Chapter 7a

The cellular Ansatz

25 january 2011
Talca

Q-tableaux

Quadratic algebra $\mathbb{Q}$

generators $\mathcal{B} = \{B_j\}_{j \in J}$

$\mathcal{A} = \{A_i\}_{i \in I}$

commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad \begin{matrix} i \in I \\ j \in J \end{matrix}$$

Lemma. In $\mathbb{Q}$ every word $w \in (\mathcal{A} \cup \mathcal{B})^*$ can be written in a unique way

$$w = \sum_{\substack{u \in \mathcal{A}^* \\ v \in \mathcal{B}^*}} c(u, v; w)\, uv$$

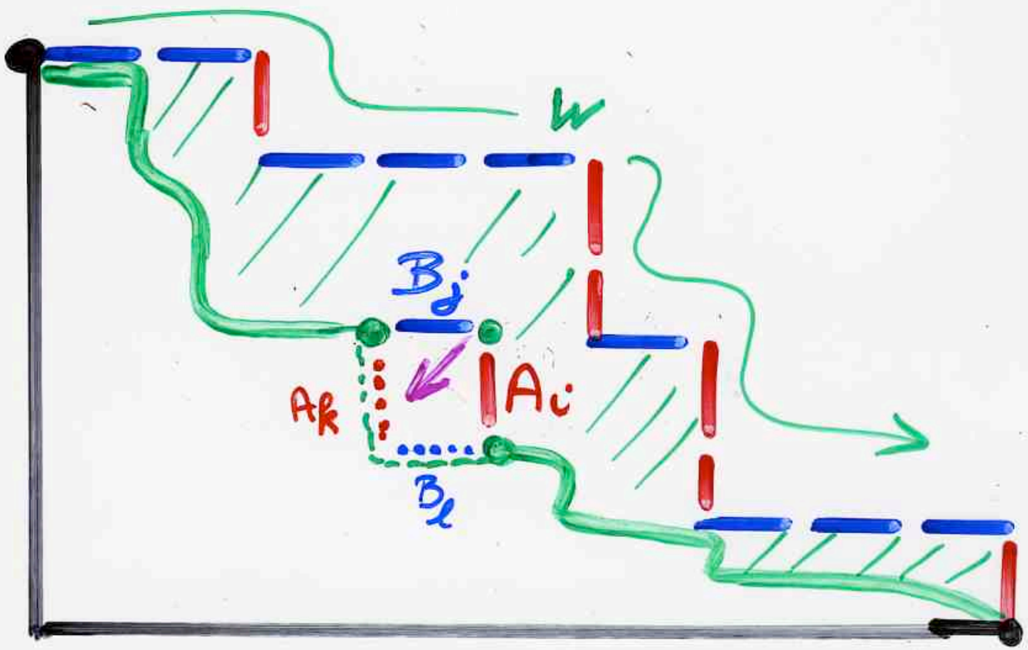
This polynomial can be obtained by successive rewriting rules :

any occurrence $B_j A_i \rightarrow \sum c_{ij}^{kl} A_k B_l$

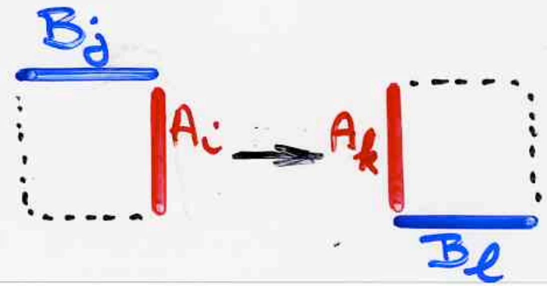
until no more such occurrence.

(Lemma) independent of the order of rewriting

Proof:



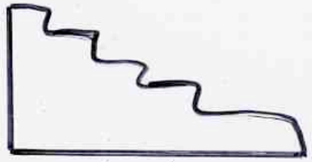
"planar" rewriting



complete

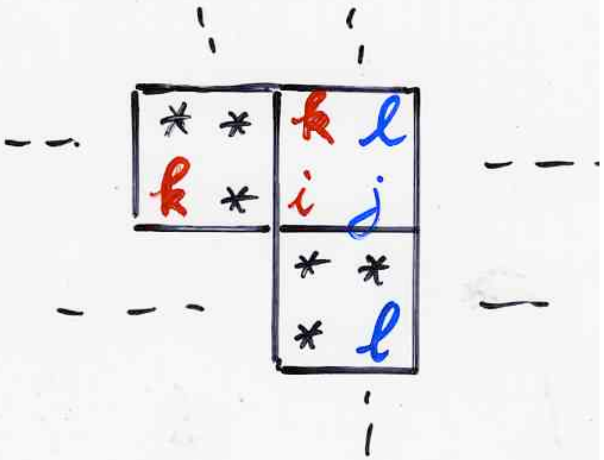
Def- Q-tableau

Ferrers diagram F



each cell $\alpha \in F$ labeled $\begin{array}{|cc|} k & l \\ i & j \end{array}$ $\quad i, k \in I \quad j, l \in J$

with "compatibility" condition:



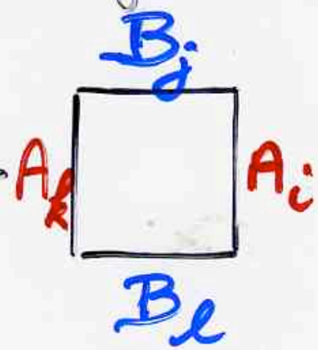
commutation relations

$$B_j A_i = \sum_{k,l} c_{ij}^{kl} A_k B_l \qquad i \in I, \; j \in J$$

complete

Def- edge-labeling of a Q-tableau T

each cell α $\begin{array}{|cc|} k & l \\ i & j \end{array}$ $\longrightarrow$

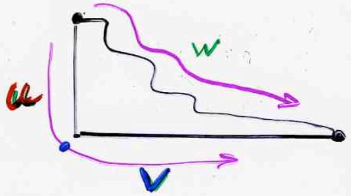


complete

Def. For T a Q-tableau

$uwb(T) \in (\mathcal{A} \cup \mathcal{B})^*$ upper word border

$lwb(T)$ lower word border



complete

Def. weight of a Q-tableau T

$$w(T) = \prod_{\substack{\text{cells} \\ \alpha \in T}} c_{ij}^{kl} \qquad \alpha = \boxed{\begin{matrix} k & l \\ i & j \end{matrix}}$$

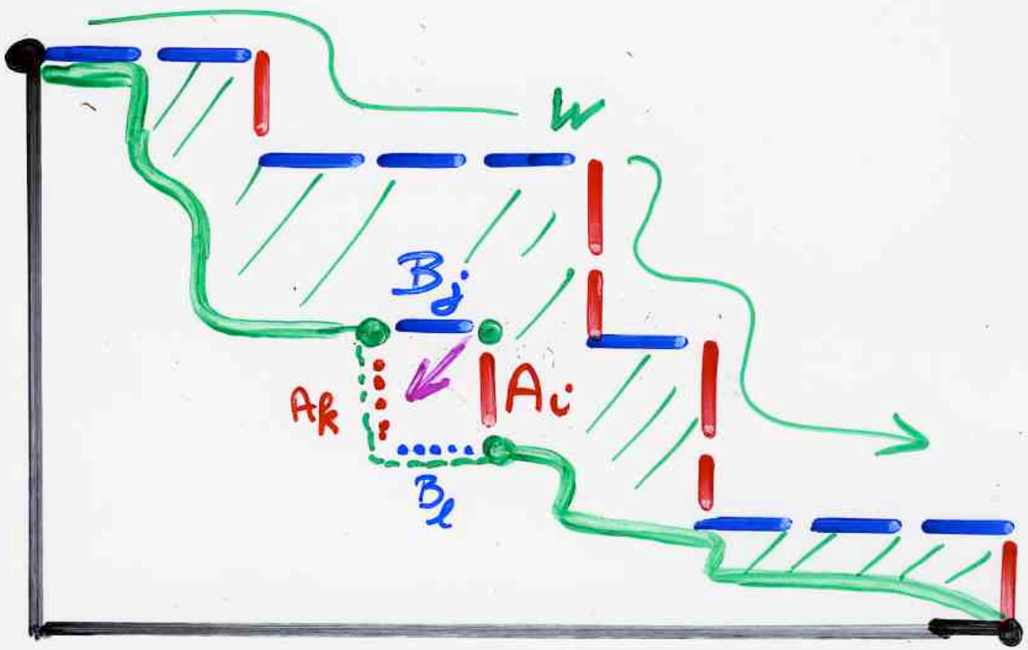
Prop For any $w \in (\mathcal{A} \cup \mathcal{B})^*$, $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$

$$c(u, v; w) = \sum_{T} w(T)$$

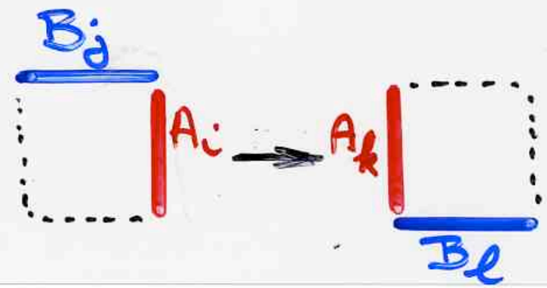
T complete Q-tableau

$\begin{cases} uwb(T) = w \\ lwb(T) = uv \end{cases}$

Proof:



"planar" rewriting



S set of labels

$$\varphi : \left\{ \begin{bmatrix} k & l \\ i & j \end{bmatrix} \right\} = R \longrightarrow S$$

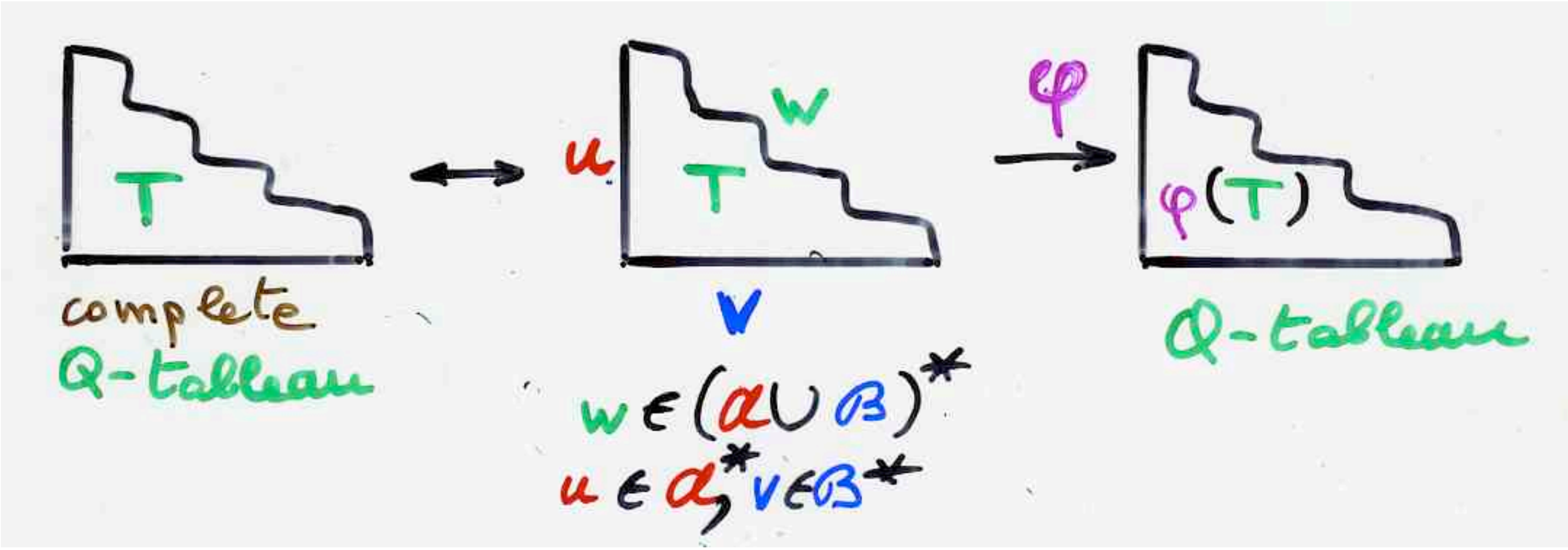
set of rewriting rules

$$B_j A_i \rightarrow C^{kl}_{ij} A_k B_l$$

such that: $\varphi\begin{pmatrix} k & l \\ i & j \end{pmatrix} = \varphi\begin{pmatrix} k' & l' \\ i' & j' \end{pmatrix} \Rightarrow \begin{cases} i \neq i' \\ j \neq j' \end{cases}$

<u>Def</u> **Q-tableau**

"image" by φ of a "complete Q-tableau"



w-compatible

w fixed

{ set of Q-tableaux w-compatible }

$\updownarrow$ bijection

{ set of complete Q-tableaux T with uwb(T) = w }

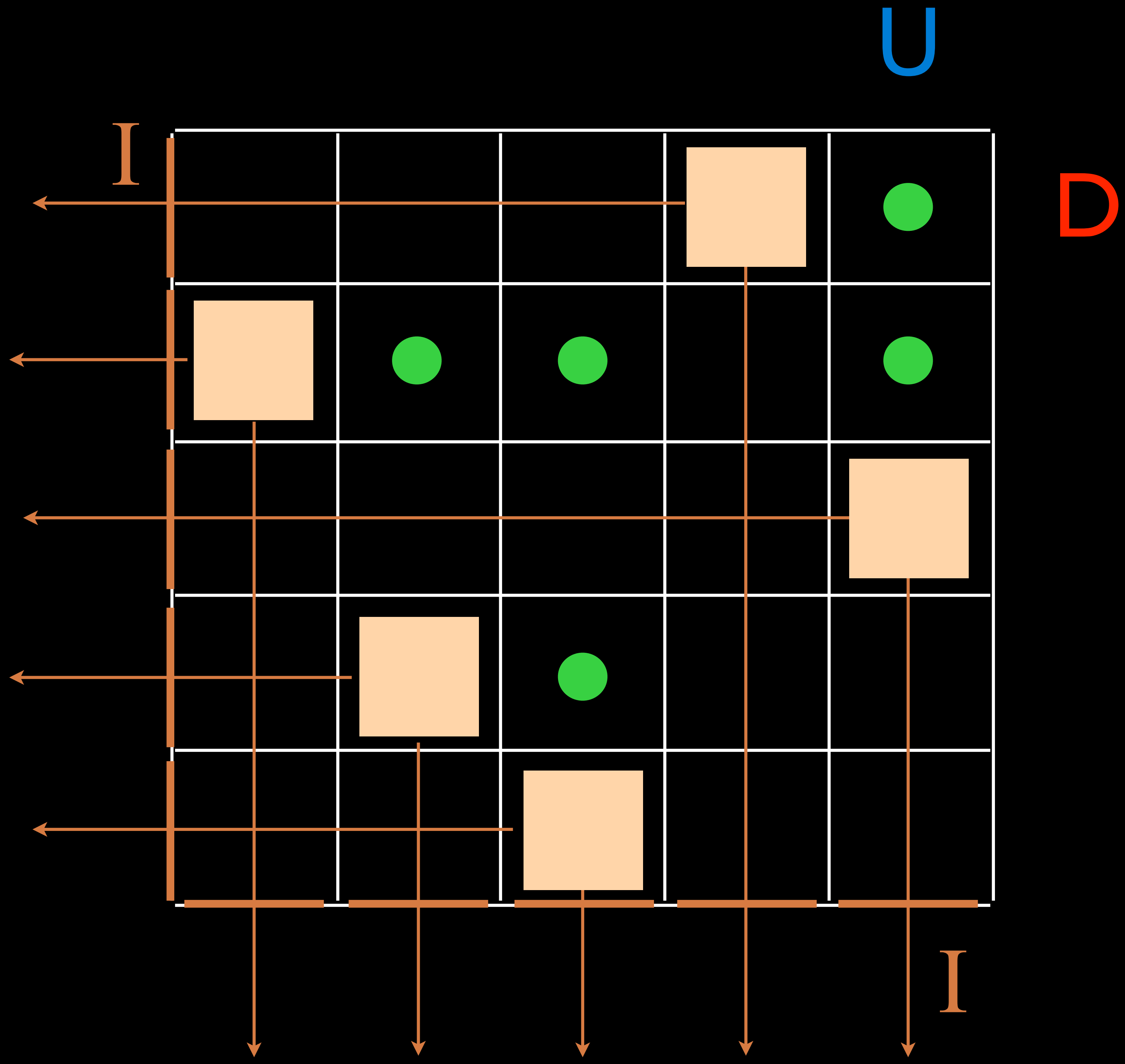
Q-tableaux: examples

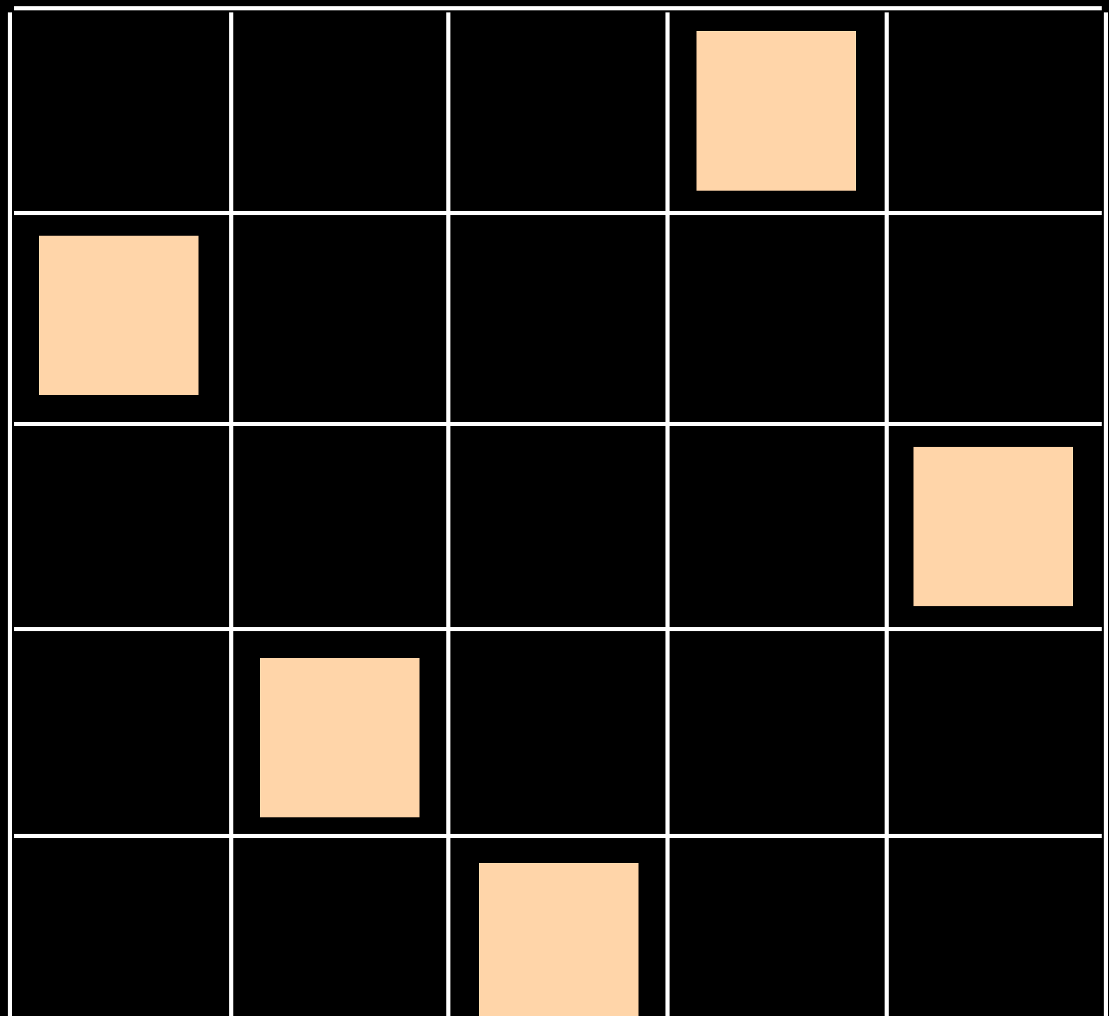
$$\begin{cases} UD = qDU + I_v I_h \\ U I_v = I_v U \\ I_h D = D I_h \\ I_h I_v = I_v I_h \end{cases}$$

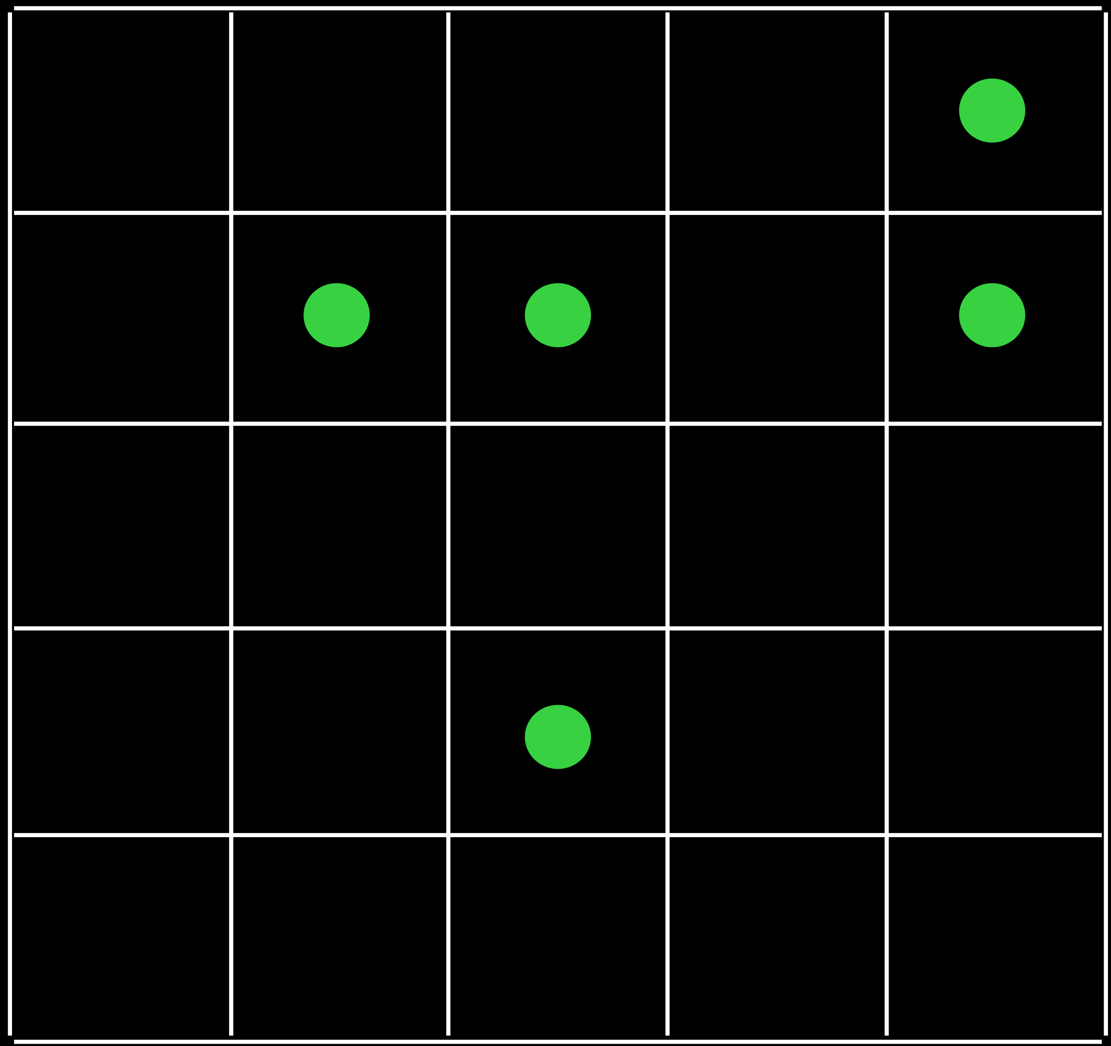
$$w = U^n D^n \qquad c(u, v; w) = n!$$
$$uv = I_v^n I_h^n$$

