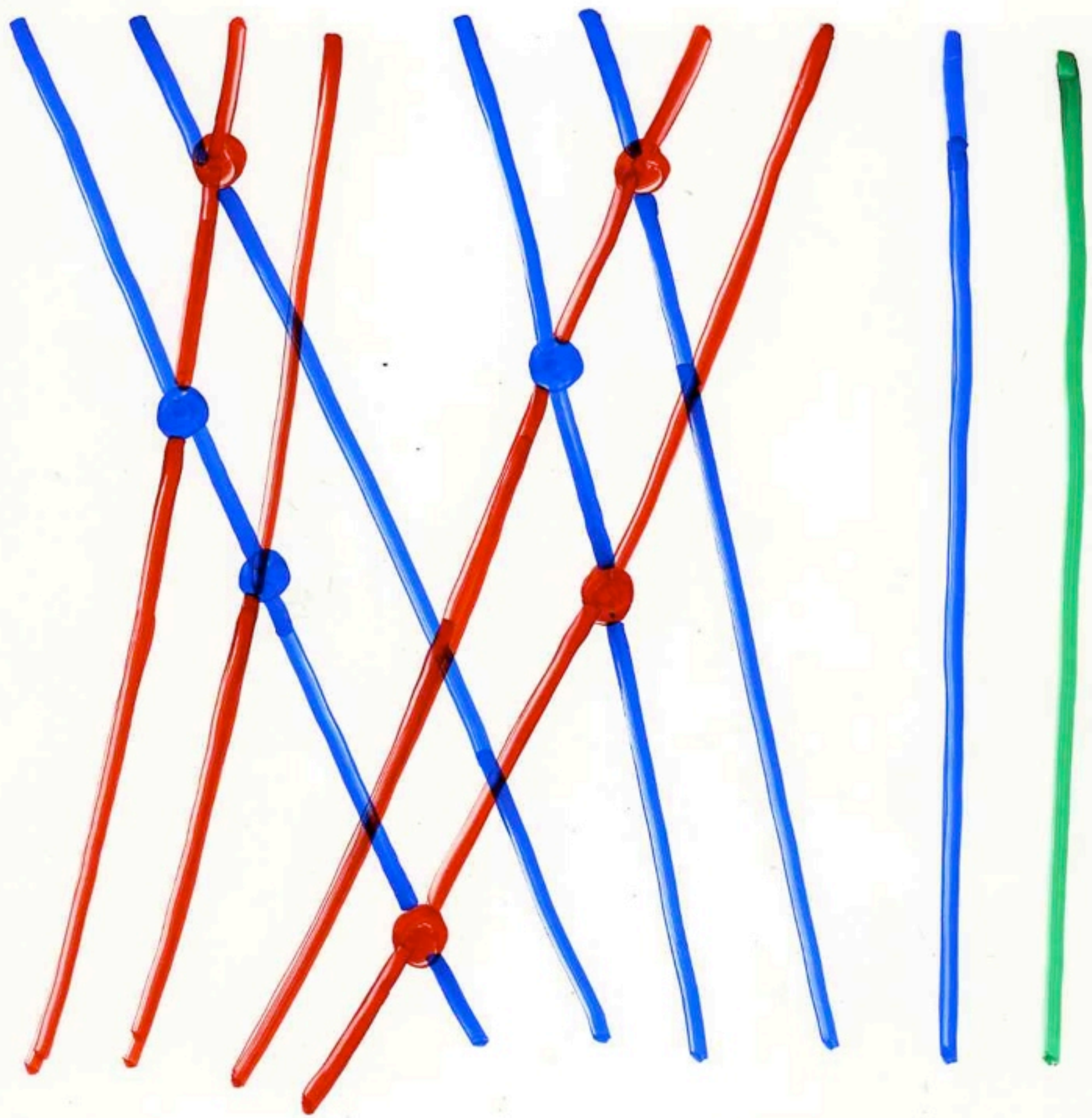

9
10
1
2
(78)
(6453)


10 1 2
(978)
(6453)