Q-tableau $\longleftrightarrow$ Permutations $\mathfrak{S}_n$

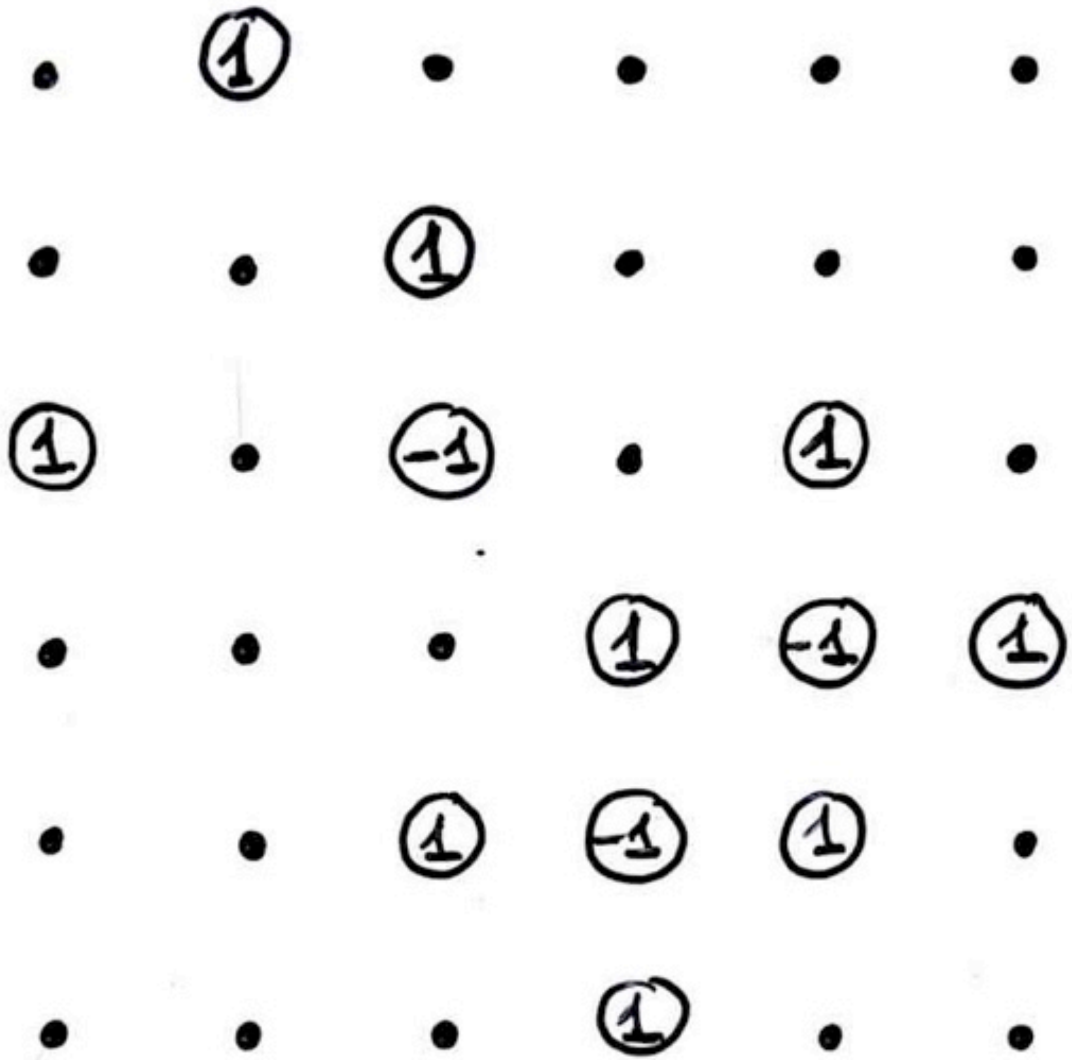
$$\begin{cases} uwb(T) = U^n D^n \\ lwb(T) = I_v^n I_h^n \end{cases}$$


U
D
I
I





ASM



Alternating sign matrices

Permutation σ

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad + \; 6$$

permutations

1 , 2 , 7 , 42 , 429 , ...

$A, A', B, B',$

commutations

$$\begin{cases} BA = AB + A'B' \\ B'A' = A'B' + AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$

$w = B^n A^n$ $\qquad$ $uv = A'^n B'^n$

$c(u, v; w)$ = number of $\text{ASM}_{n \times n}$

complete Q-tableau $\xleftrightarrow{\text{bijection}}$ $\text{ASM}_{n\times n}$

$Q(B, B', A, A')$

$uwb = B^n A^n$

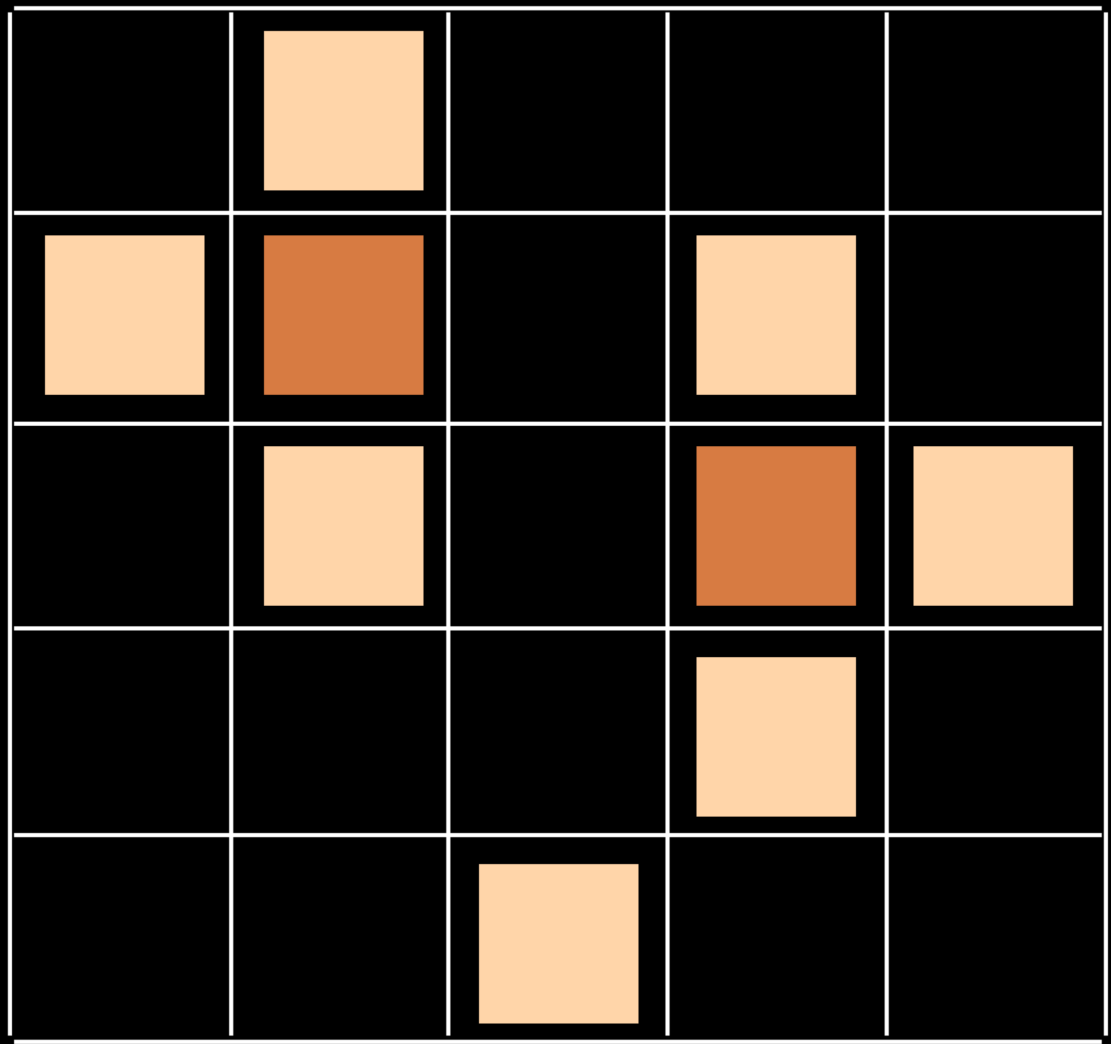
$lwb = A'^n B'^n$

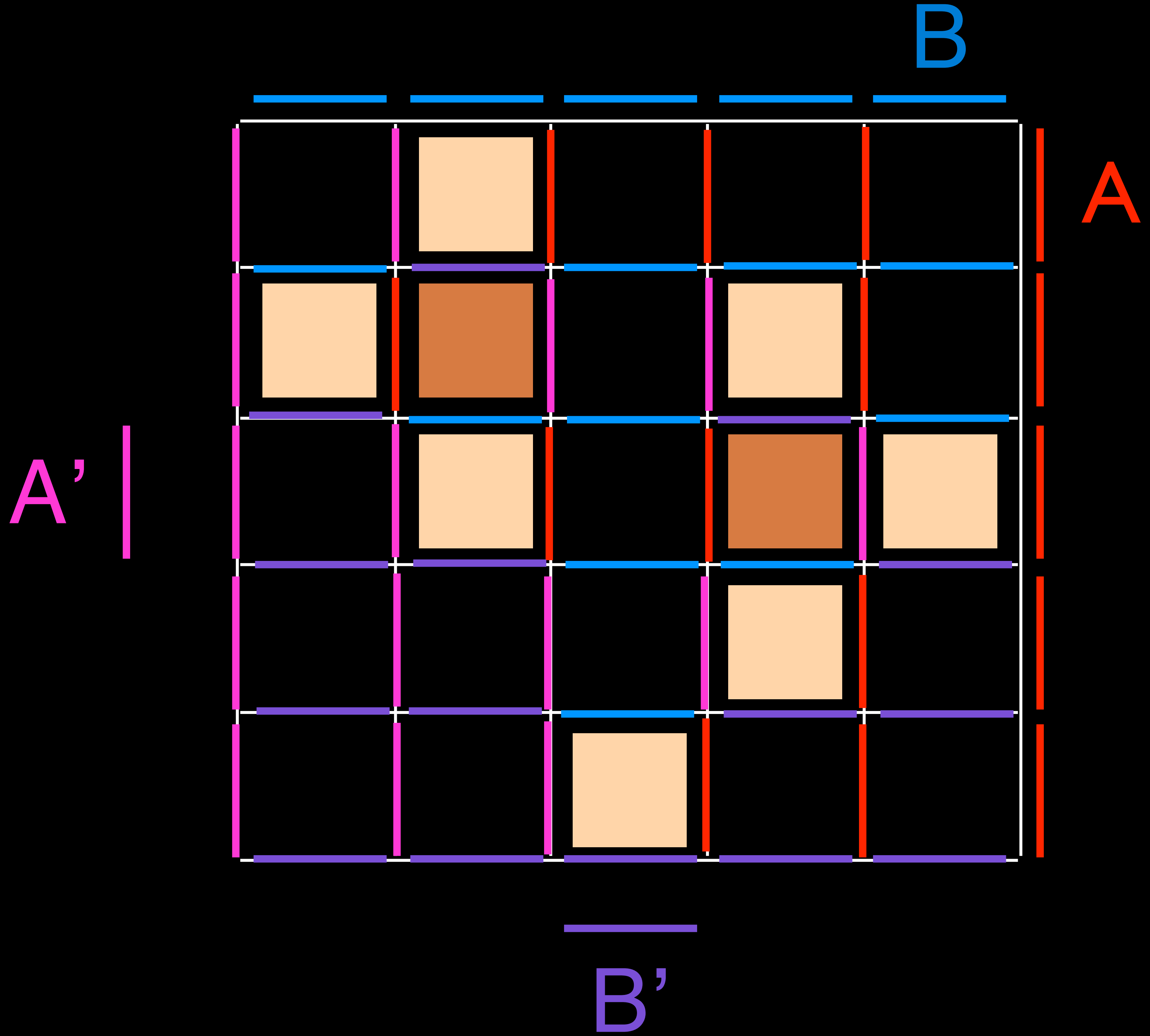
$$BA = q_{\circ\circ} AB + t_{\circ\circ} A'B'$$

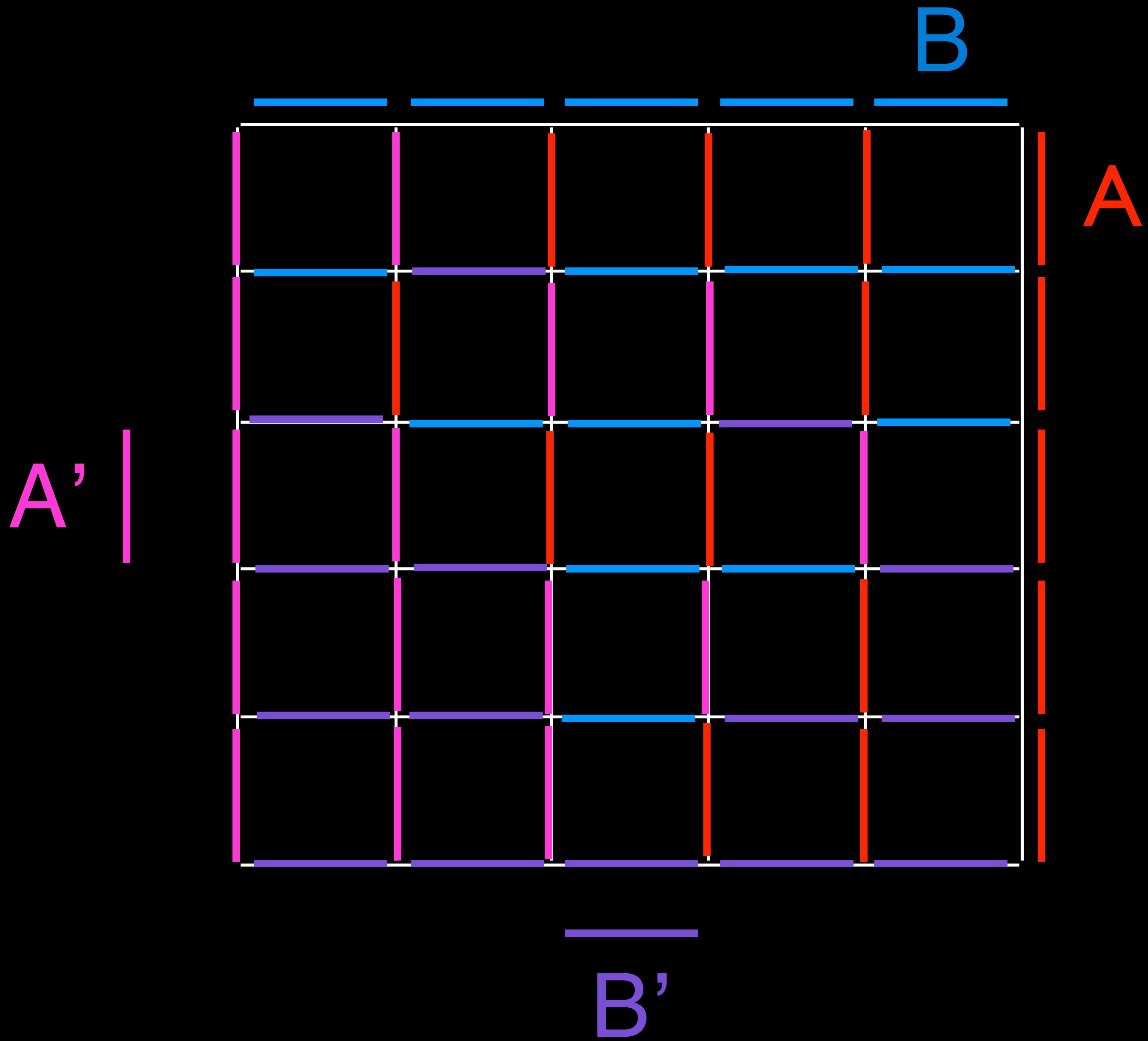
$$B'A' = q_{\bullet\bullet} A'B' + t_{\bullet\bullet} AB$$

$$B'A = q_{\bullet\circ} AB'$$

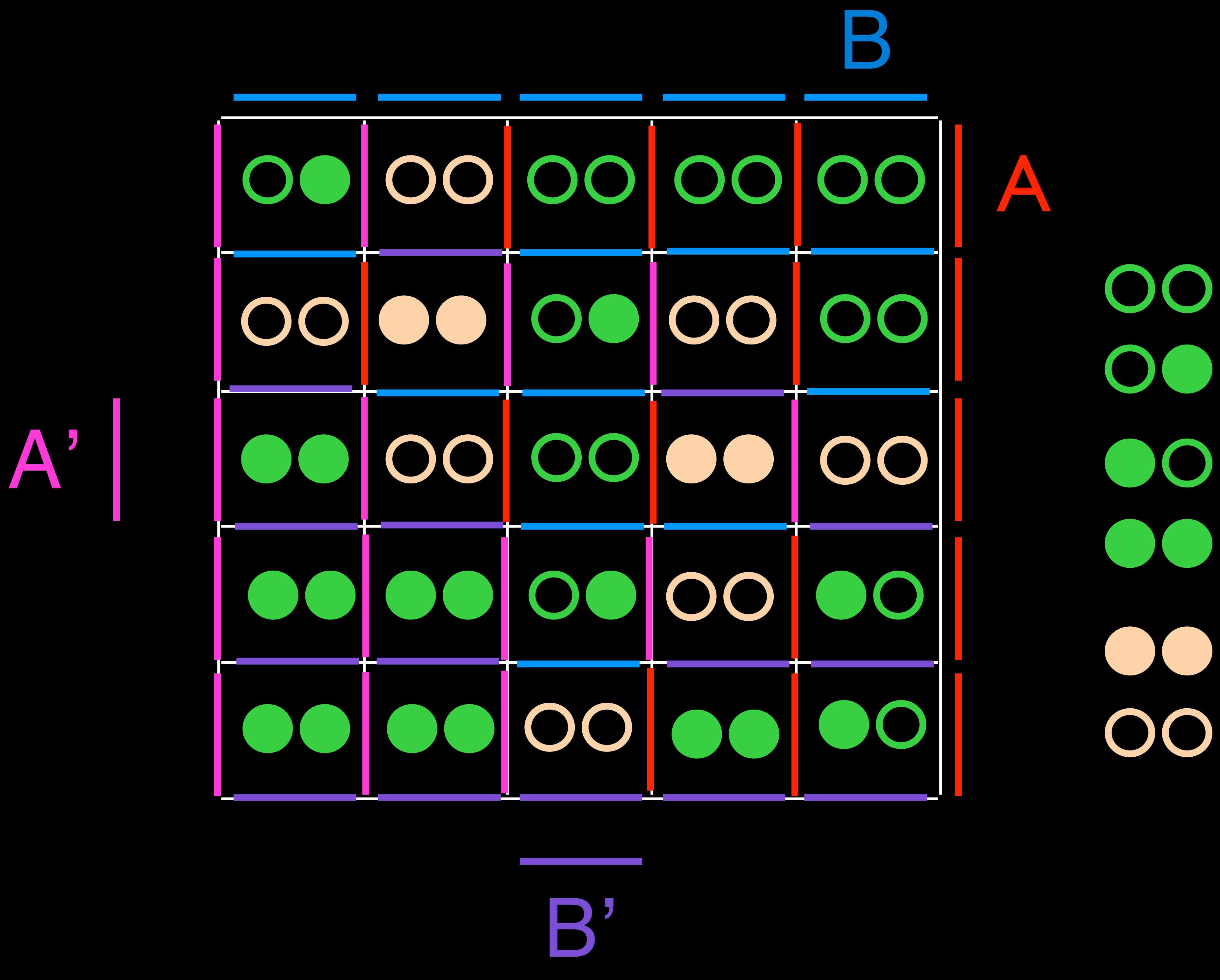
$$BA' = q_{\circ\bullet} A'B$$

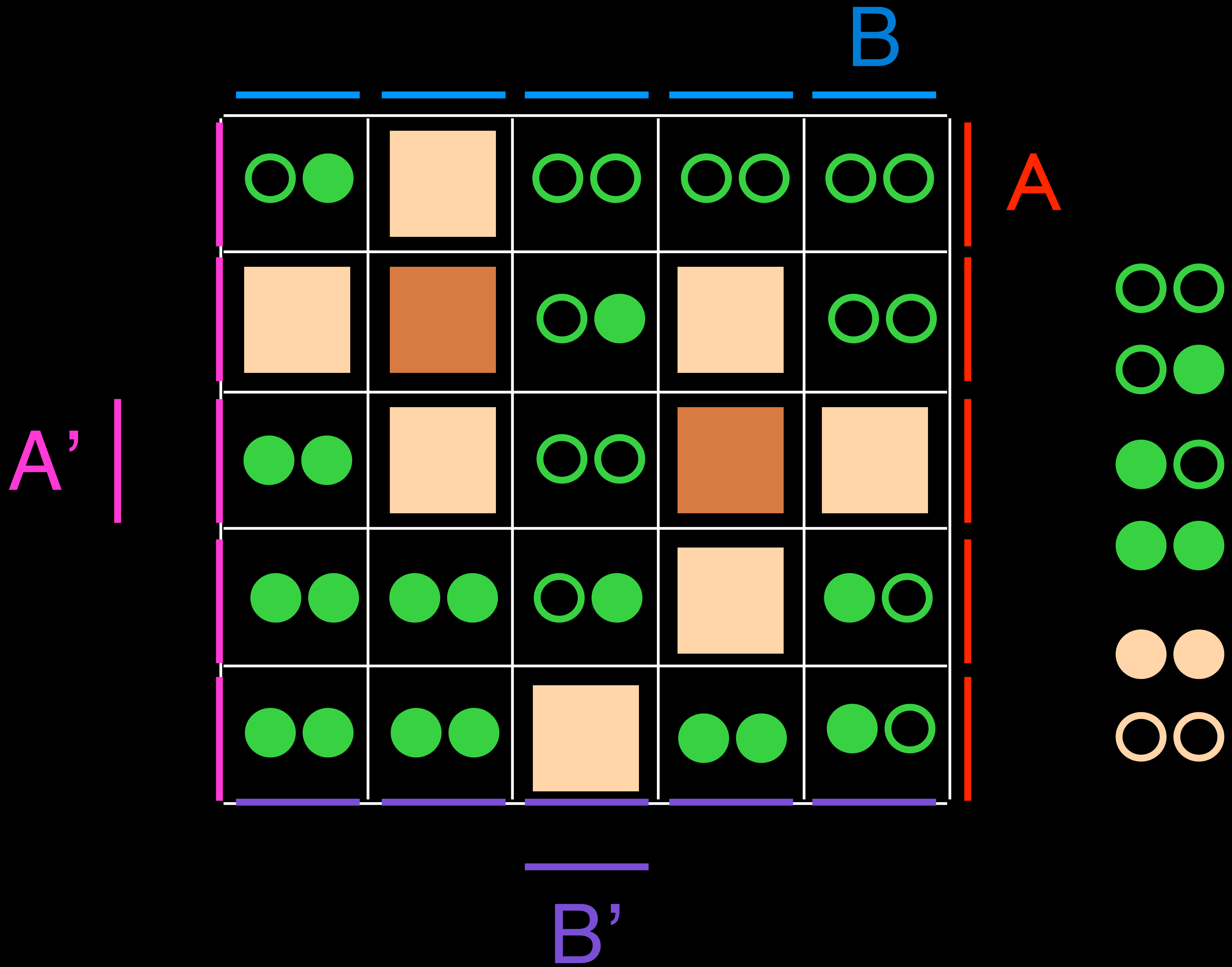


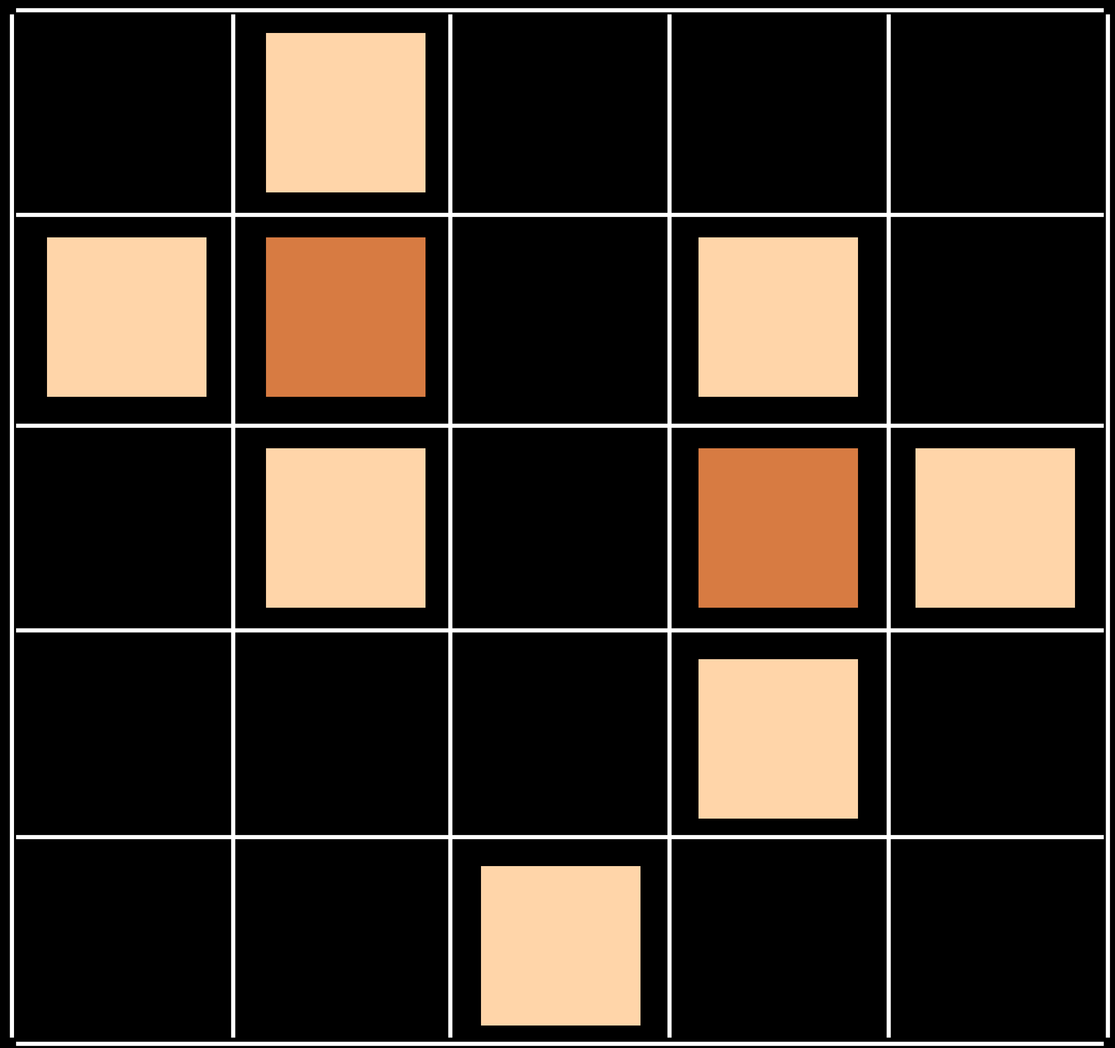

B
A
A'
B'



B
A
A'
B'


B
A
A'
B'


B
A
A’
B’

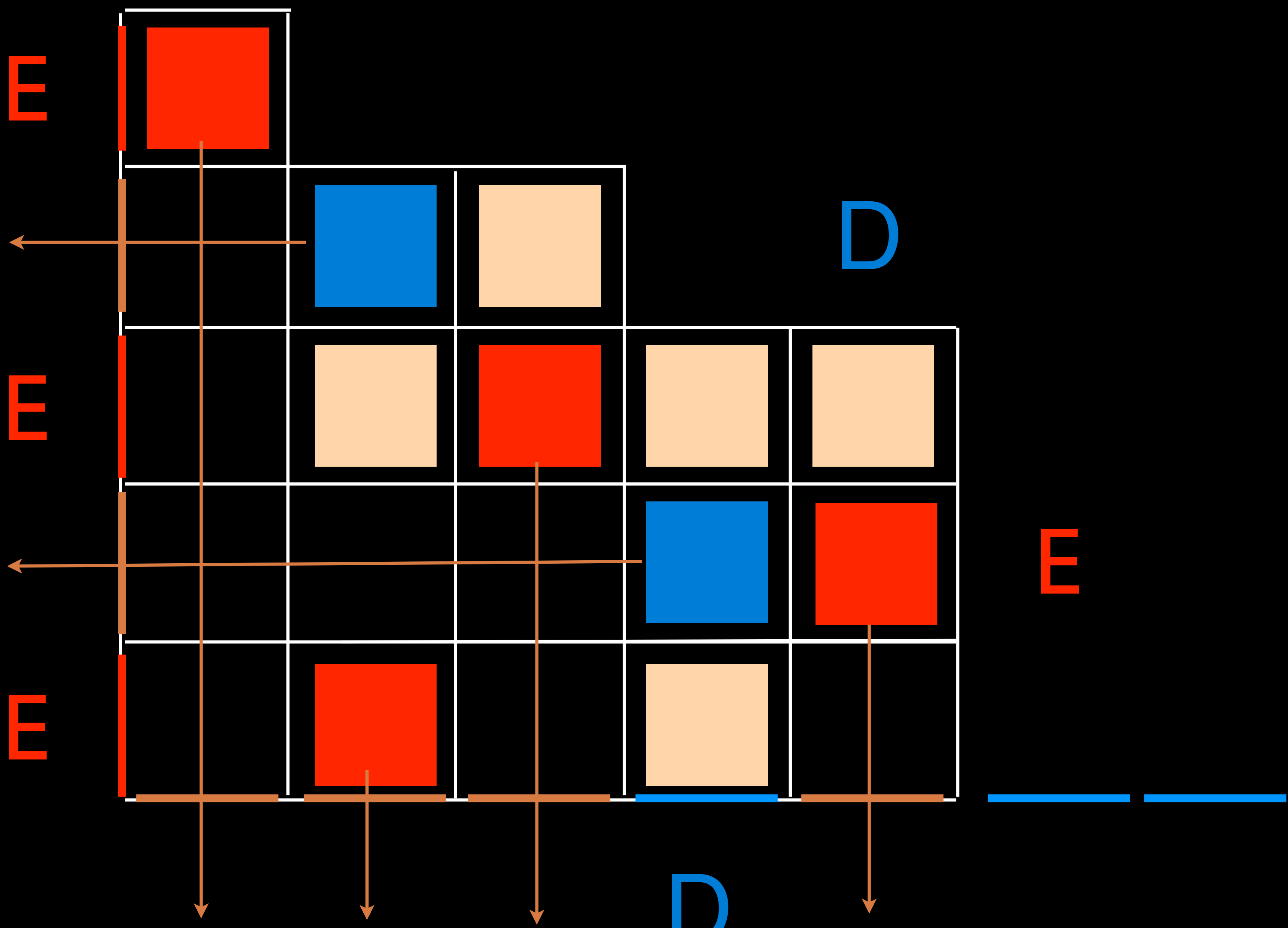


$$DE = qED + EI_h + I_v D$$

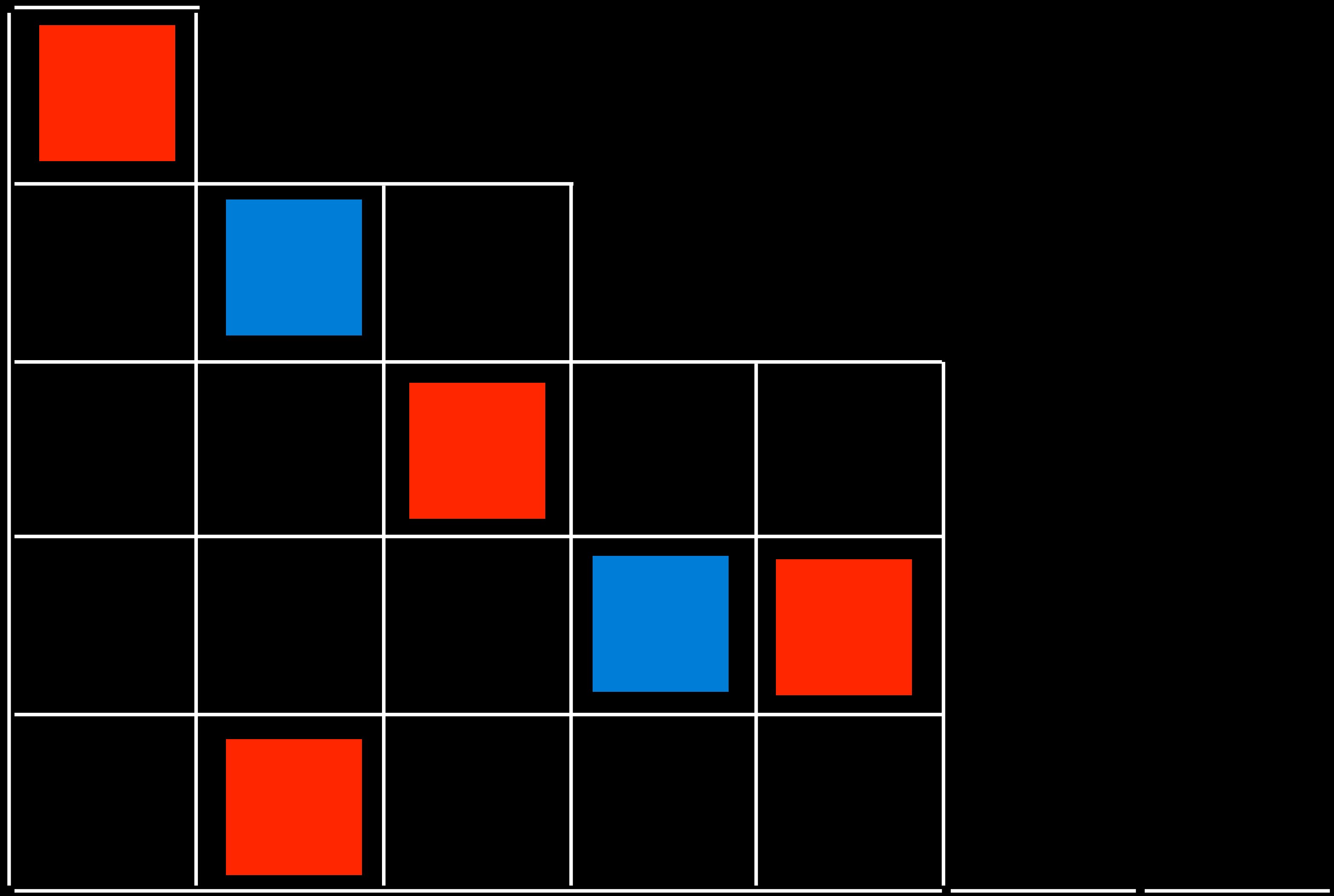
$$DI_v = I_v D$$

$$I_h E = E I_h$$

$$I_h I_v = I_v I_h$$


E
D
E
E
E
D

alternative tableau



bijection

Q-tableau ⟷ alternative tableaux

Q PASEP algebra

$$\begin{cases} BA = qAB + dAY + dA_\bullet Y_\bullet + dXB + dX_\bullet B_\bullet \\ B_\bullet A_\bullet = qA_\bullet B_\bullet + dAY + dA_\bullet Y_\bullet + dXB + dX_\bullet B_\bullet \\ B_\bullet A = qAB_\bullet + dAY + dA_\bullet Y_\bullet + dXB + dX_\bullet B_\bullet \\ BA_\bullet = qA_\bullet B + dAY + dA_\bullet Y + dXB + dX_\bullet B_\bullet \end{cases}$$

$$\begin{cases} BX = qXB \\ BX_\bullet = qX_\bullet B \\ B_\bullet X = qXB_\bullet \\ B_\bullet X_\bullet = qX_\bullet B_\bullet \end{cases} \qquad \begin{cases} YA = qAY \\ Y_\bullet A = qAY_\bullet \\ YA_\bullet = qA_\bullet Y \\ Y_\bullet A_\bullet = qA_\bullet Y_\bullet \end{cases}$$

$$\begin{cases} YX = qXY \\ YX_\bullet = qX_\bullet Y \\ Y_\bullet X = qXY_\bullet \\ Y_\bullet X_\bullet = qX_\bullet Y_\bullet \end{cases}$$

The quadratic algebra
for the tableaux interpreting
the moments of the
Askey-Wilson polynomials

S. Corteel, L. Williams
(2007) (2008) (2009) (2010)

S. Corteel, R. Stanley, D. Stanton, L. Williams (2010)

partition function Z
as a sum of certain weights of «staircase tableaux»
or as the moments of the Askey-Wilson polynomials

alternating tableaux with two colors for the blue and for the red cells,
plus two colors for each edges of the border of the Ferrers diagram

The number of alternating tableaux with two colors for the blue and for the red cells is $2^n n!$

Planar automaton

Def. planar automaton $\mathcal{P}$

- 3 finite sets
  - $\mathcal{B}$ horizontal alphabet
  - $\mathcal{A}$ vertical alphabet
  - S planar labels (state)

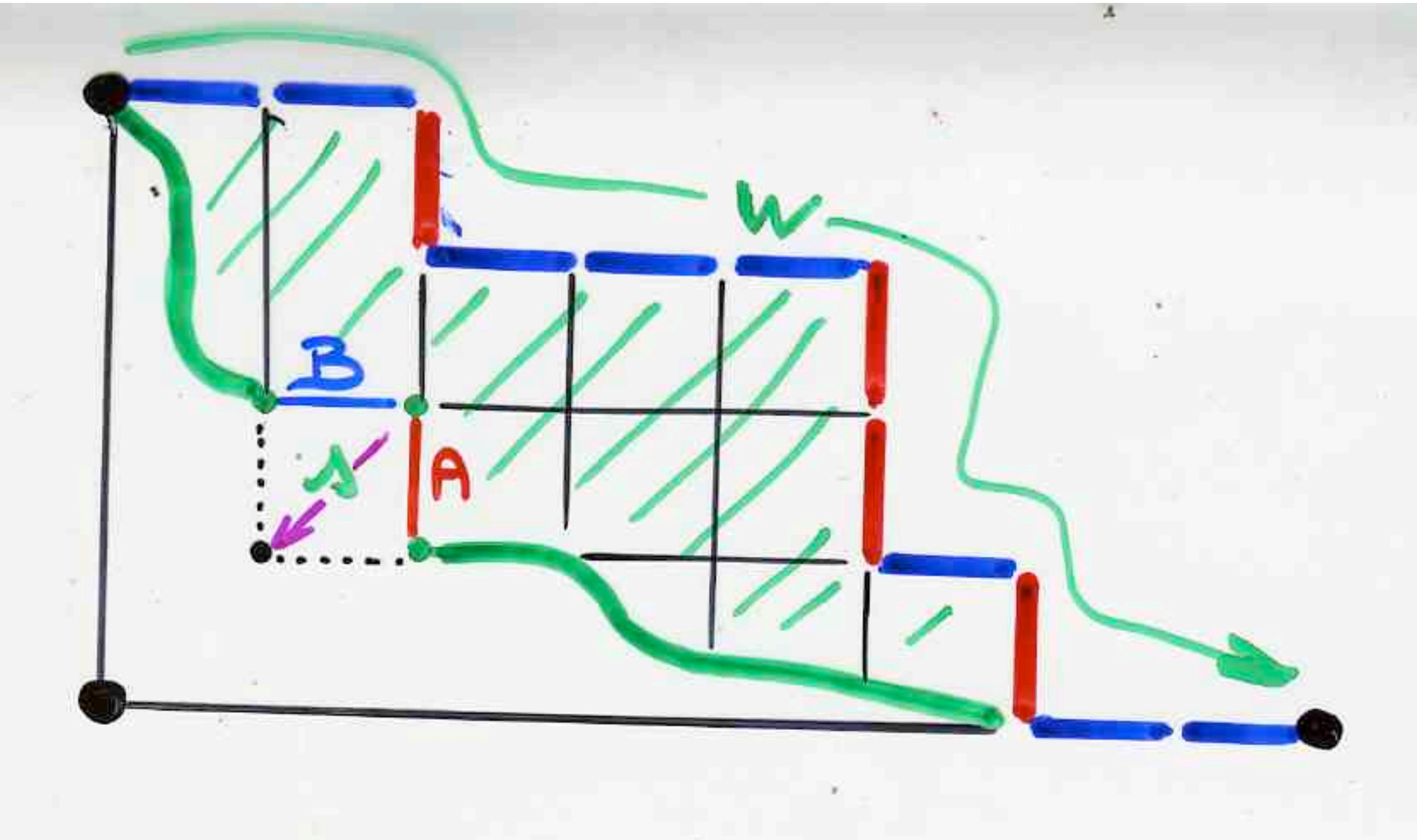
- θ (partial) transition function

$$(s, B, A) \xrightarrow{\theta} (B', A') \quad \text{or} \quad \emptyset$$

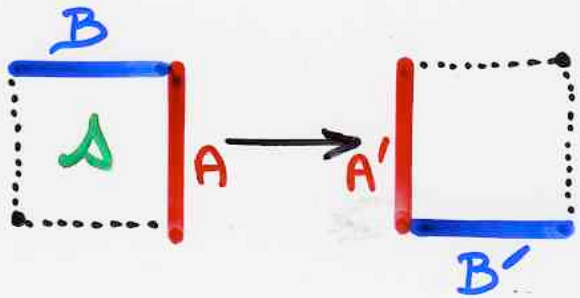
$s \in S$; $B, B' \in \mathcal{B}$; $A, A' \in \mathcal{A}$

- $w \in (\mathcal{A} \cup \mathcal{B})^*$ initial word
- uv , $u \in \mathcal{A}^*$, $v \in \mathcal{B}^*$ final word

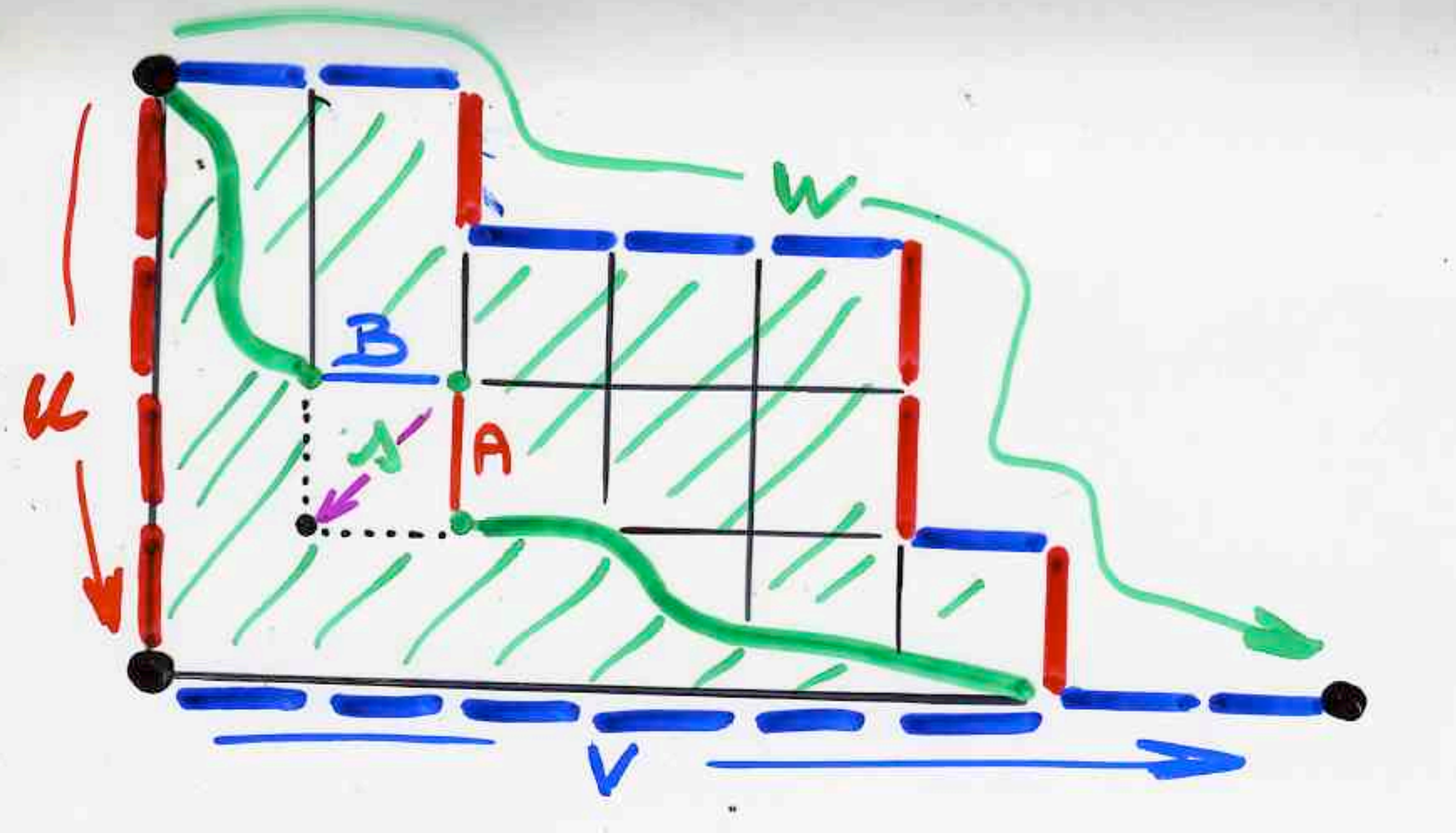
Def. tableau T accepted by a planar automaton P = (S, B, A, θ, w, uv)



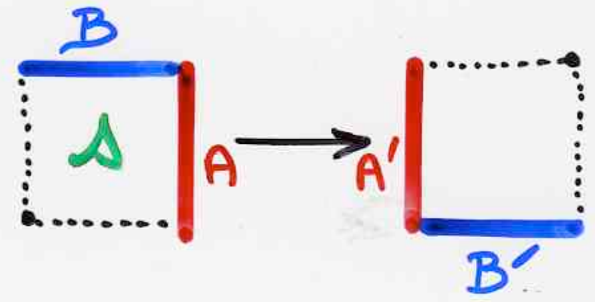
planar rewriting



Def. tableau T accepted by a planar automaton $P = (S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



planar rewriting



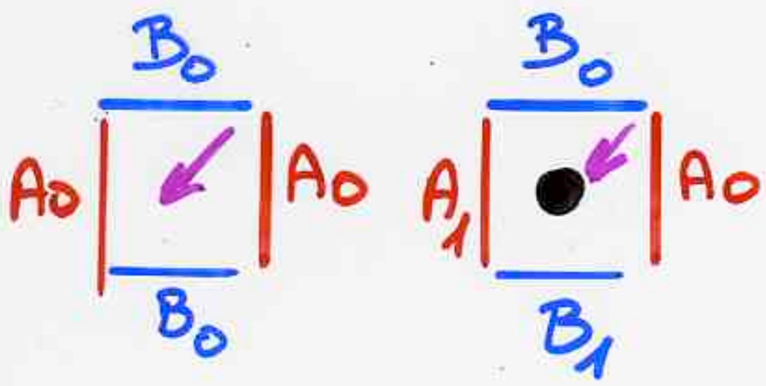
The "RSK planar automaton"