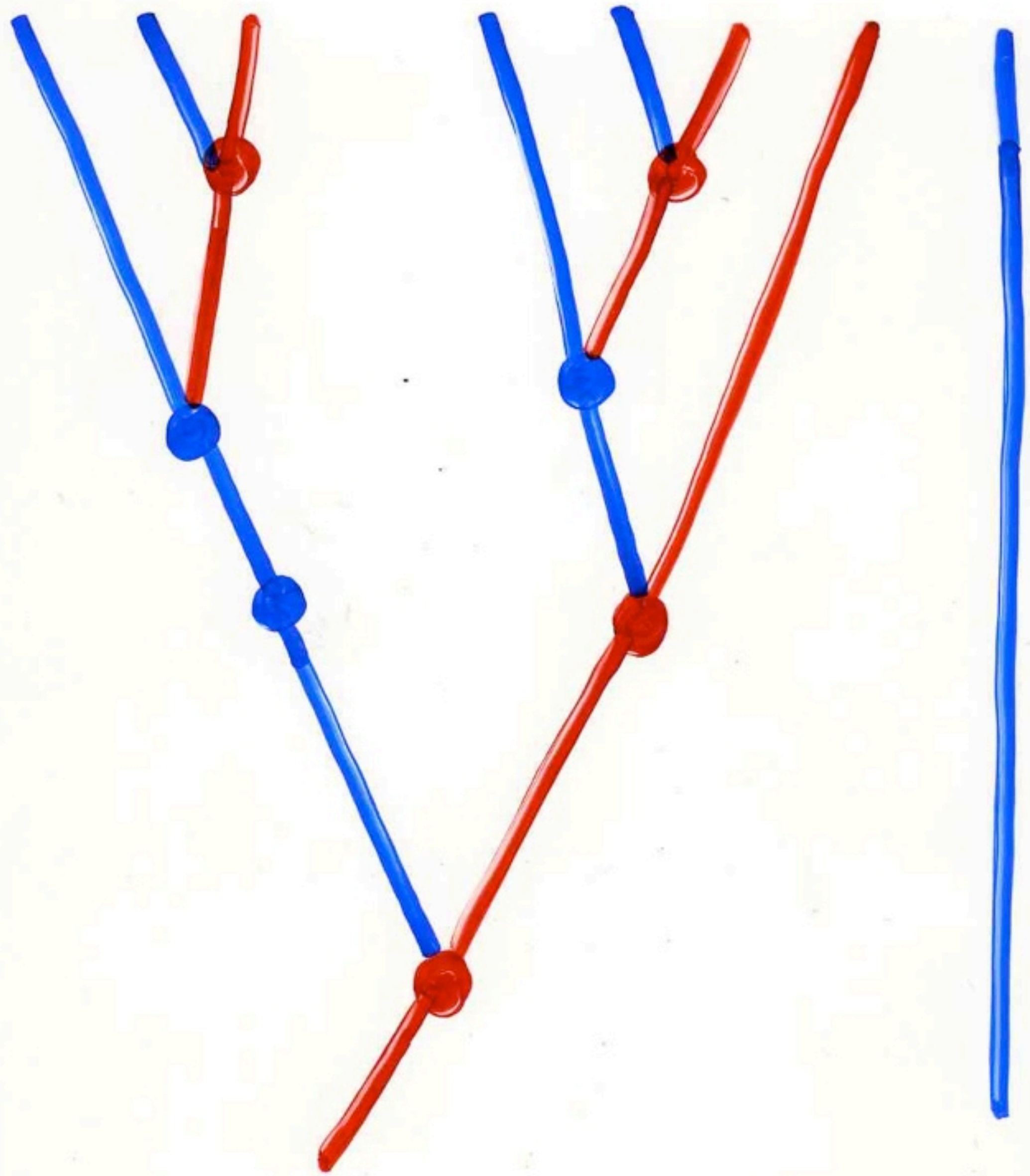

1
2
(6453)
(978X)

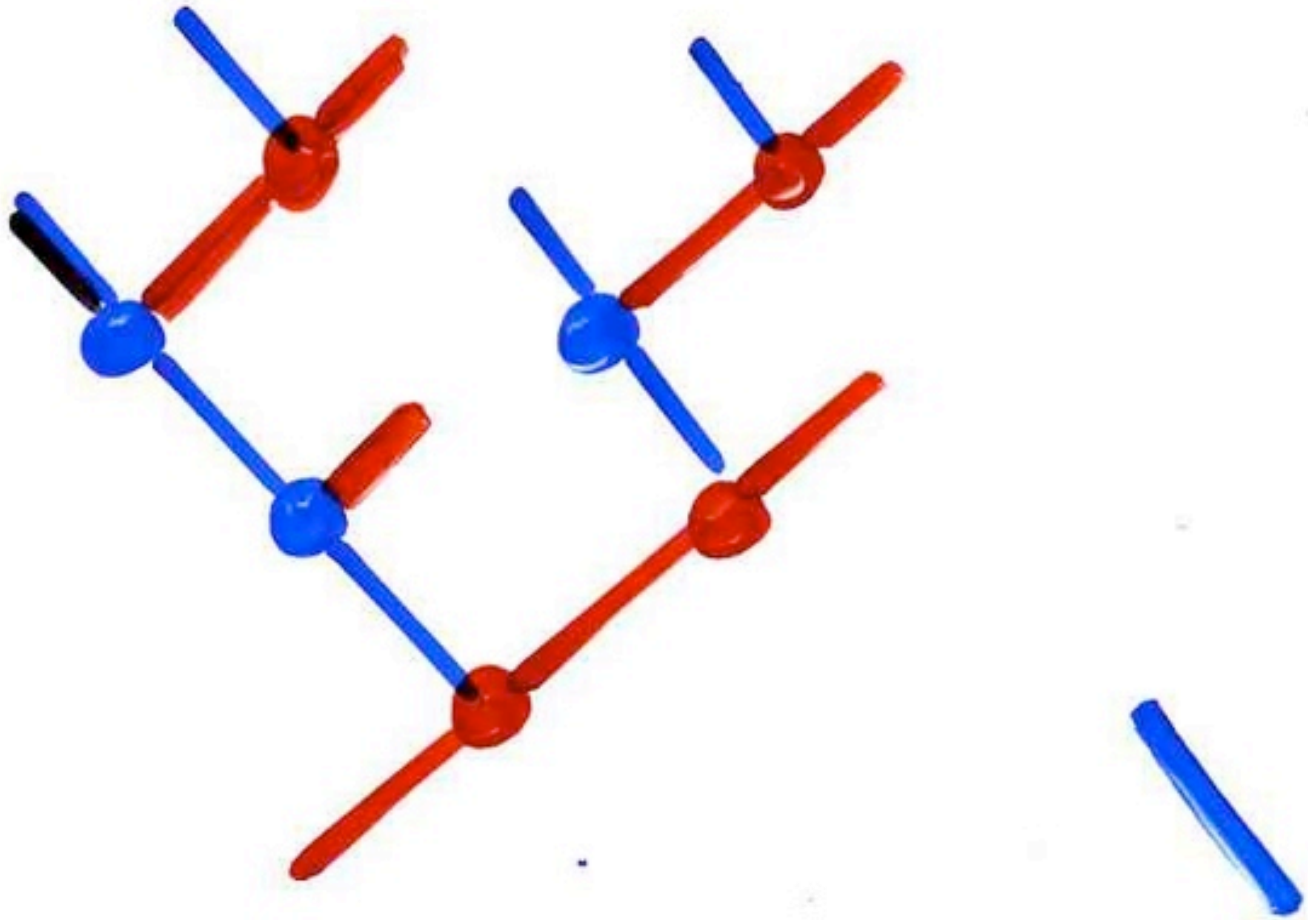


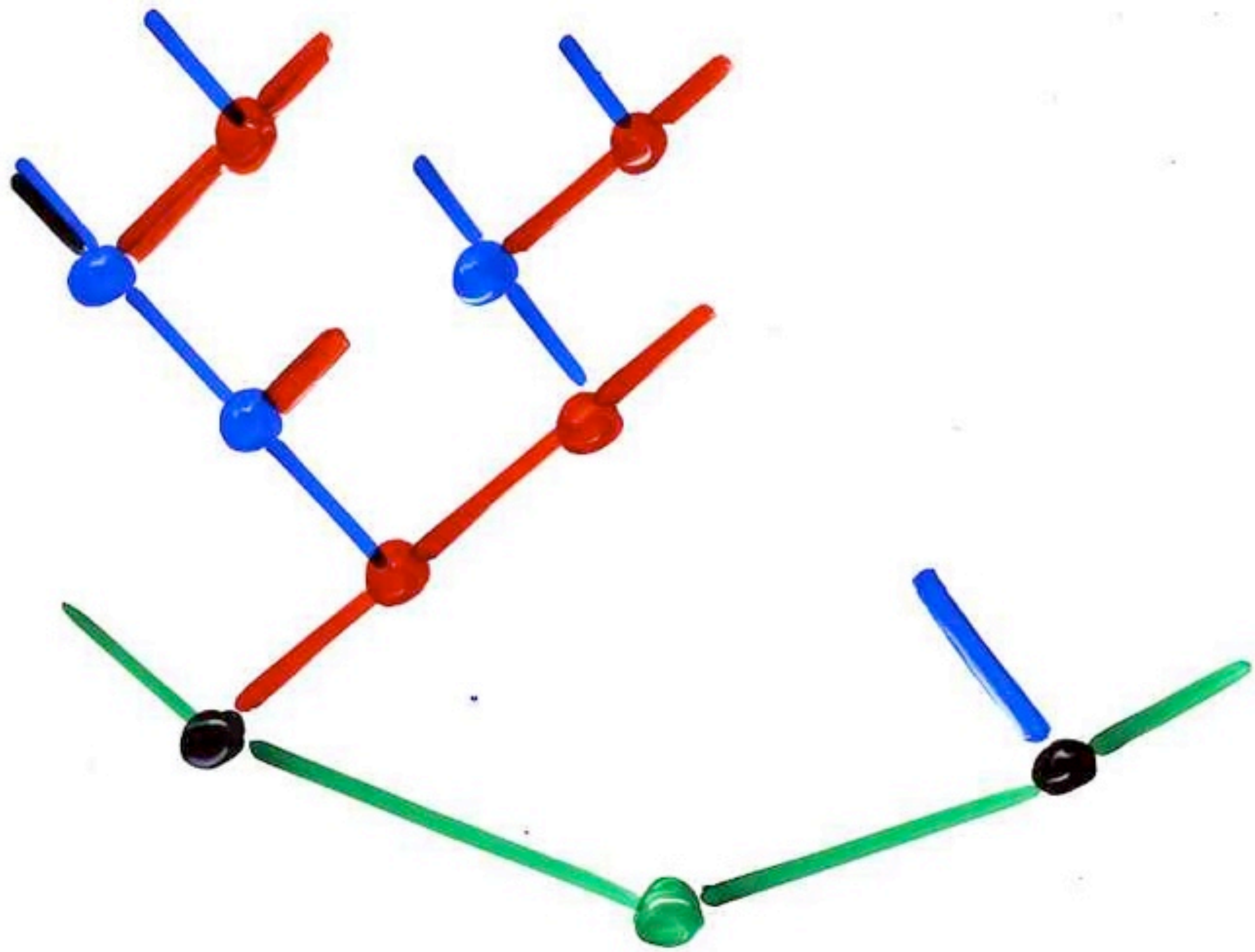
(6453978X)