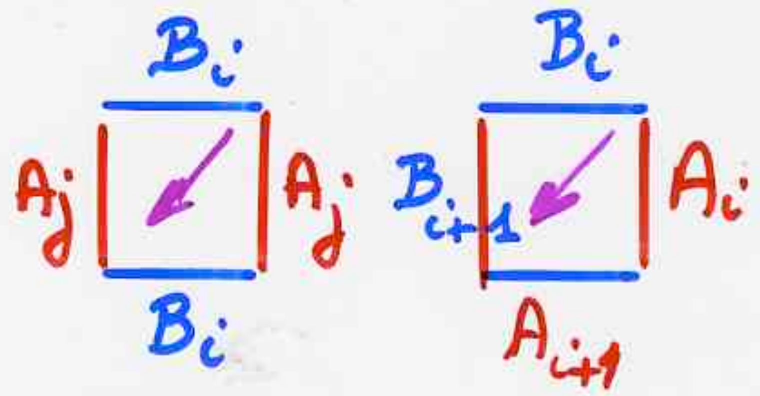
$\mathcal{B} = \{B_0, B_1, \ldots B_k\}$

$\mathcal{A} = \{A_0 \quad A_1 \quad A_k\}$

$w \in \{B_0, A_0\}^*$

$S = \{\square, \boxdot\}$





$i \neq j$

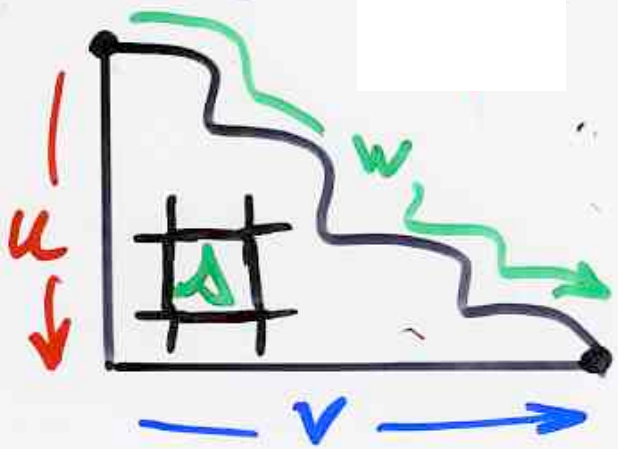
equivalence

Q-tableaux $\longleftrightarrow$ tableaux accepted by a planar automaton

Q quadratic algebra

φ

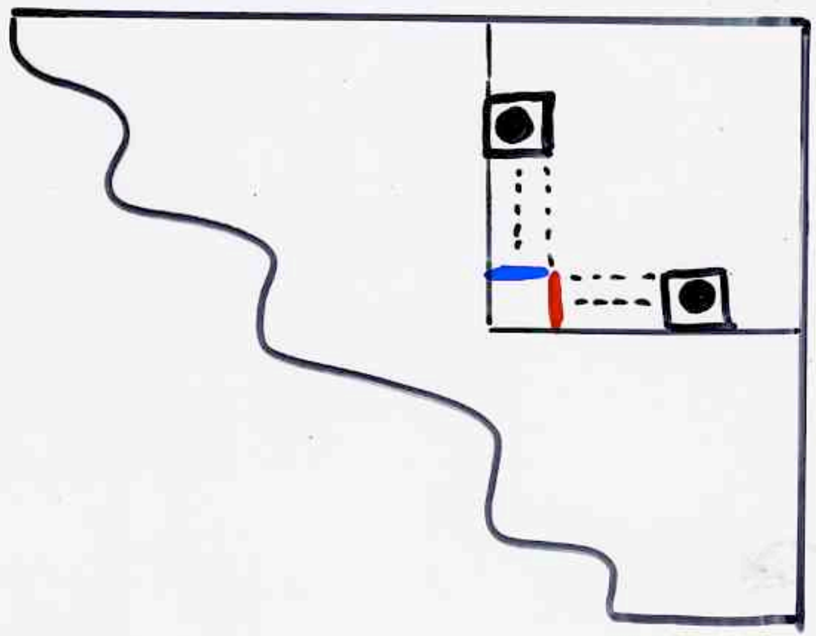
$\mathcal{P} = (S, \mathcal{B}, \mathcal{A}, \theta, w, uv)$



$$BA = \sum_{s \in S} A'B' \qquad (B', A') = \theta(s, B, A)$$

«Figures»
accepted by planar automata ?

- surjectivity in diagrams



L - diagrams

(┘- , ┐-)

Bijections between pattern-avoiding fillings of Young diagrams

Josuat-Vergès (2008)

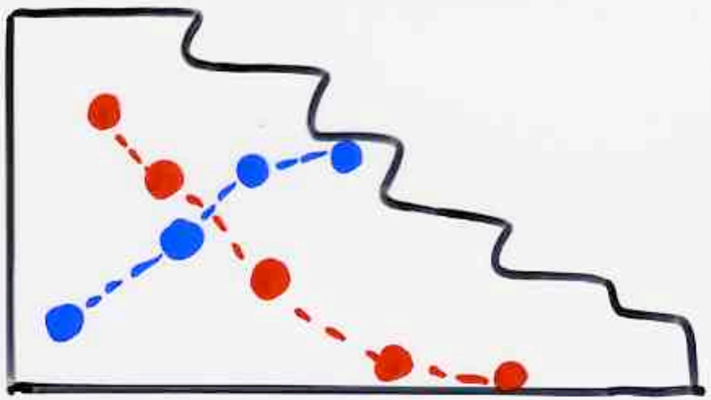
⅂-diagrams

X-diagrams

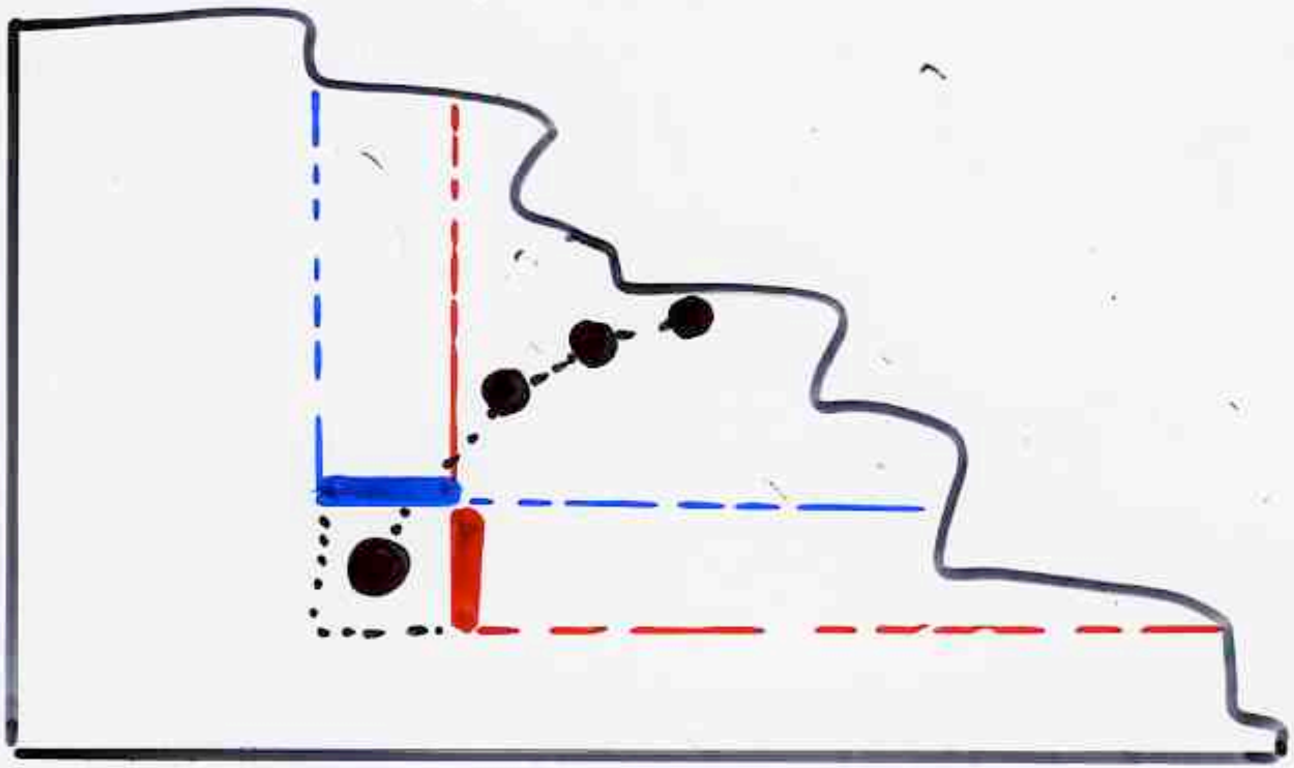
?

increasing / decreasing chains in fillings of Ferrers shapes

(Jonsson, 2005) (Krattenthaler, 2006)
(Backelin, West, Xin, 2005) (Bousquet-Mélou, Steingrímsson, 2005) ....



increasing / decreasing subsequences (chains)



The 8-vertex algebra
(or Z - algebra)

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + t_{\bullet\circ}\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + t_{\circ\bullet}\, A B_\bullet \end{cases}$$

$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

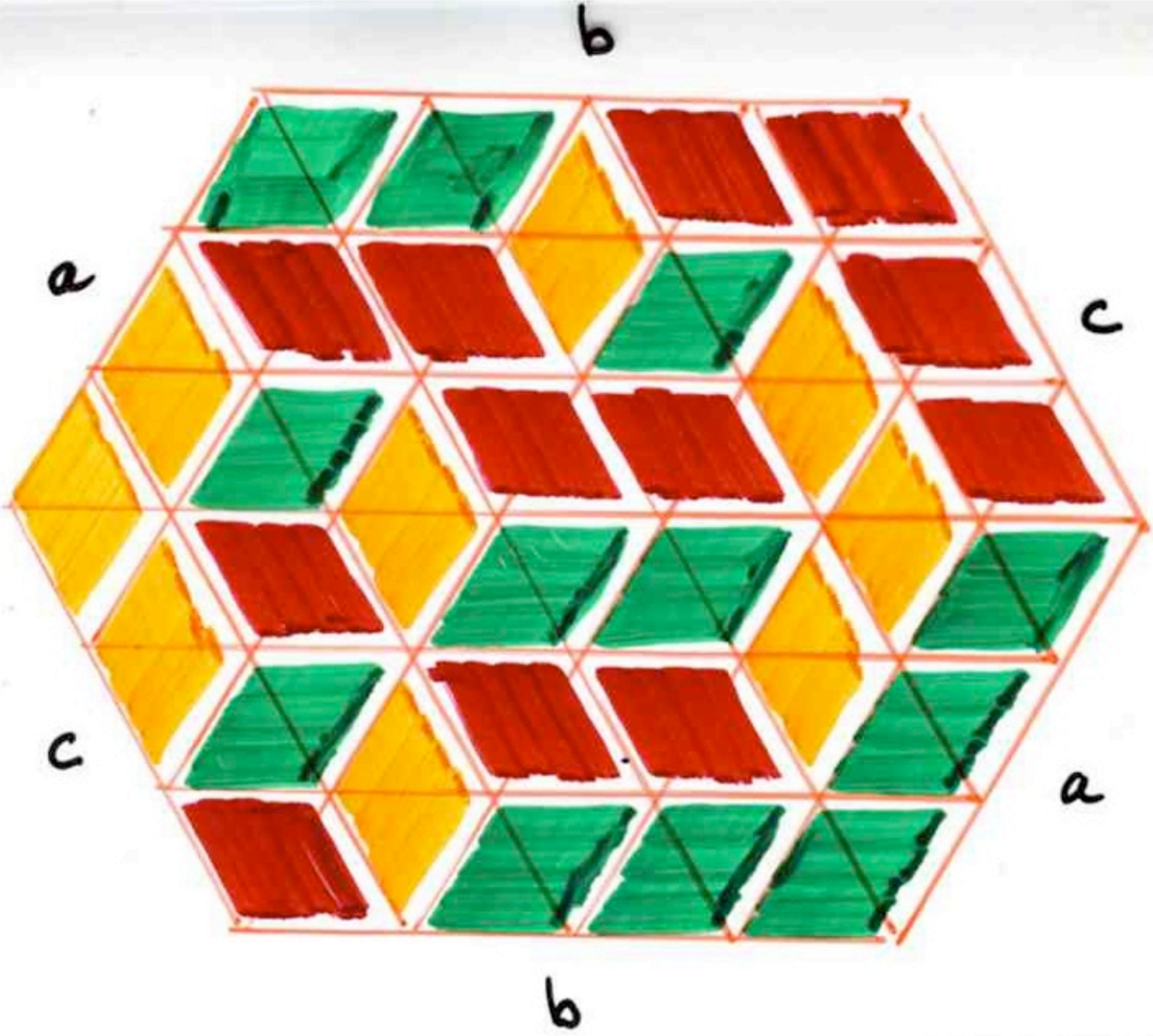
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$

$$w = B^n A^n \qquad uv = A_\bullet^n \, B_\bullet^n$$

$c(u, v ; w)$ = nb of ASM $n \times n$

example:
rhombus tilings


b
a
c
c
a
b

$$
\begin{cases} t_{\bullet\circ} = t_{\circ\bullet} = 0 \\ q_{\bullet\bullet} = 0 \end{cases} \qquad \text{(ASM)}
$$

Rhombus tilings

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

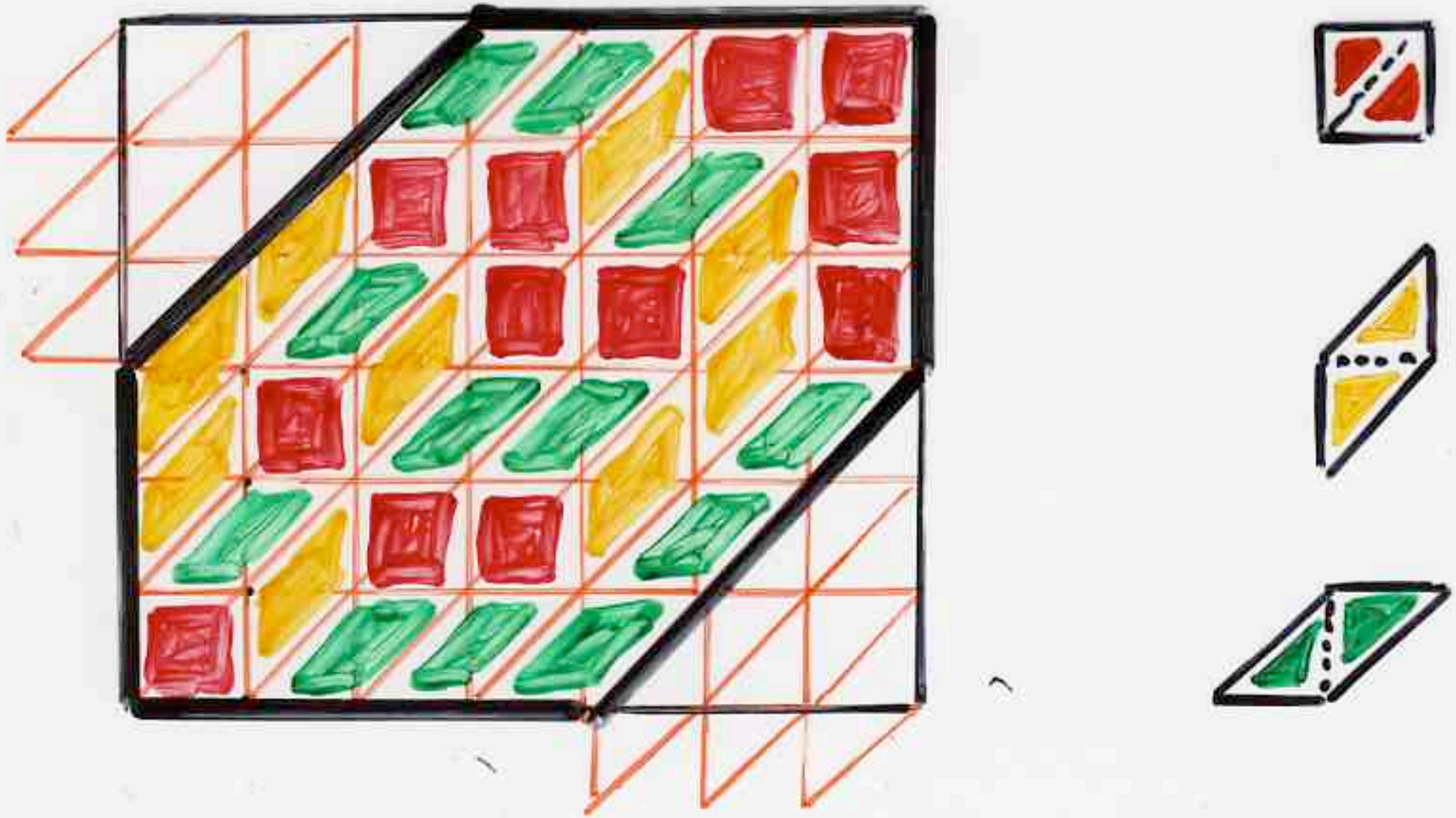
$$
\begin{cases}
B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\
B_\bullet A = q_{\bullet\circ} \, A B_\bullet + \bigcirc \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet
\end{cases}
$$

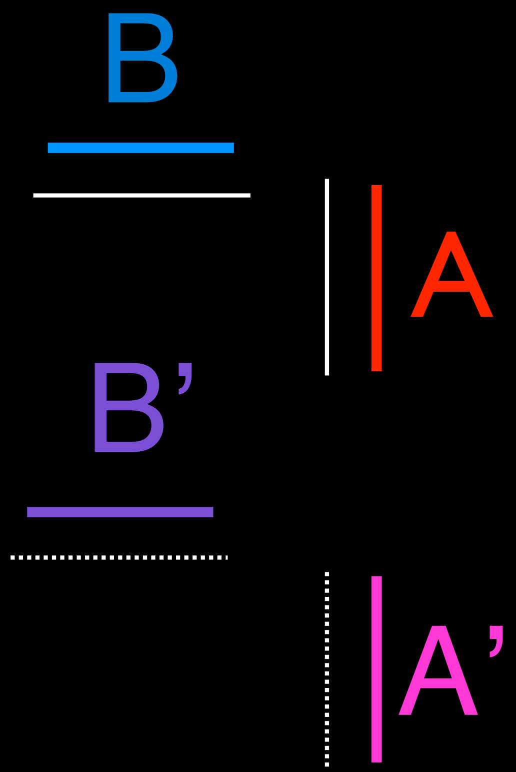
$A, A', B, B',$

commutations

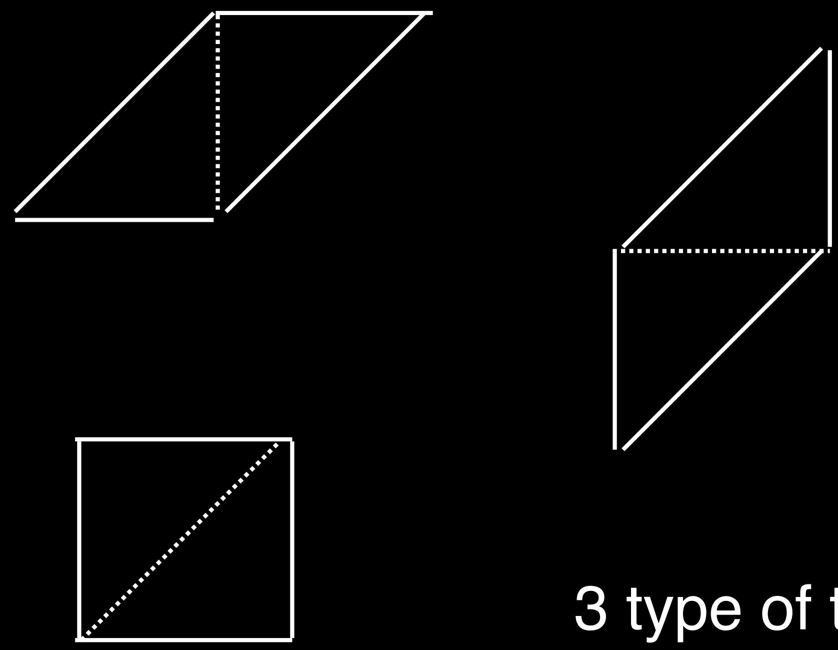
$$\begin{cases} BA = AB + A'B' \\ B'A' = AB \end{cases}$$

$$\begin{cases} B'A = AB' \\ BA' = A'B \end{cases}$$



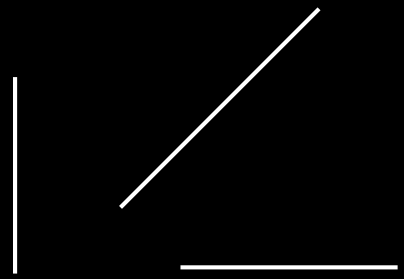


coding of the edges
for tilings
of the triangular lattice

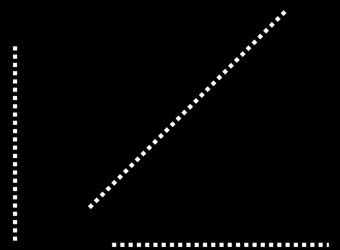


3 type of tiles

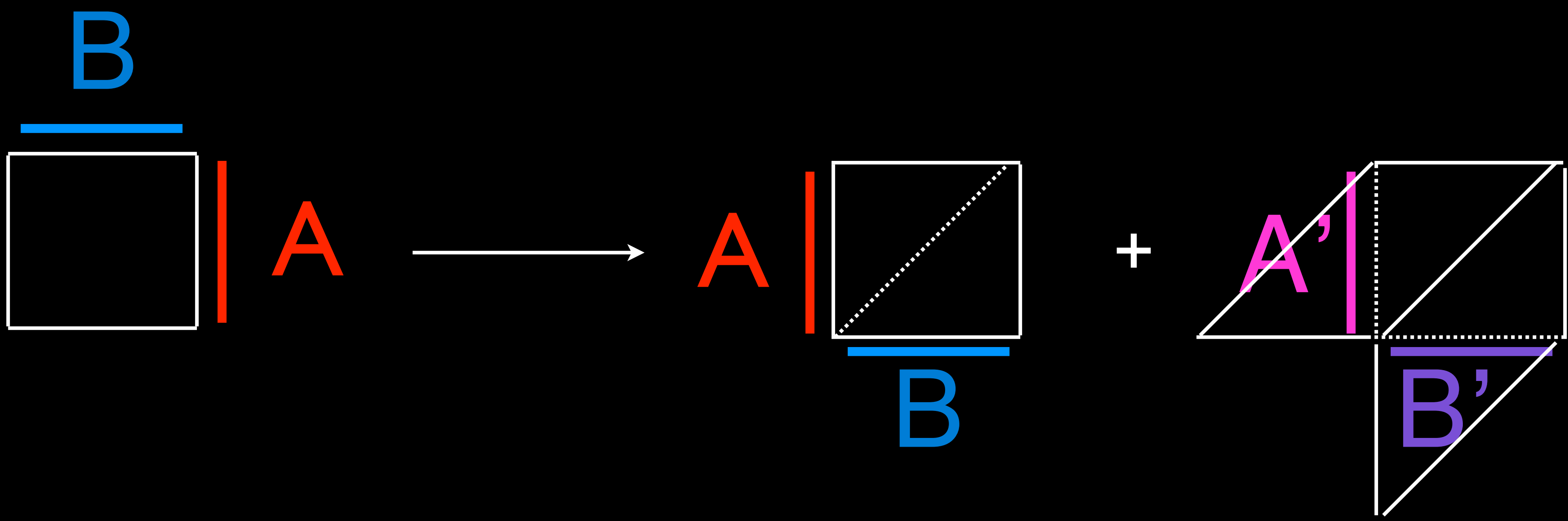
border of a tile



inside a tile



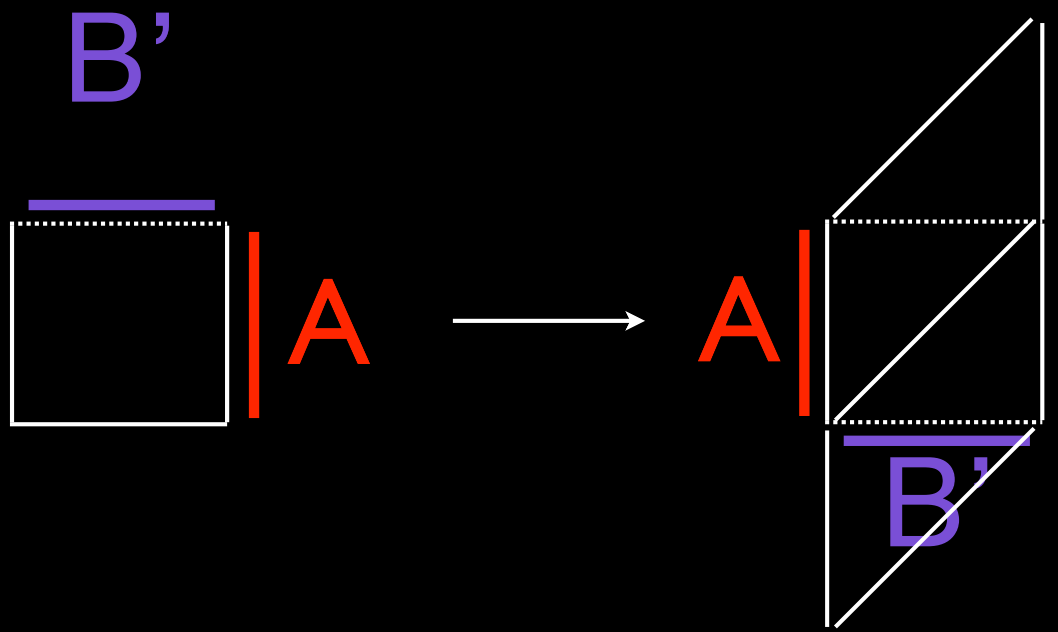
“rewriting rules” for tilings of the triangular lattice



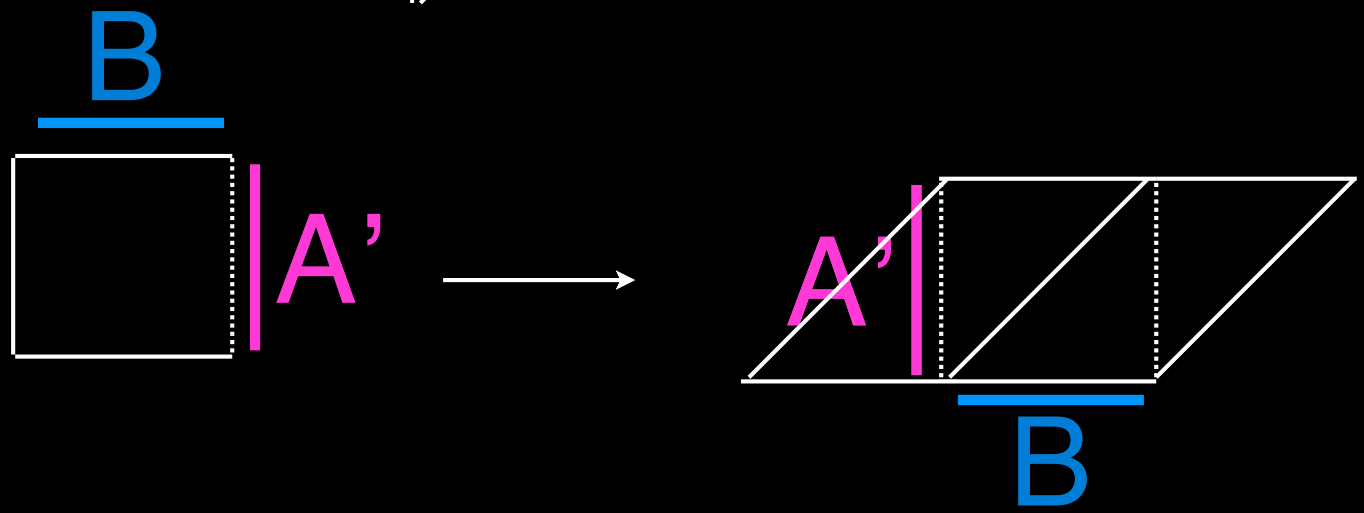
$$BA = AB + A'B'$$

$$B'A' = AB$$

"rewriting rules" for tilings of the triangular lattice



B' A = A B'



B A' = A' B

“rewriting rules” for tilings of the triangular lattice

$$BA = AB + A'B'$$

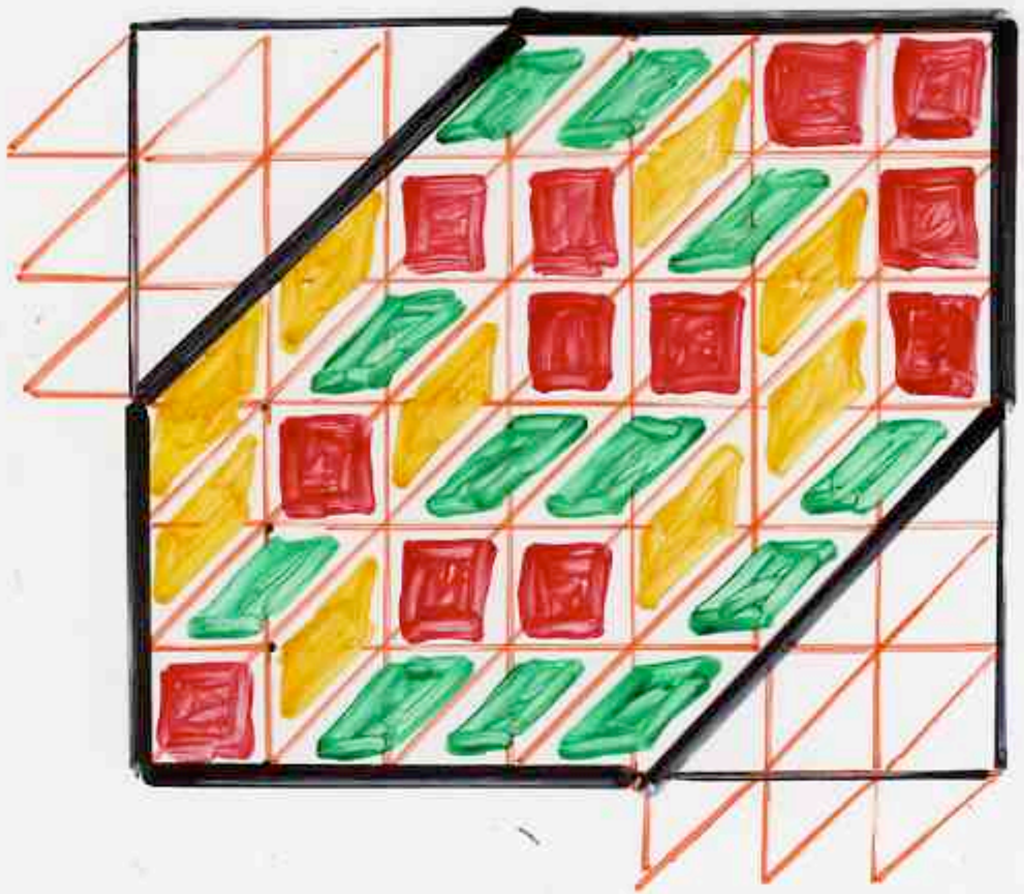
$$B'A' = AB$$

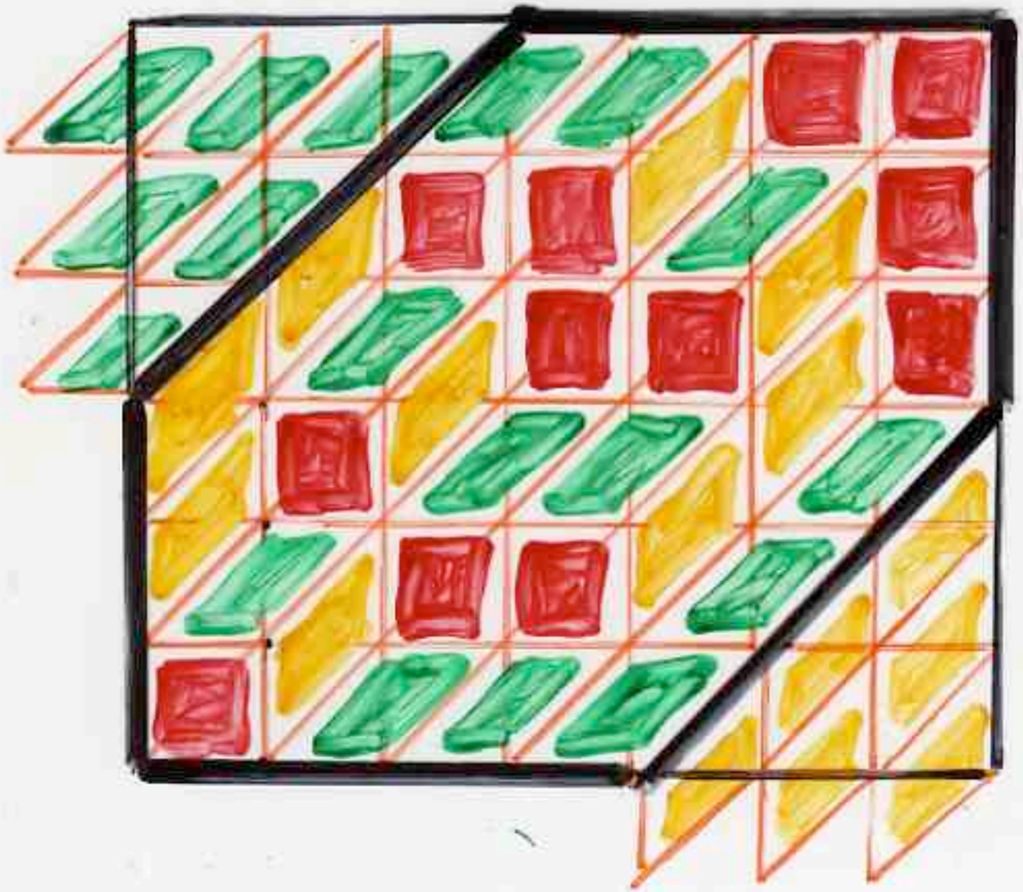
$$B'A = AB'$$

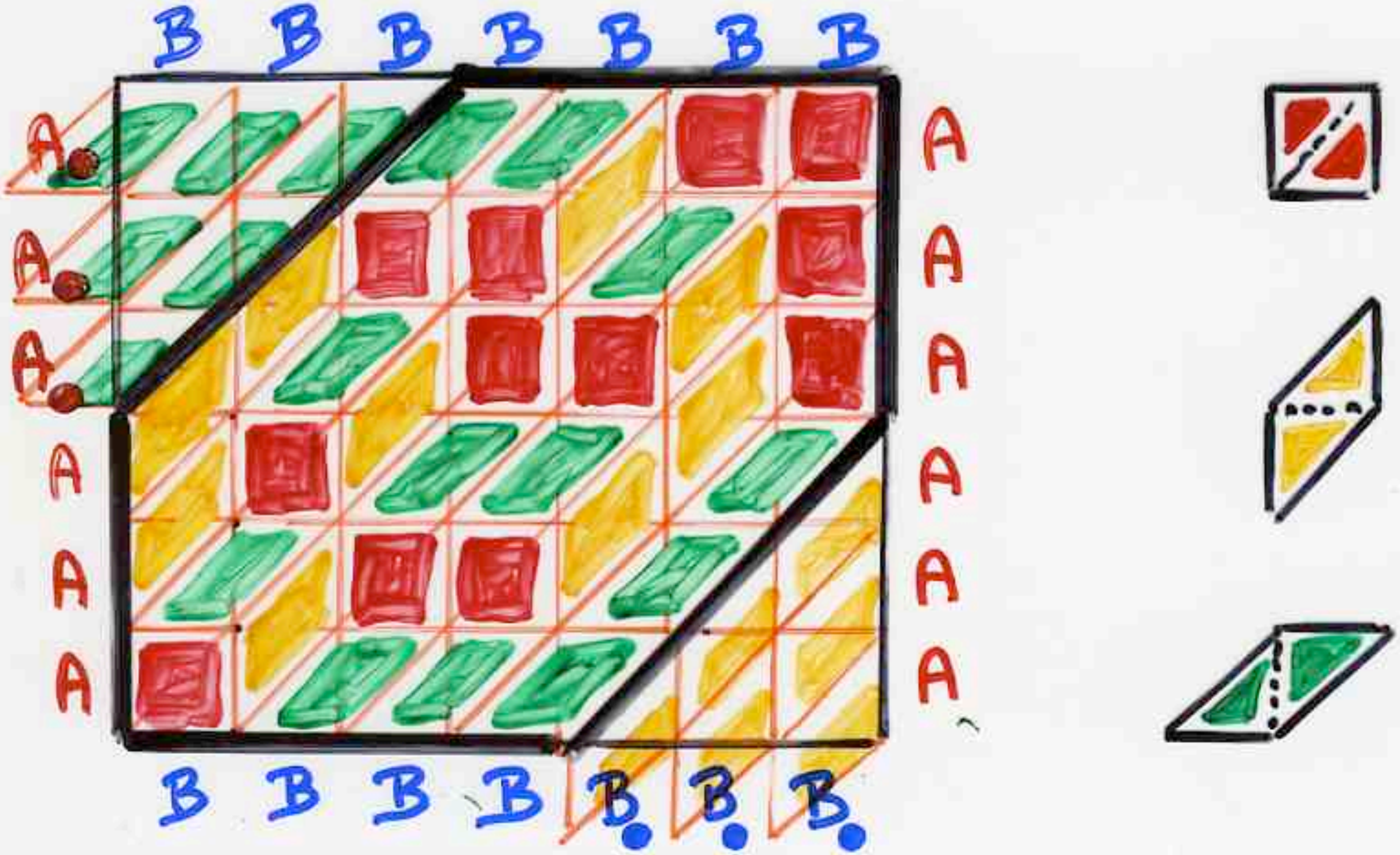
$$BA' = A'B$$

same as for ASM , except the rewriting rule

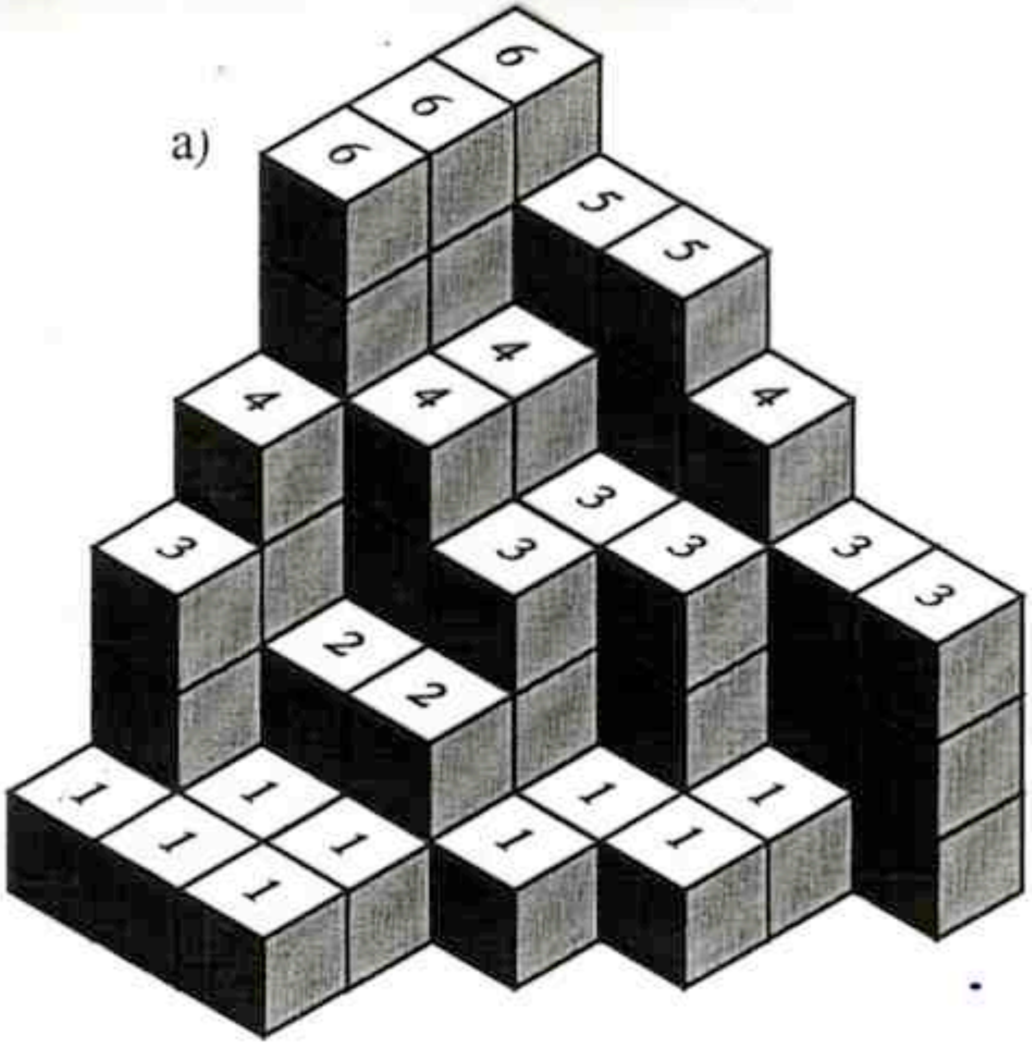
$B'A' \longrightarrow A'B'$ is forbidden




B B B B B B B
A A A A A A
A A A A A A
B B B B B B B

a)



b)

6 5 5 4 3 3

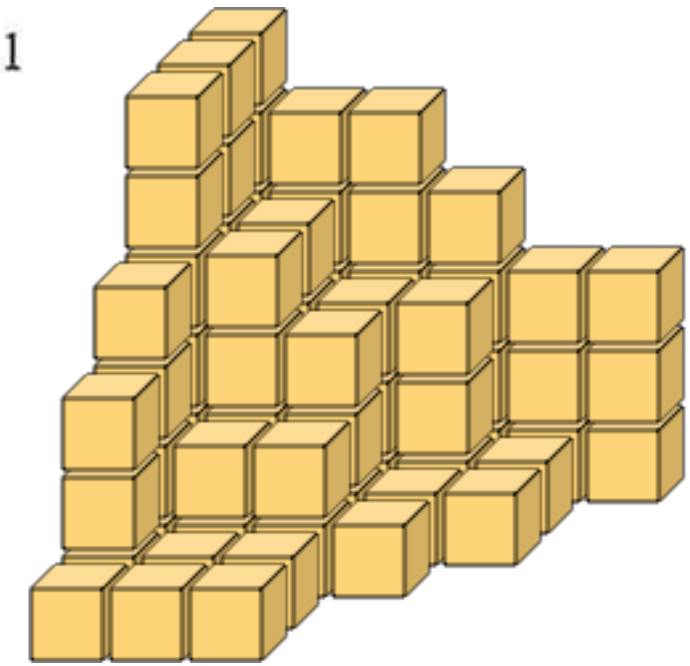
6 4 3 3 1

6 4 3 1 1

4 2 2 1

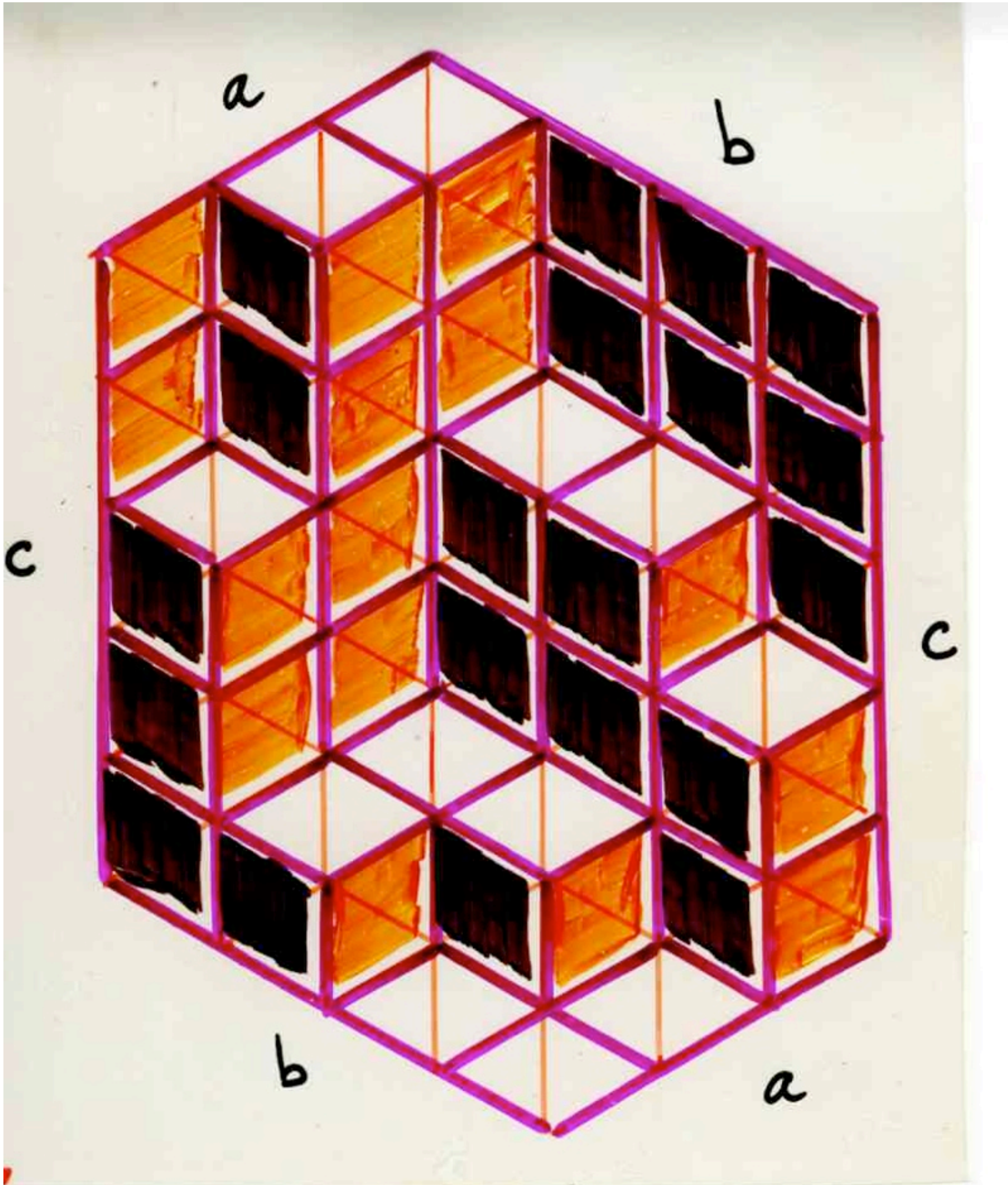
3 1 1

1 1 1



example:
plane
partitions
in a box

(MacMahon
formula)