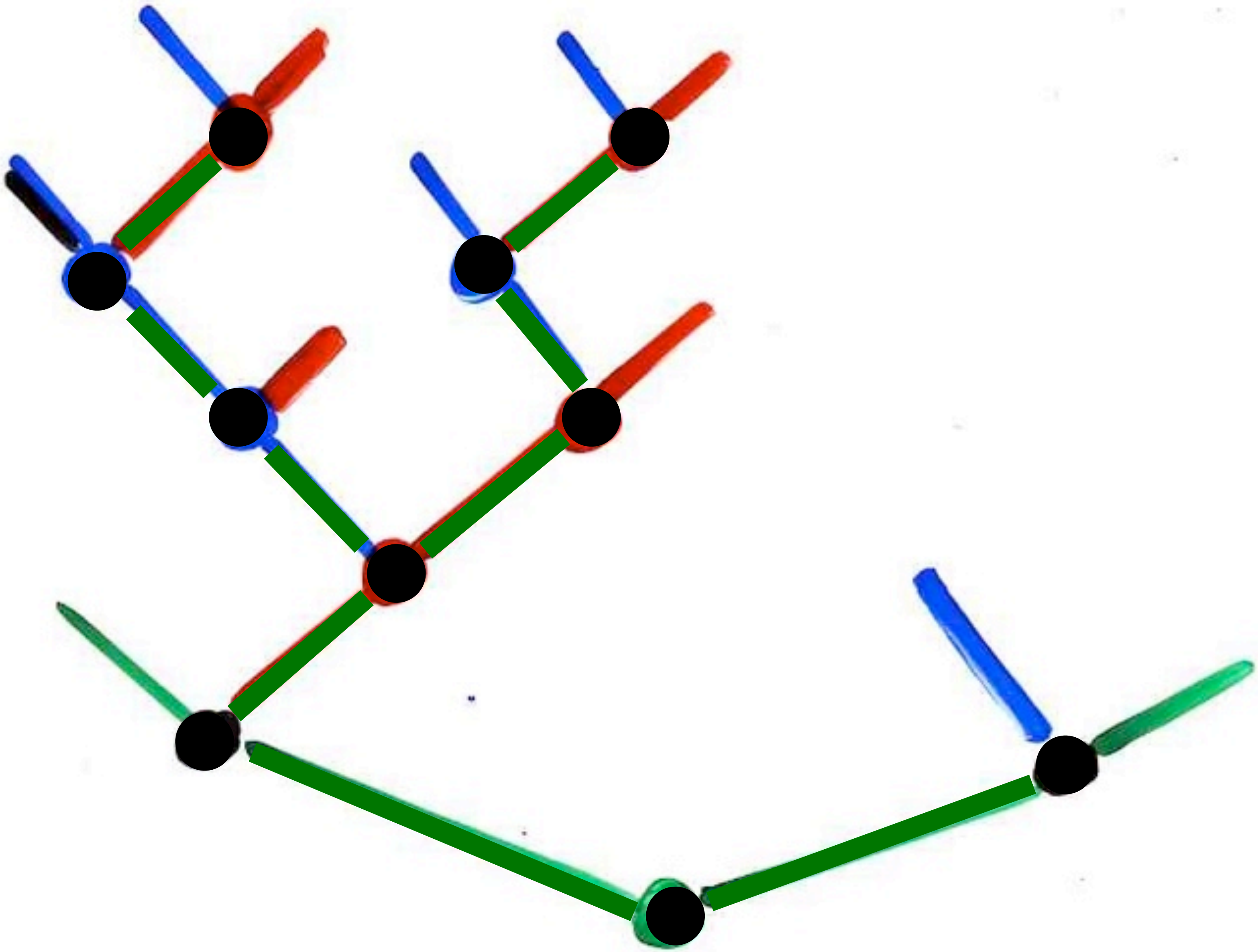


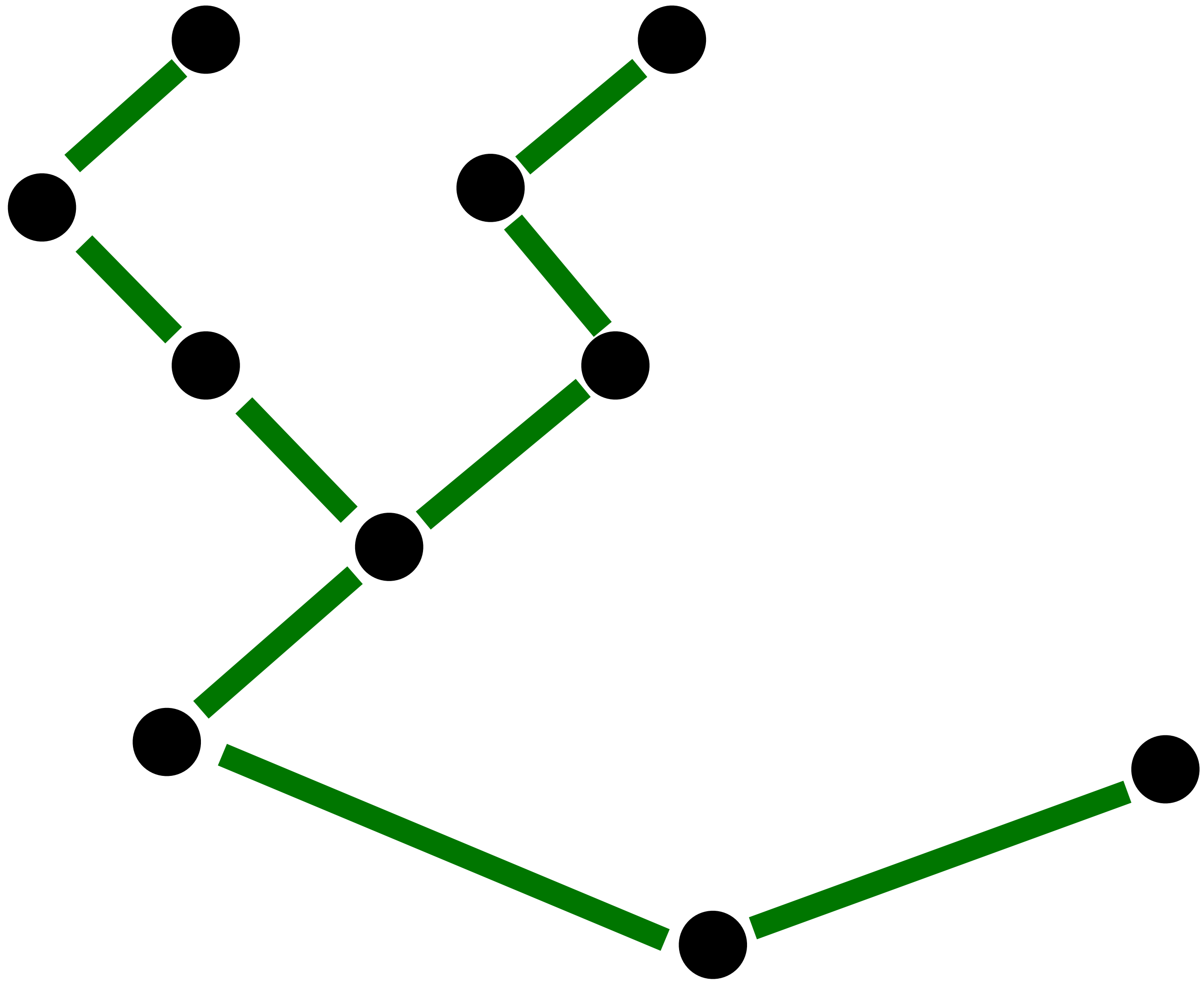
(6453978X) 1 2

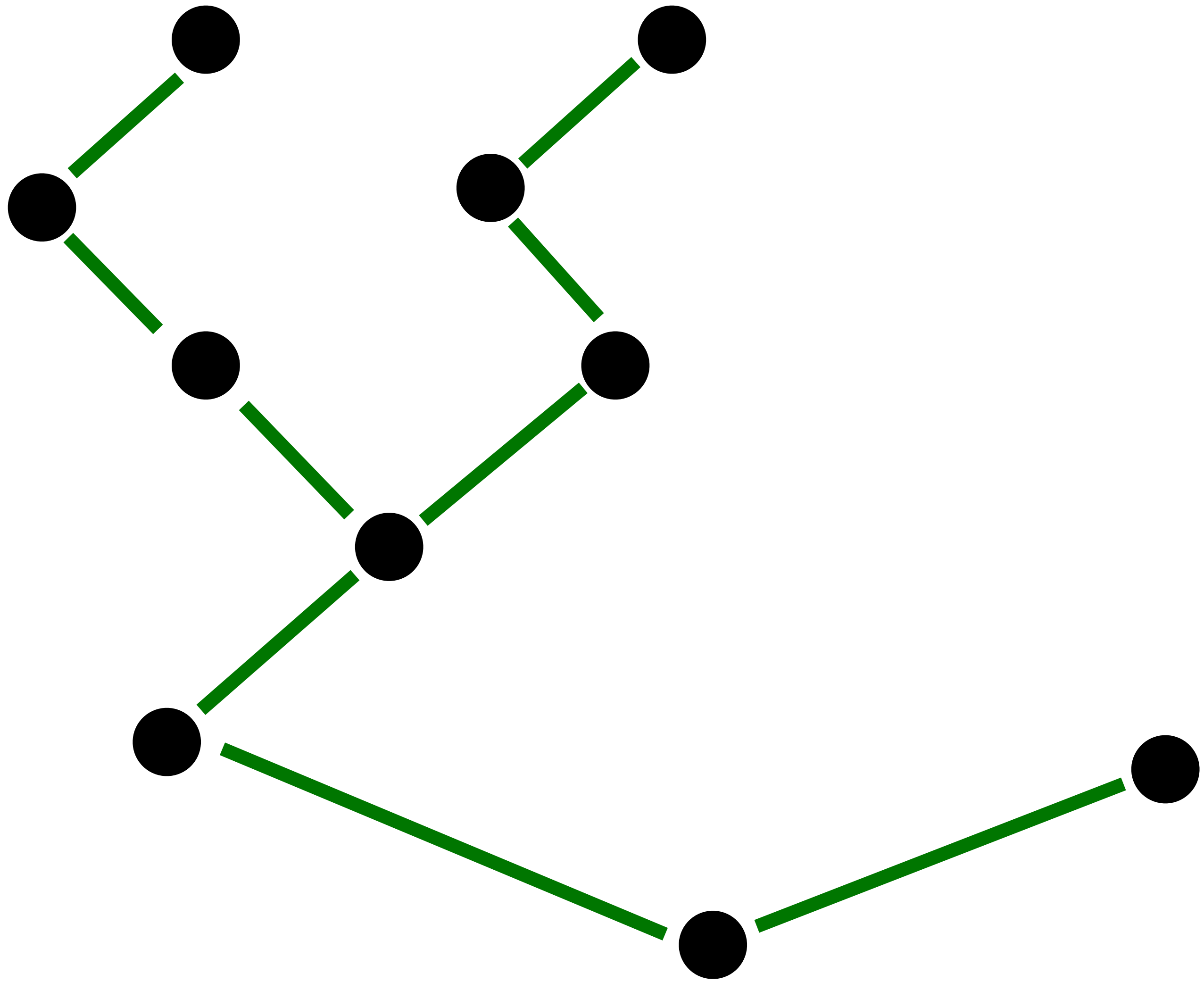












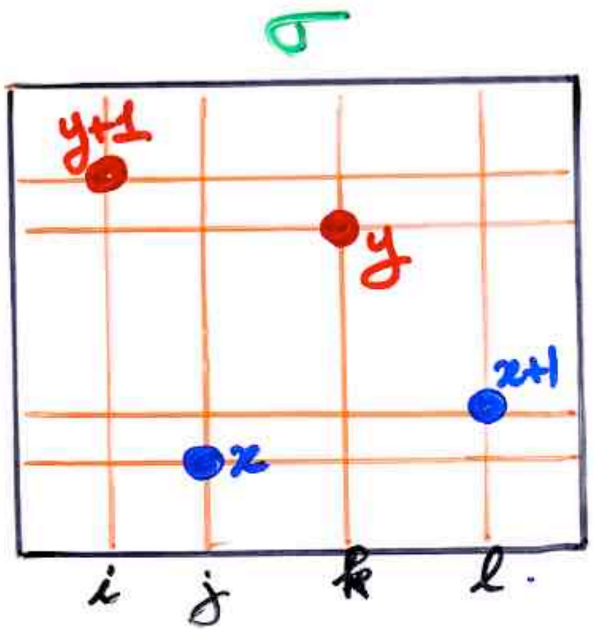