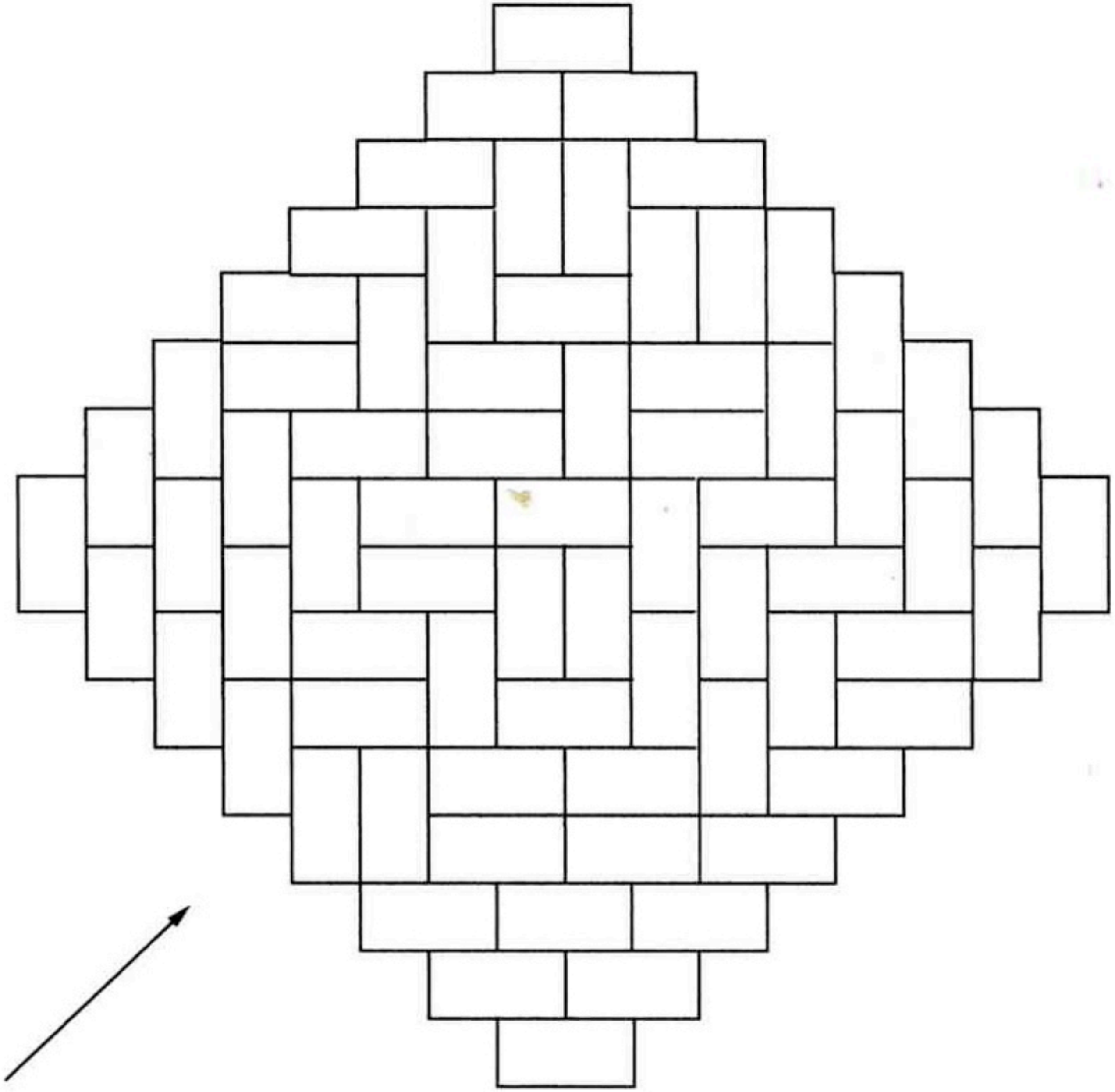


$$\prod_{\substack{1 \leq i \leq a \\ 1 \leq j \leq b \\ 1 \leq k \leq c}} \frac{i+j+k-1}{i+j+k-2}$$

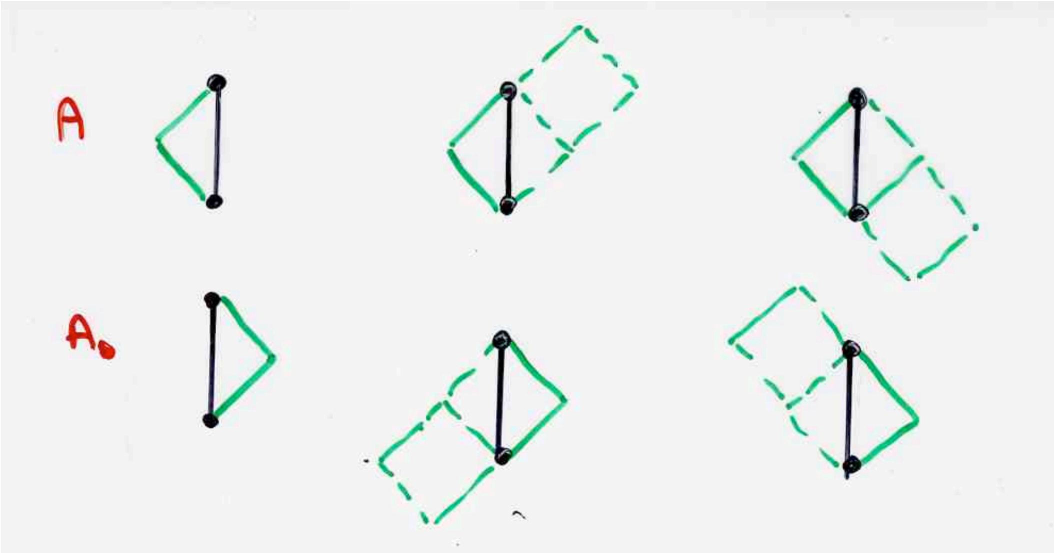
example:
Aztec diamond

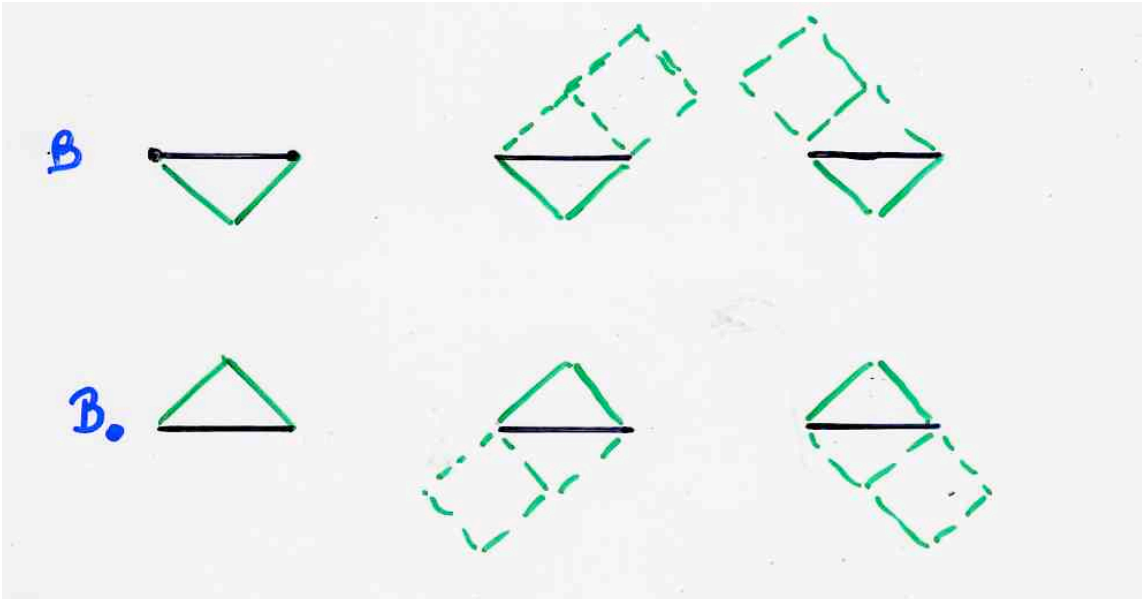
$2^{n(n-1)/2}$

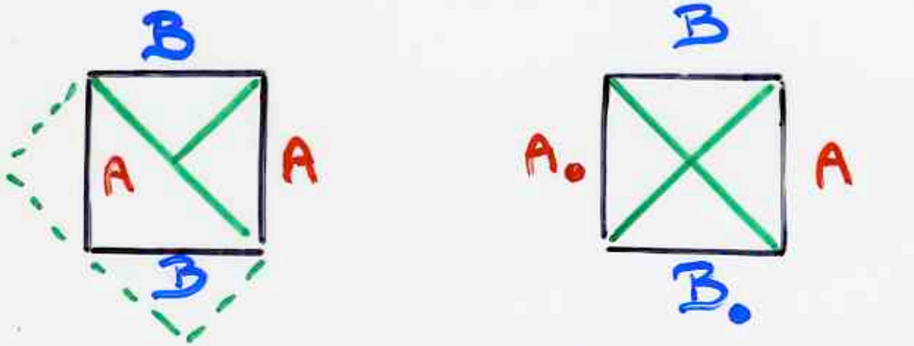
$A_n(2)$



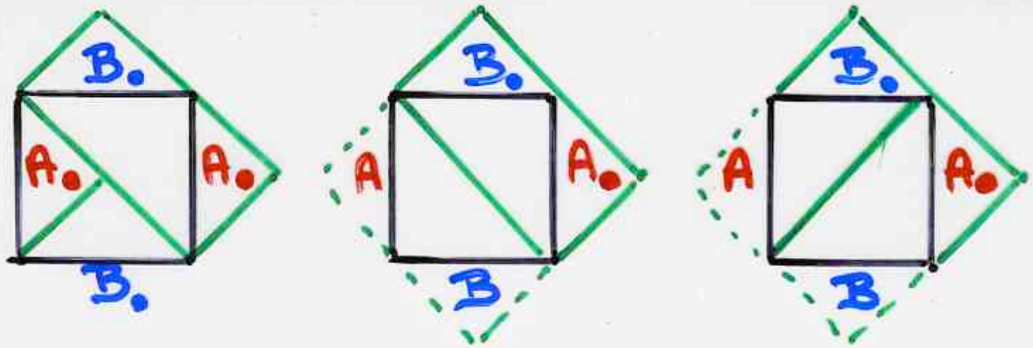
Elkies,
Kuperberg,
Larsen,
Propp
(1992)


A
A•


B
B•



$$B A = A B + A_\bullet B_\bullet$$



$$B_\bullet A_\bullet = A_\bullet B_\bullet + 2 A B$$

$$B_\bullet A = A B_\bullet \qquad B A_\bullet = A_\bullet B$$

Aztec tilings

$$t_{\bullet\circ} = t_{\circ\bullet} = 0 \qquad \text{(ASM)}$$

$$t_{\bullet\bullet} = 2 \qquad (\text{nb of } -1 \text{ in ASM})$$

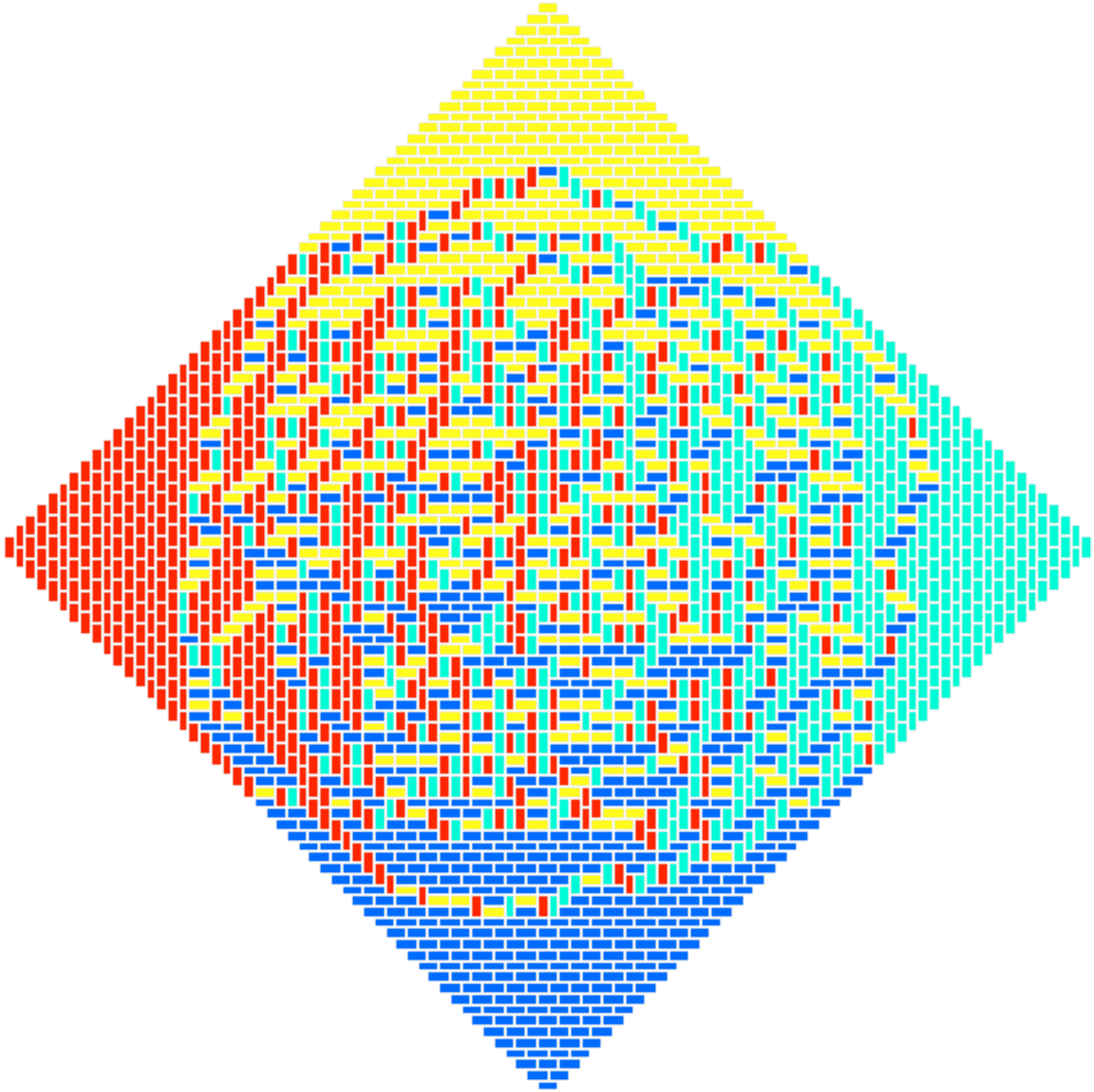
The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

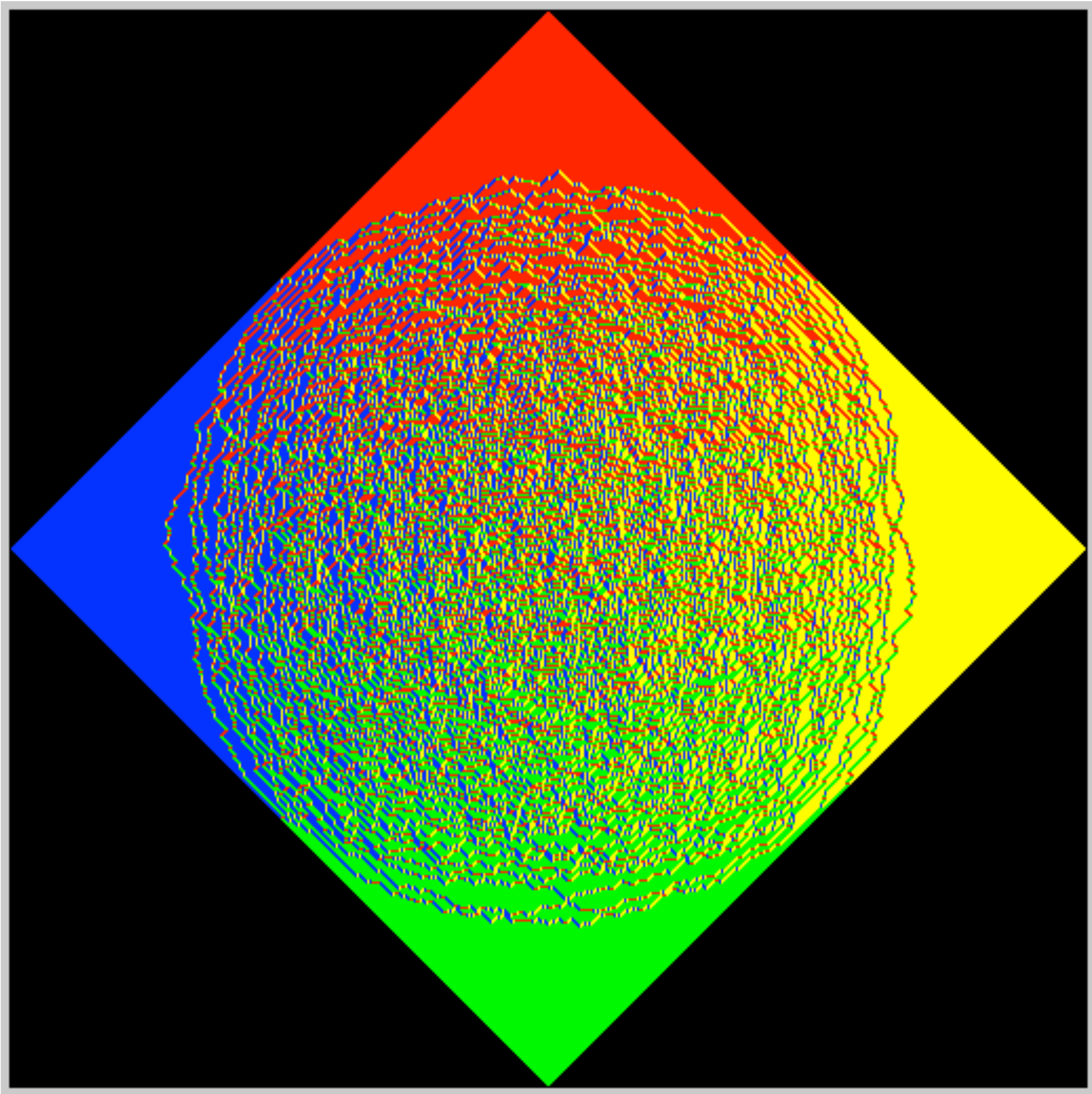
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} BA = q_{\circ\circ} AB + t_{\circ\circ} A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet} A_\bullet B_\bullet + 2 \, AB \\ B_\bullet A = q_{\bullet\circ} A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$