Bernardi permutations

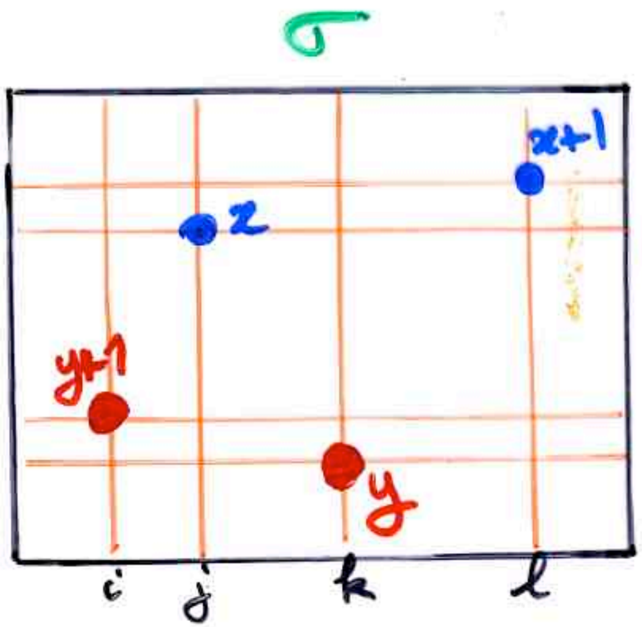
Permutations

with no subsequence of the type

.... $(y+1)$ ... x ... y .. $(x+1)$ ....

ex: σ = 6 4 5 3 9 7 8 (10) 1 2





Permutations

with no subsequence of the type

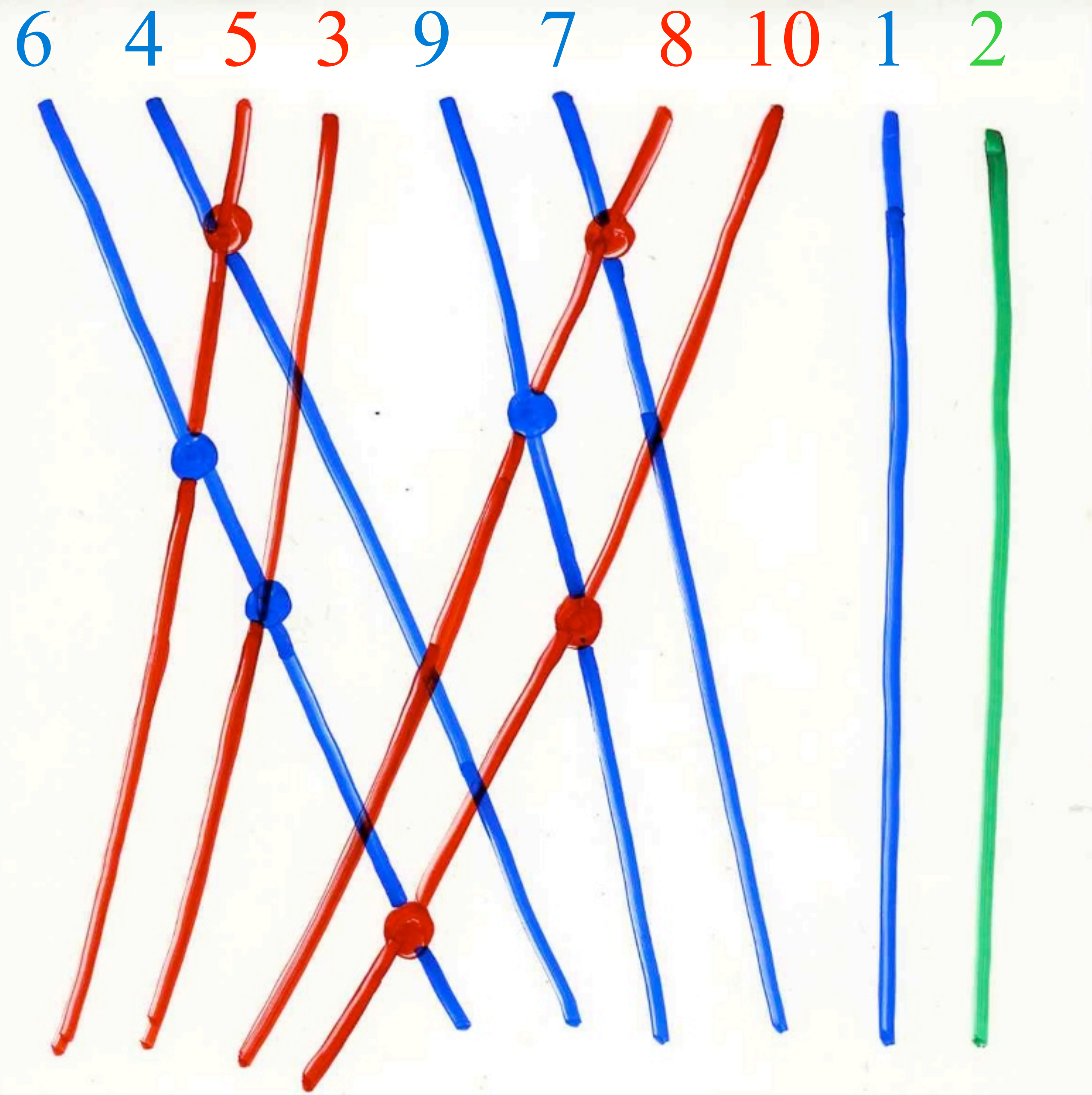
$$\cdots\ (y+1)\ \cdots\ x\ \cdots\ y\ \cdots\ (x+1)\ \cdots$$