random
Aztec
tilings

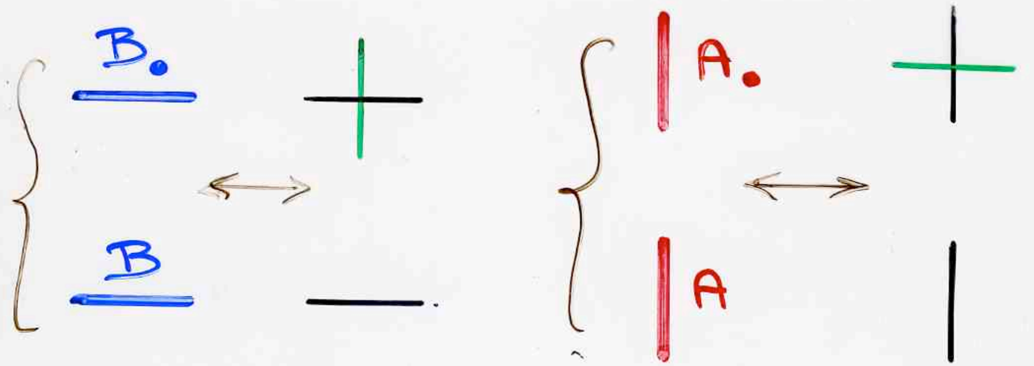


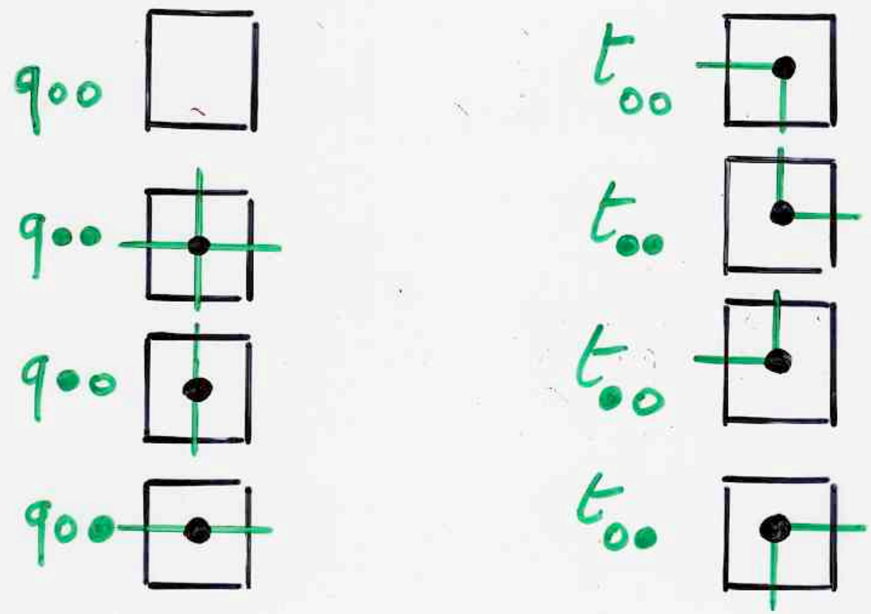
the «artic circle» theorem



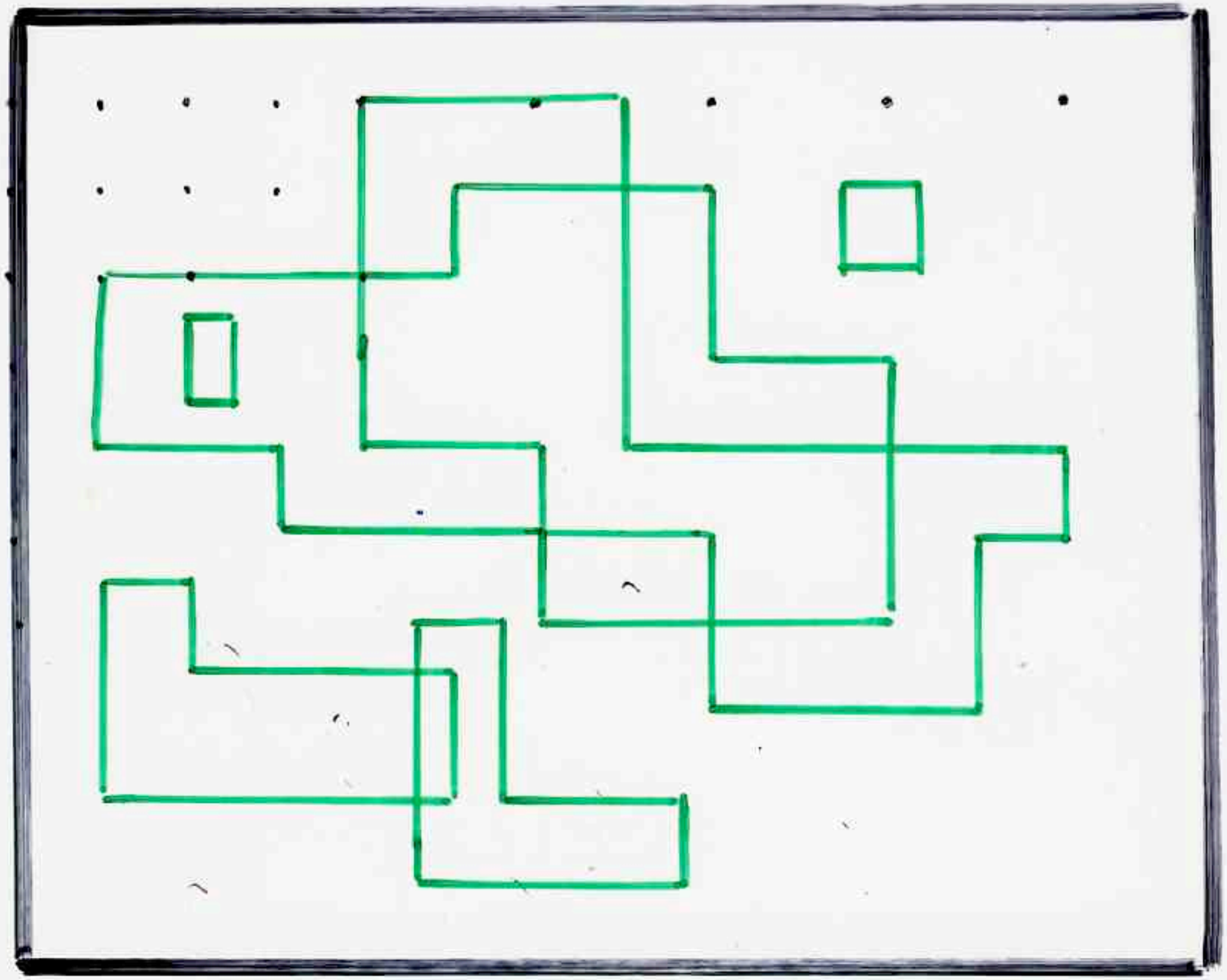
geometric interpretation of Z-tableaux

geometric interpretations of Z-tableaux





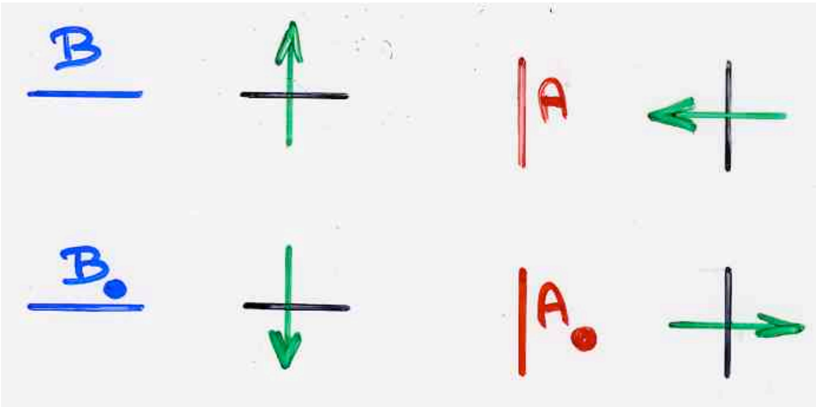
8 - vertex model



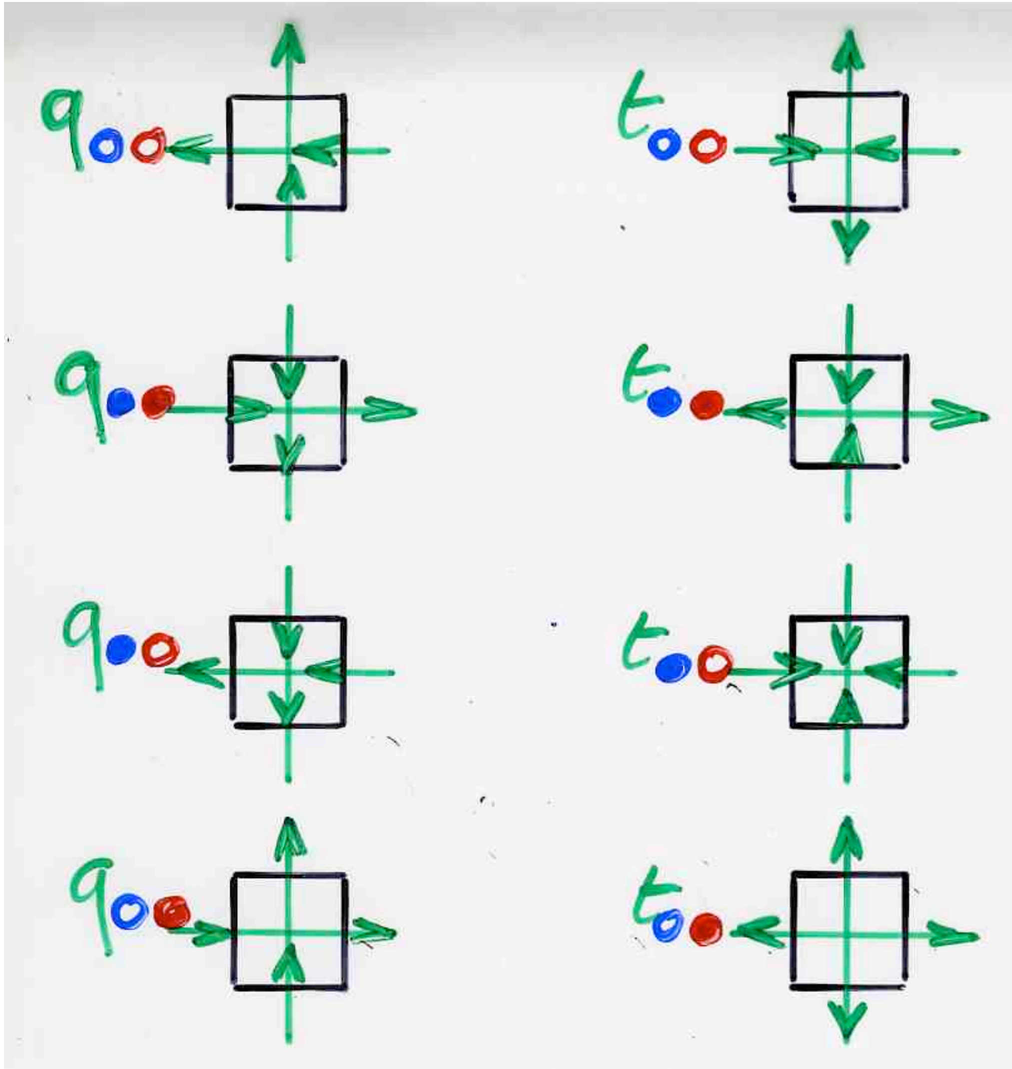
"closed" graph

Ising model

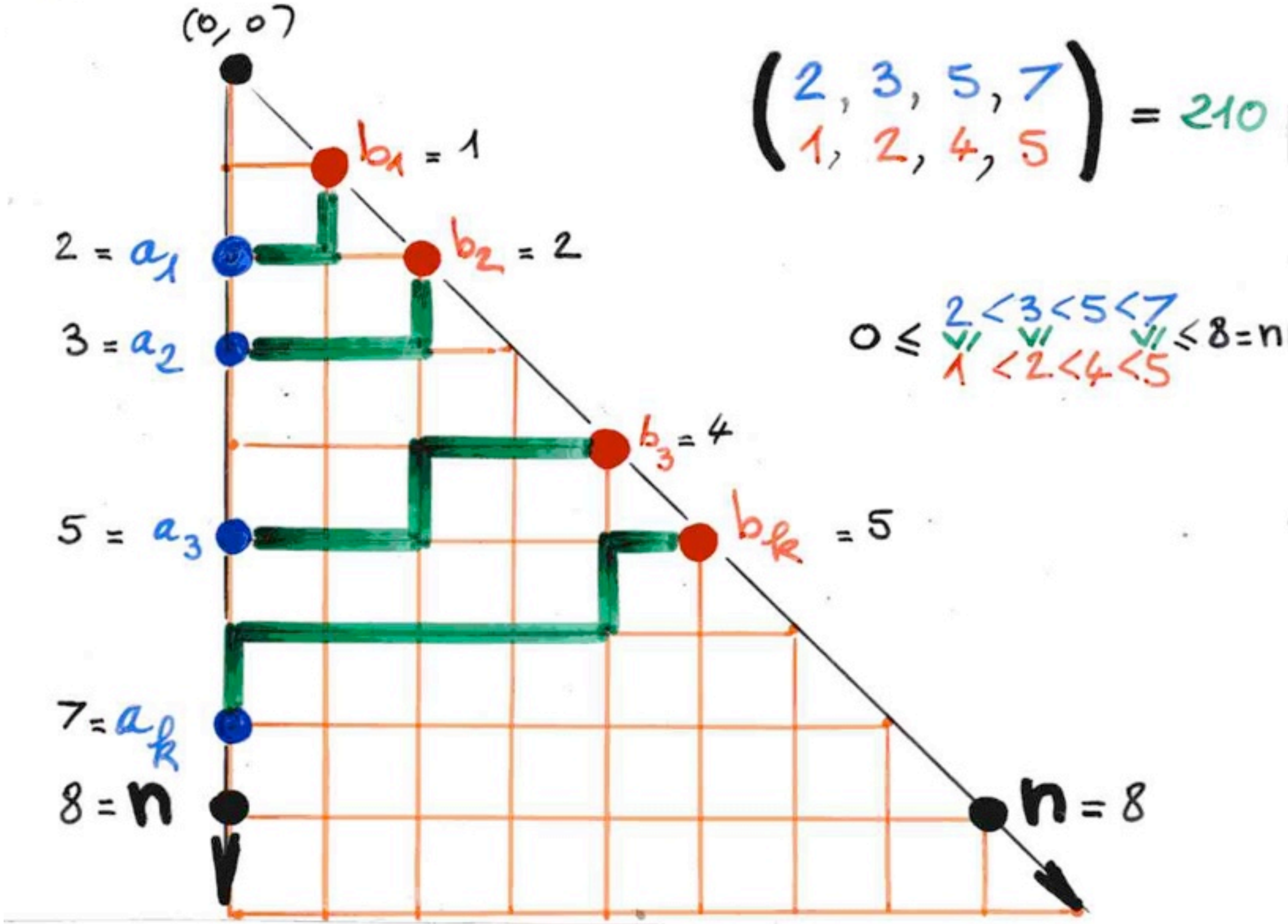
$$w = B^m A^n$$
$$uv = A^n B^m$$



8 - vertex model



non-intersecting paths



example: binomial determinant

non intersecting paths

$$\begin{cases} q_{\bullet\bullet} = 0 \\ t_{\bullet\circ} = t_{\circ\bullet} = 0 \end{cases}$$

(ASM)
(osc. paths)

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$
8 parameters $q_{\dots}$, $t_{\dots}$

$$\begin{cases} B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\ B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\ B_\bullet A = q_{\bullet\circ} \, A B_\bullet + \bigcirc \, A_\bullet B \\ B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet \end{cases}$$

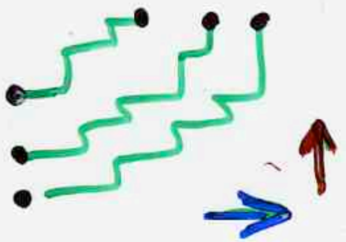
bijection
plane partition
non-intersecting paths

The quadratic algebra Z

4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\dots}$, $t_{\dots}$

$$
\begin{cases}
B A = q_{\circ\circ} \, A B + \bigcirc \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + \bigcirc \, A B \\
B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet
\end{cases}
$$



$$
\begin{cases}
t_{\circ\circ} = 0 \\
q_{\bullet\bullet} = t_{\bullet\bullet} = 0
\end{cases}
$$

$A \leftrightarrow A_\bullet$

exchanging

$$
\begin{cases}
t_{\circ\bullet} = 0 \\
q_{\bullet\circ} = t_{\bullet\circ} = 0
\end{cases}
$$

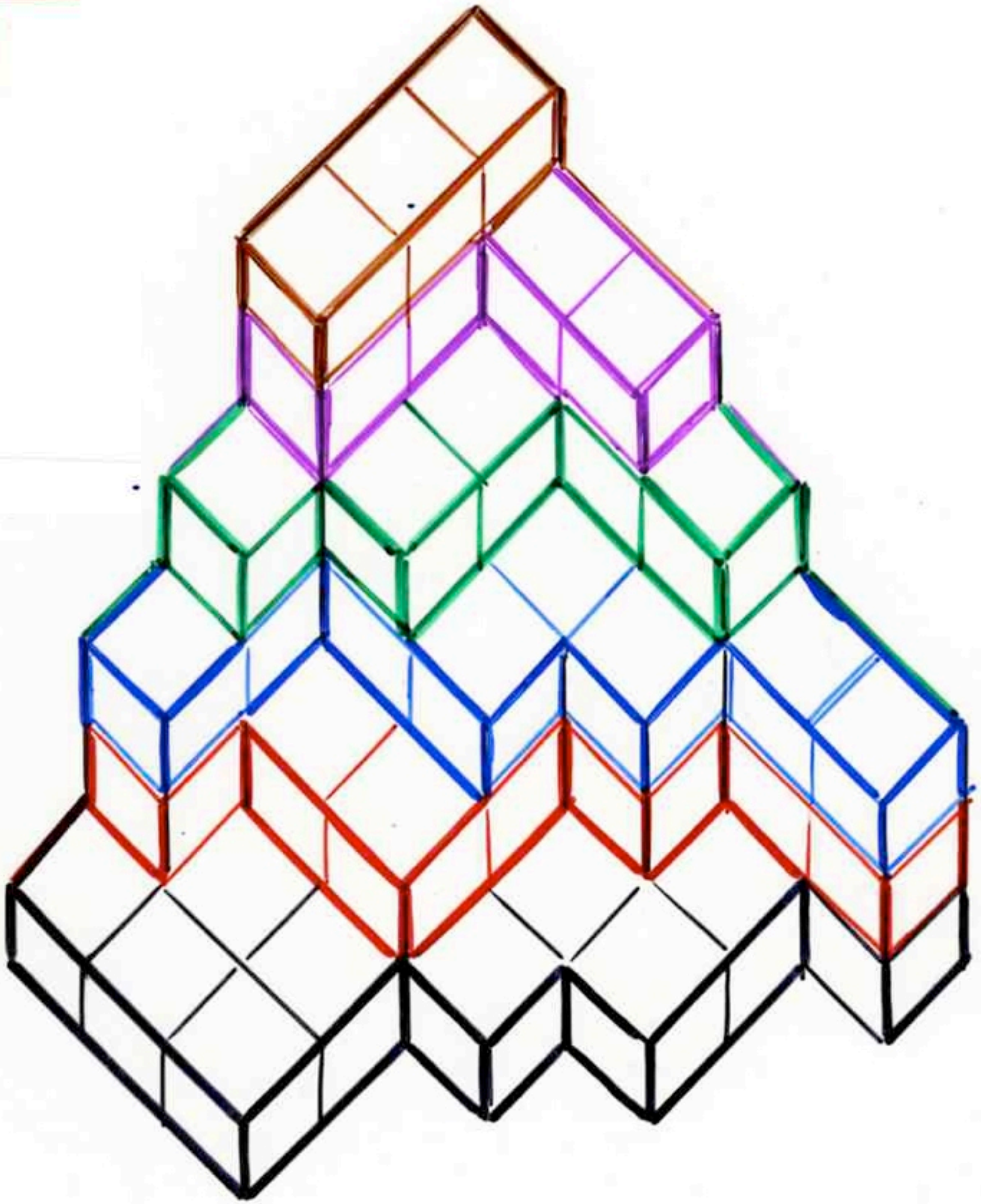
The quadratic algebra Z

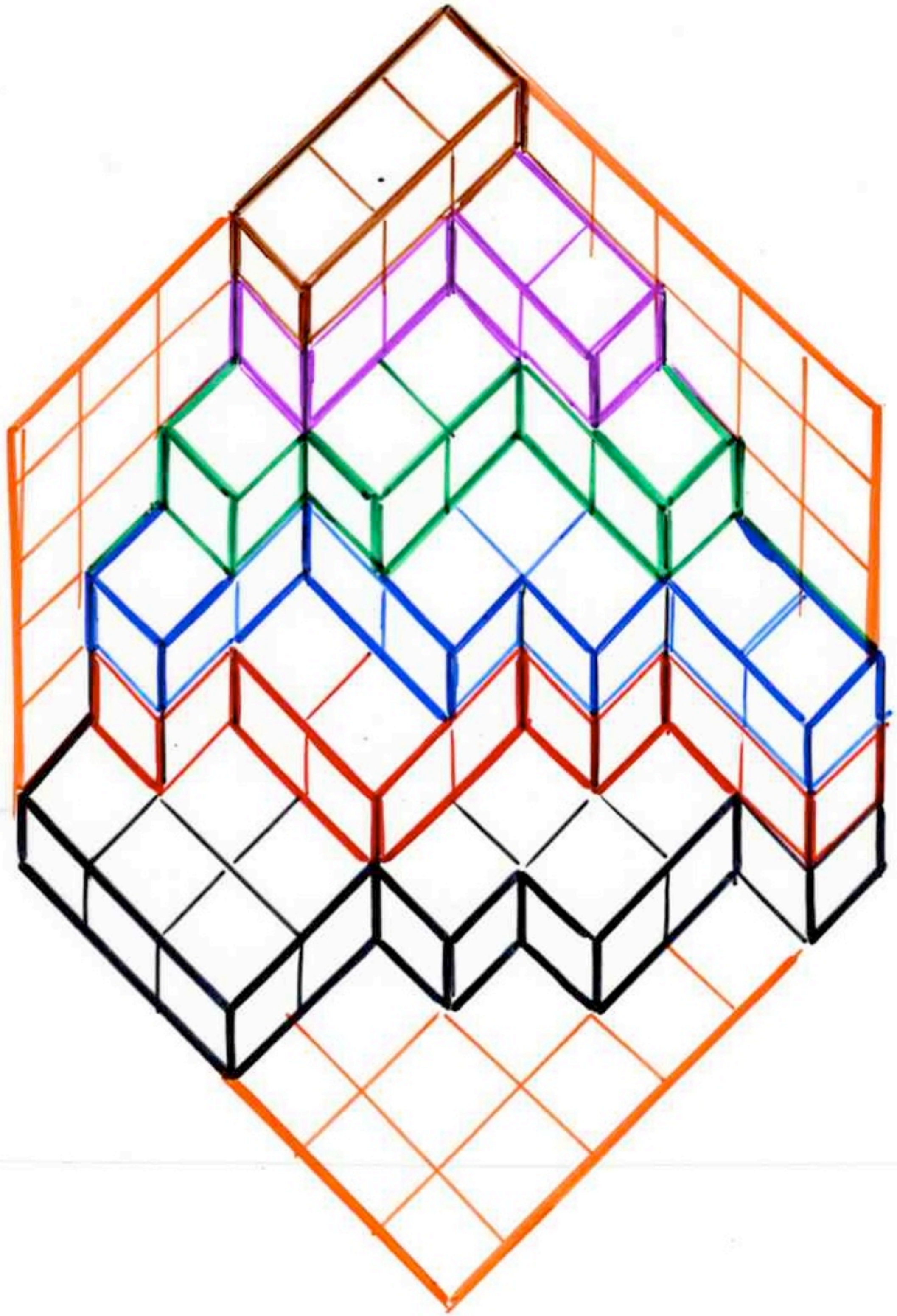
4 generators $B_\bullet \; A_\bullet \; B \; A$

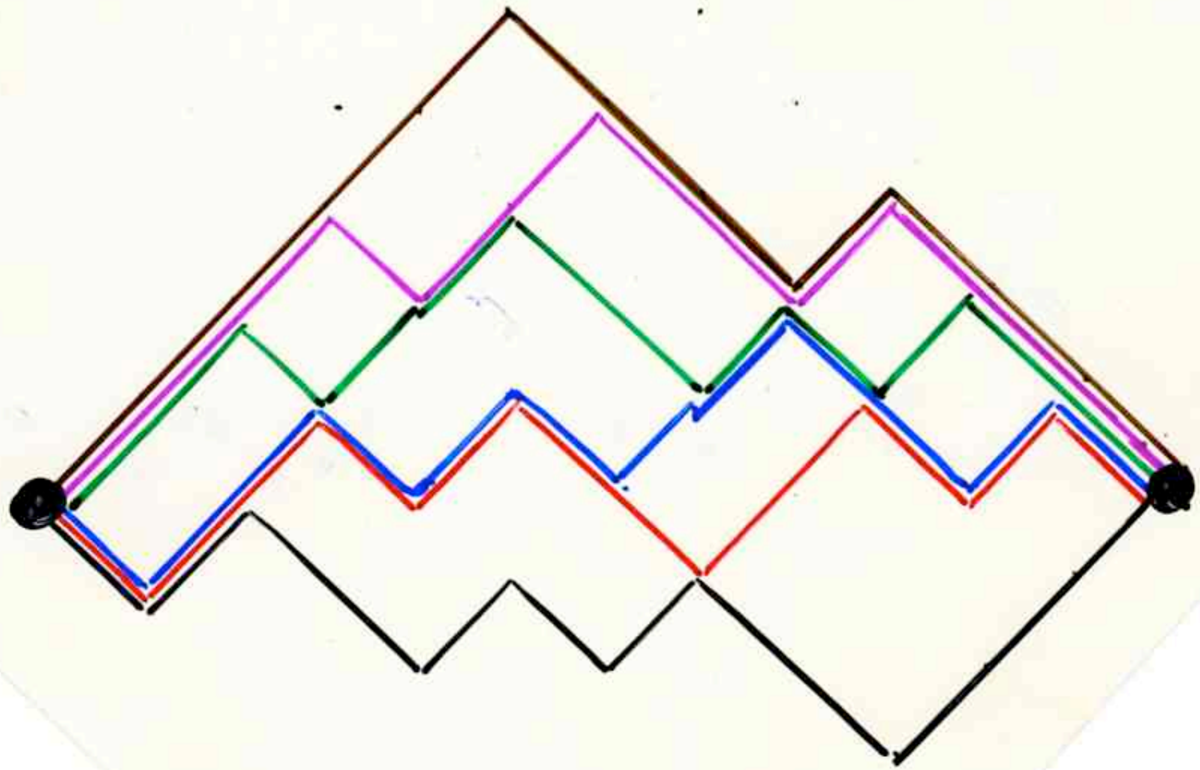
8 parameters $q_{\dots}$, $t_{\dots}$

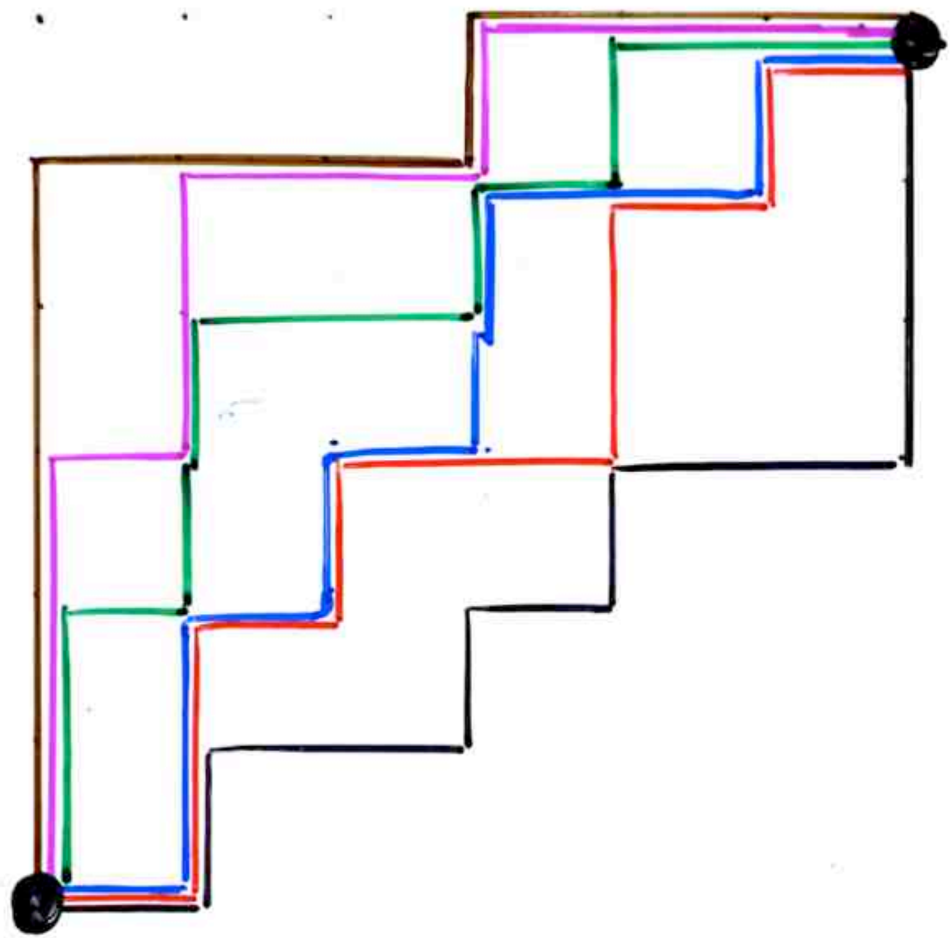
$$
\begin{cases}
B A = q_{\circ\circ} \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet} \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\
B_\bullet A = \bigcirc \, A B_\bullet + \bigcirc \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + \bigcirc \, A B_\bullet
\end{cases}
$$