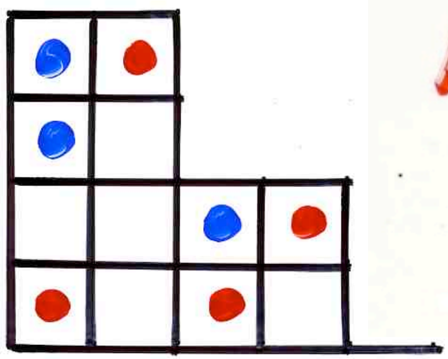
ex: $\sigma = 6\ 4\ 5\ 3\ 9\ 7\ 8\ (10)\ 1\ 2$

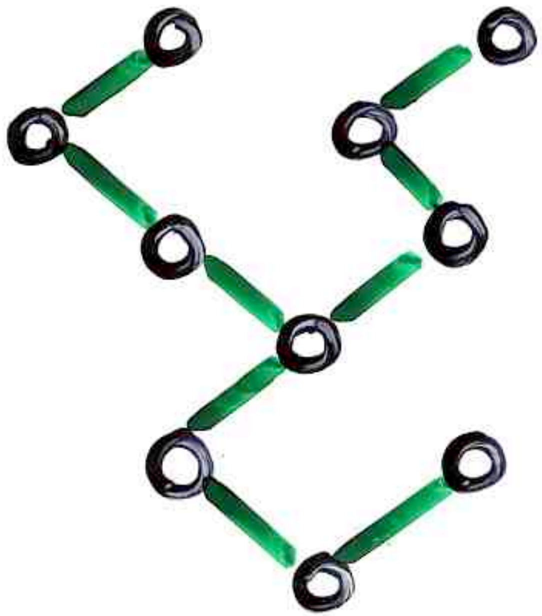
Prop. (O. Bernardi, 2008)
The number of such permutations on n elements is C_n Catalan number

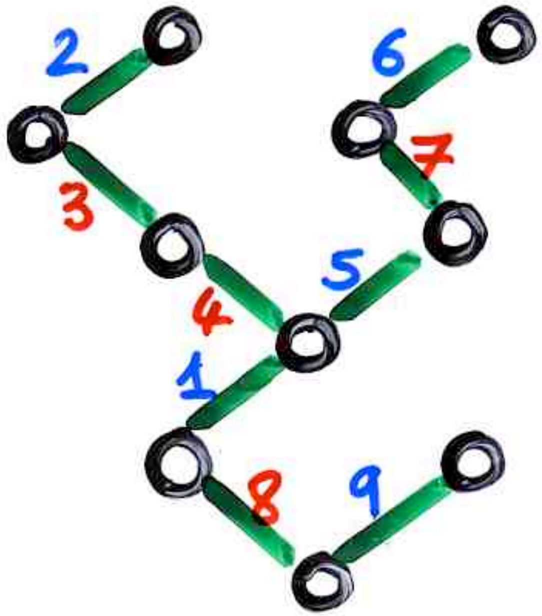
Lemma. $\sigma \Longleftrightarrow T$ alternating tableau
(σ permutation)

T has no crossing
$\Longleftrightarrow$ σ has no subsequence of type
$(y+1) \cdots x \cdots y \cdots (x+1)$


6 4 5 3 9 7 8 10 1 2

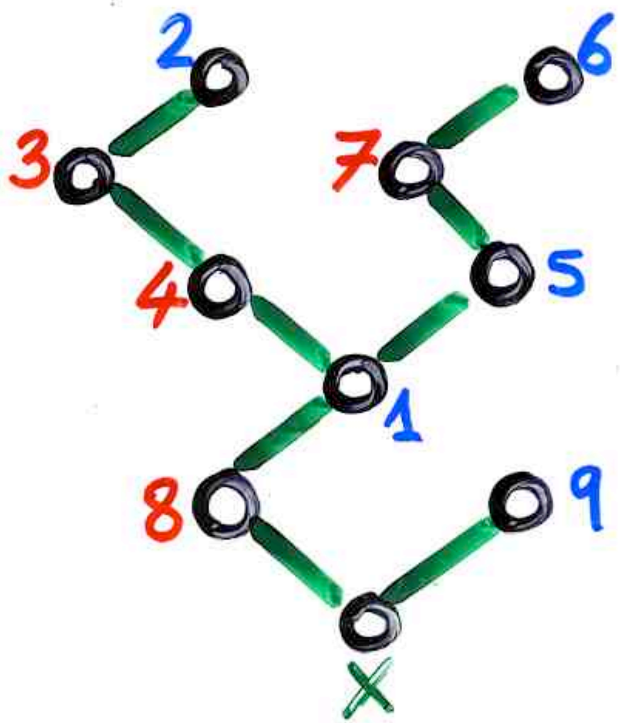


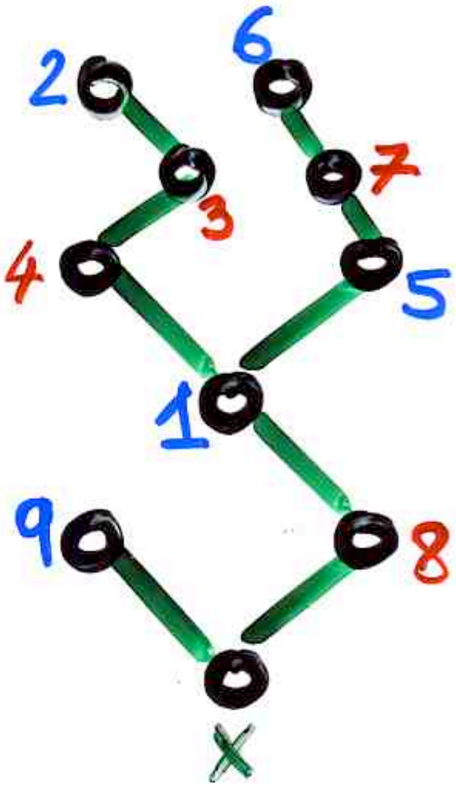




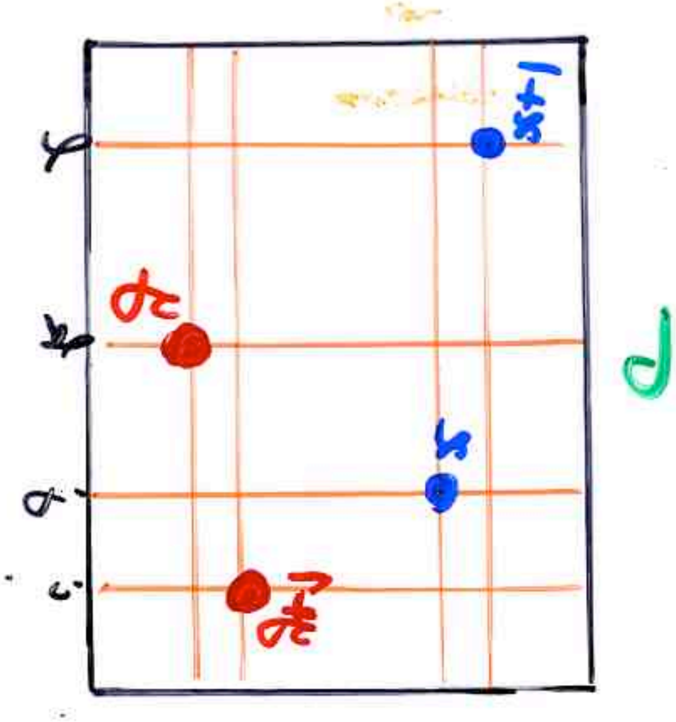
2
6
3
7
5
4
1
8
9

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 6 & 4 & 5 & 3 & 9 & 7 & 8 & X & 1 & 2 \end{pmatrix}$$

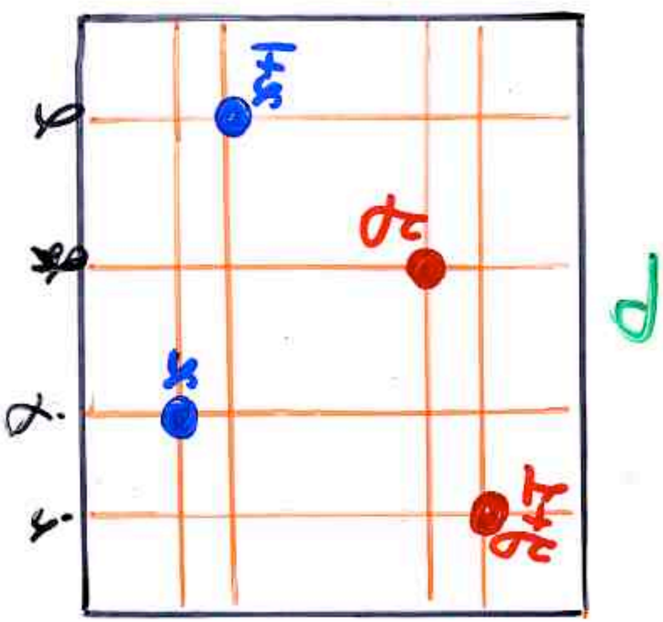




$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & X \\ 9 & X & 4 & 2 & 3 & 1 & 6 & 7 & 5 & 8 \end{pmatrix}$$



31- 24



24 - 31