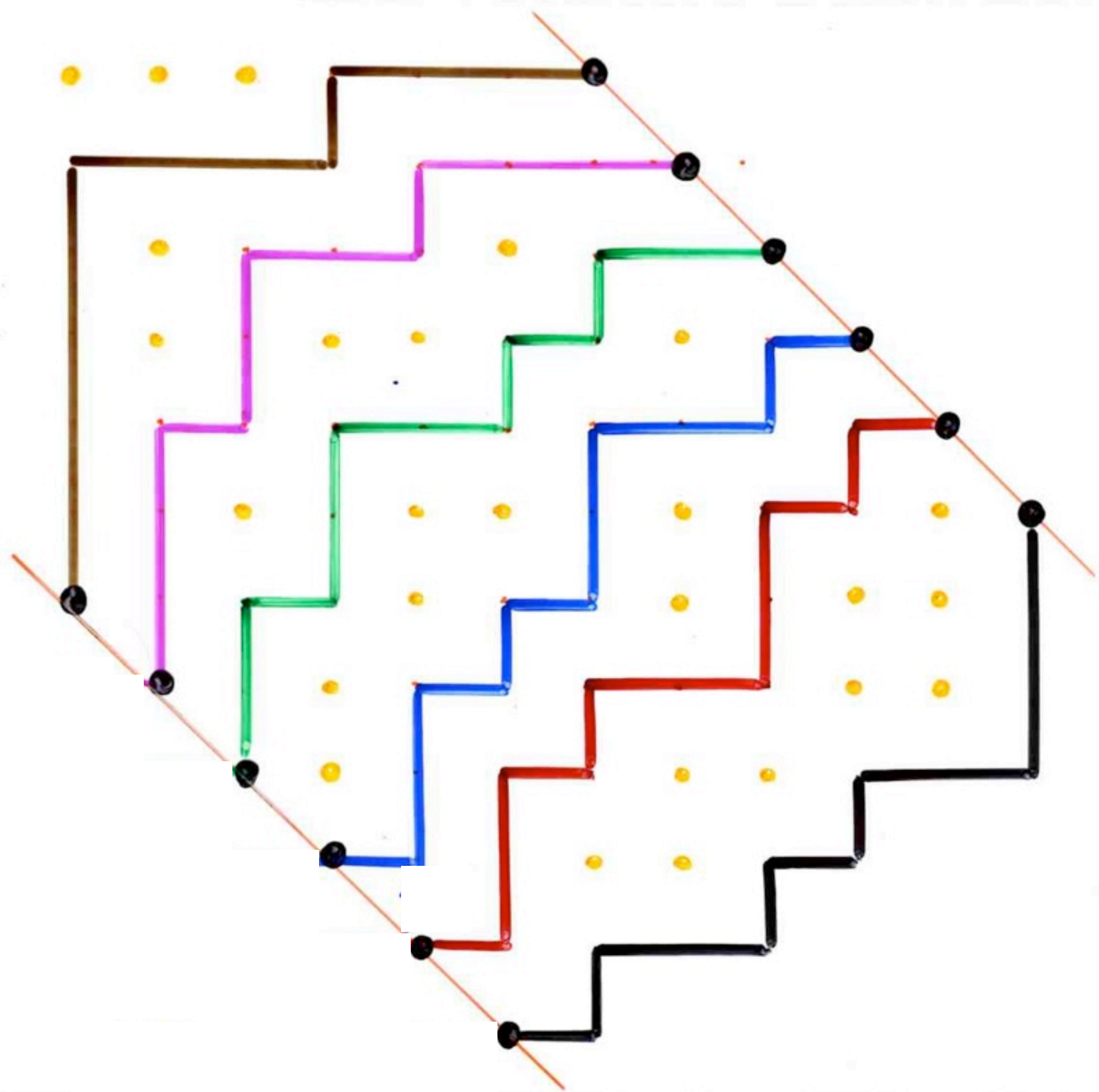
6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

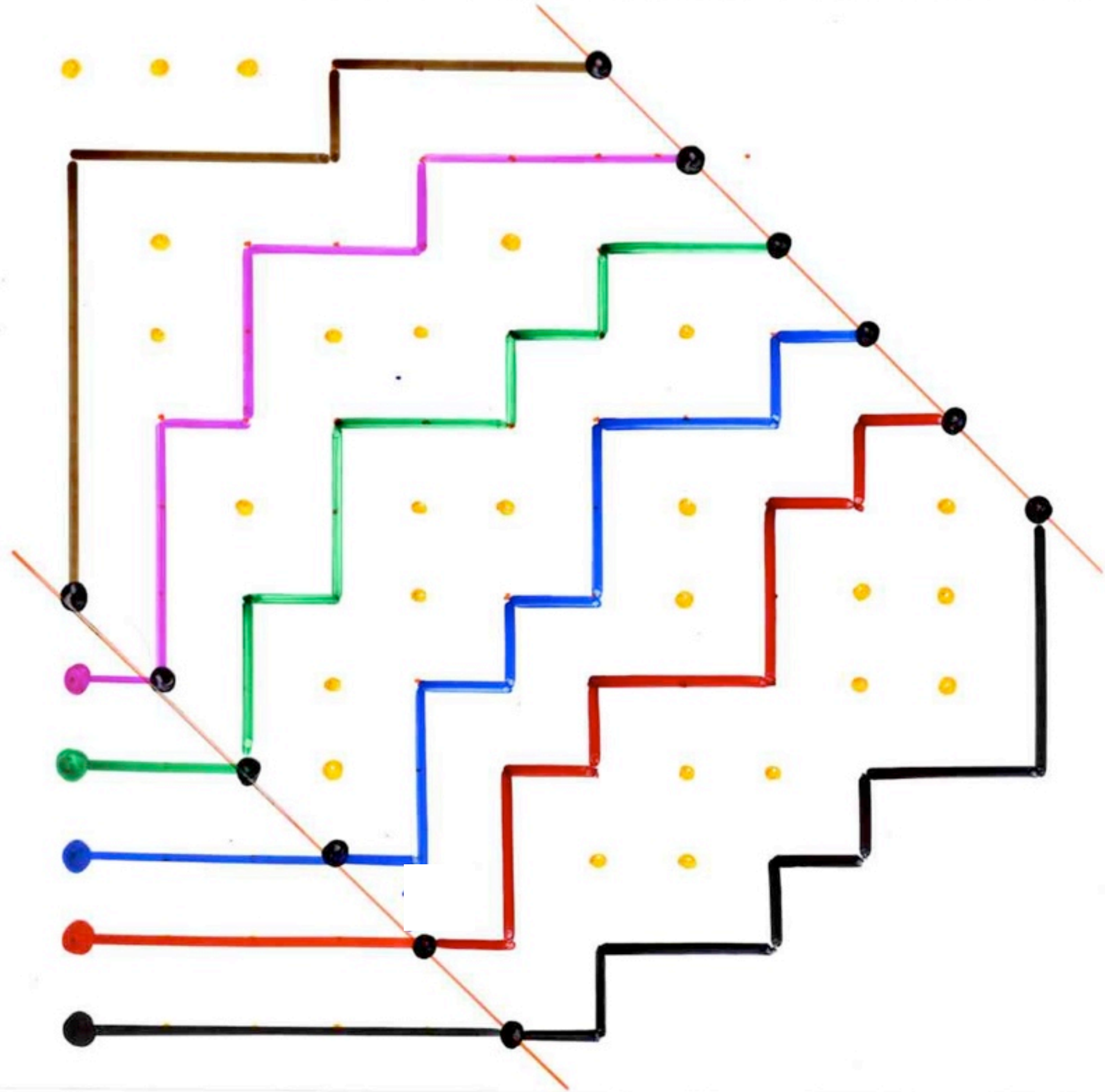




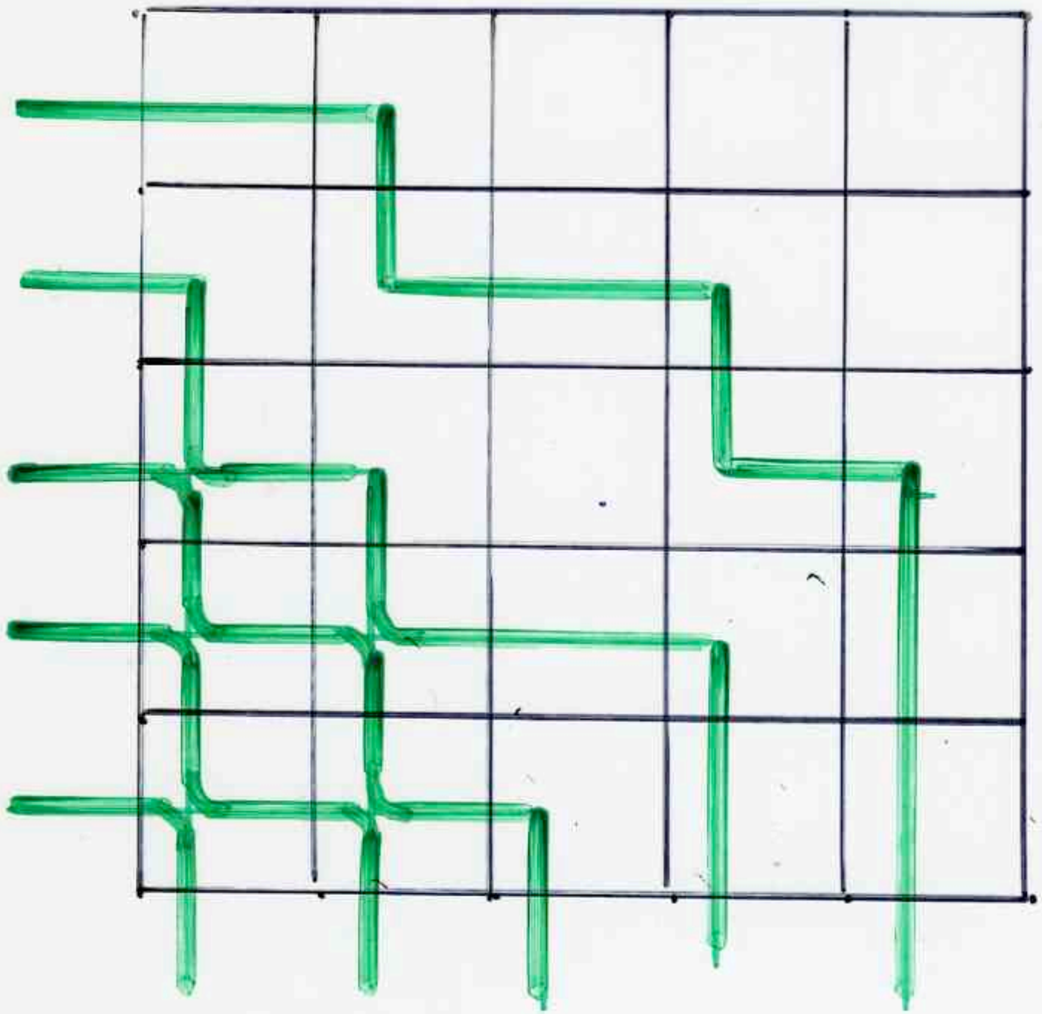








osculating paths



osculating paths

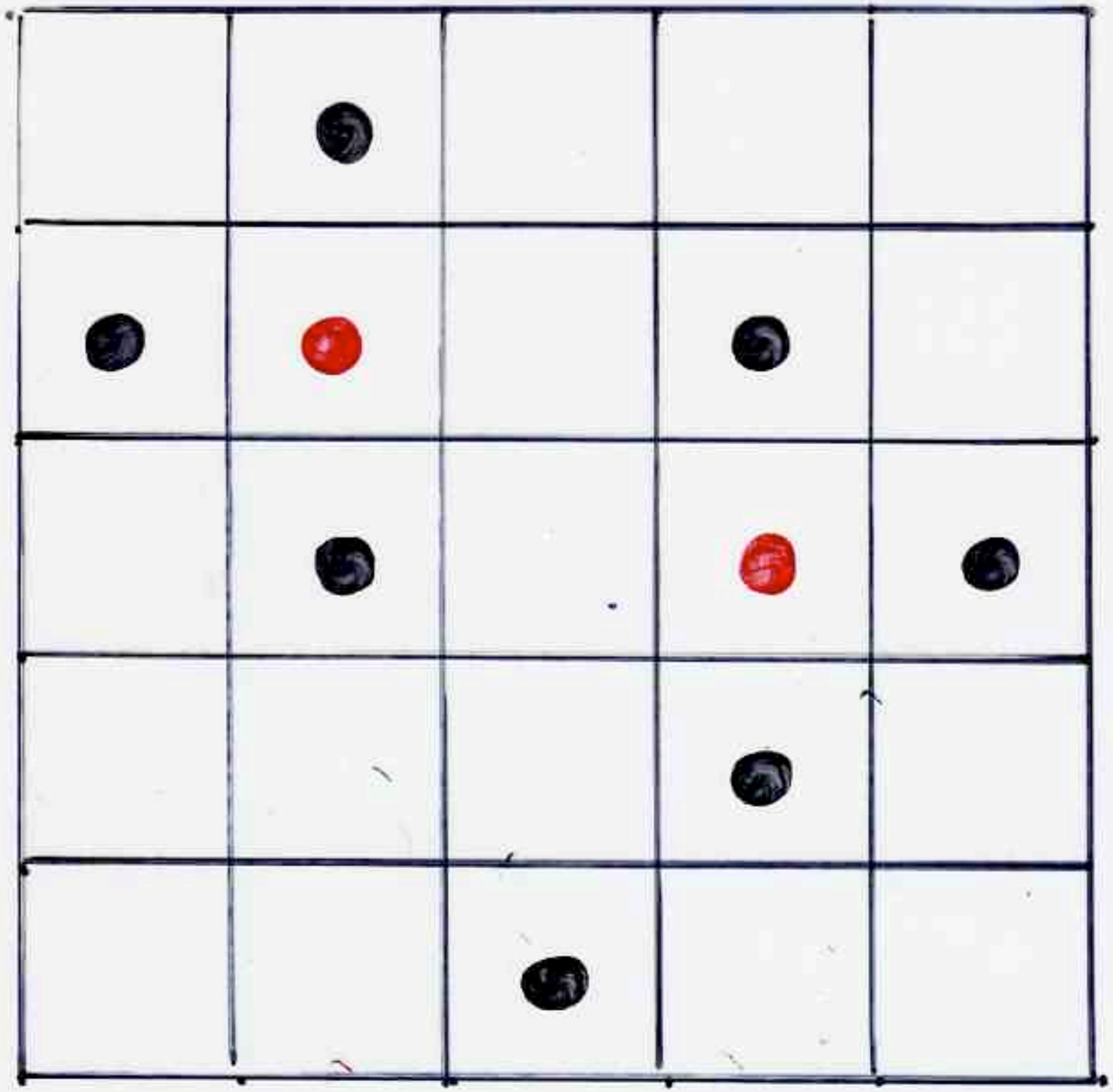
$$t_{\bullet\circ} = t_{\circ\bullet} = 0$$

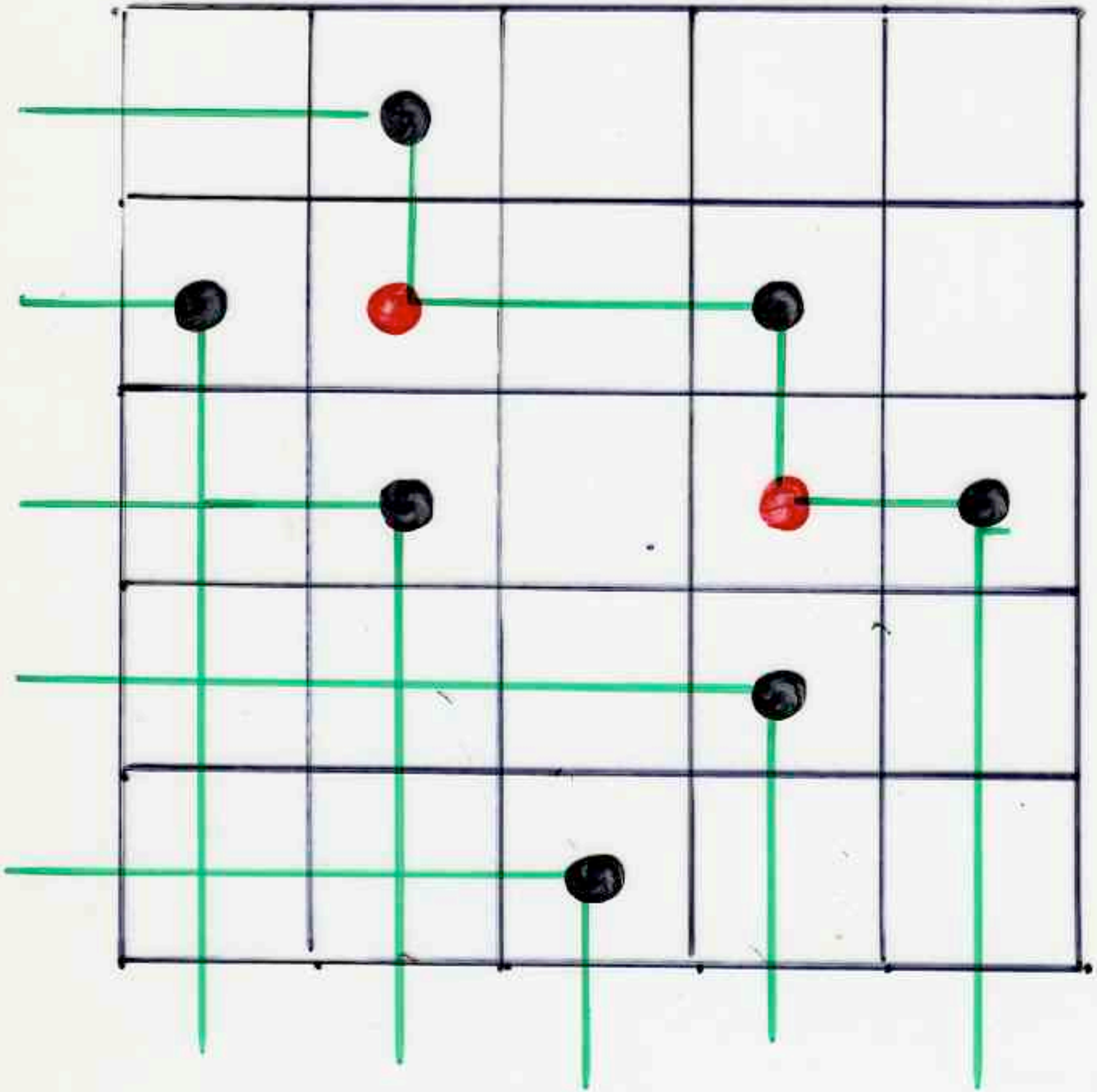
The quadratic algebra Z

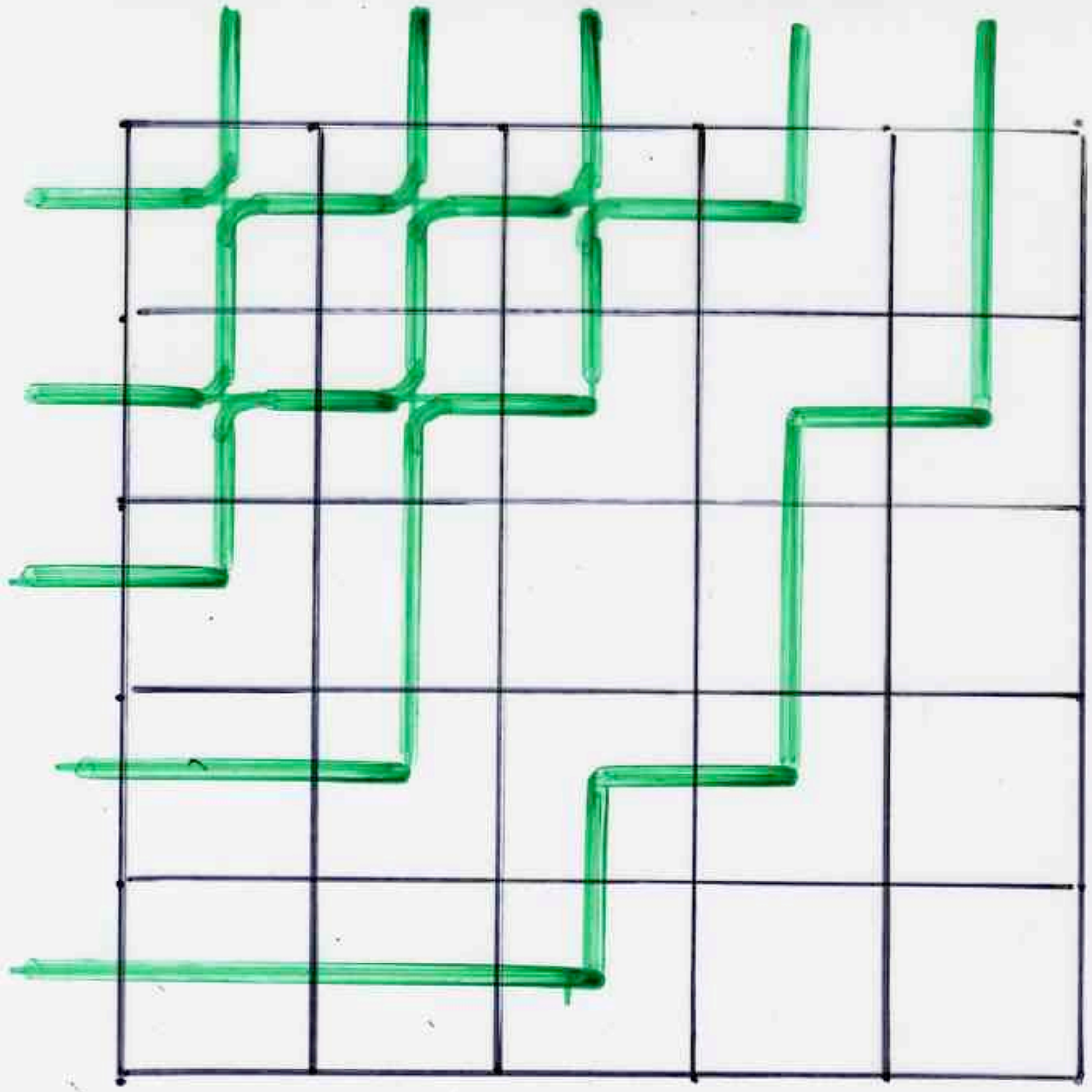
4 generators $B_\bullet \; A_\bullet \; B \; A$

8 parameters $q_{\cdots}$, $t_{\cdots}$

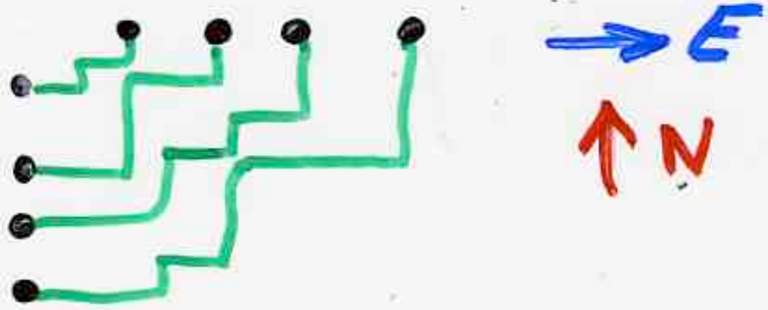
$$\begin{cases} BA = q_{\circ\circ}\, AB + t_{\circ\circ}\, A_\bullet B_\bullet \\ B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + t_{\bullet\bullet}\, AB \\ B_\bullet A = q_{\bullet\circ}\, A B_\bullet + \bigcirc\, A_\bullet B \\ B A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\, A B_\bullet \end{cases}$$







osculating paths



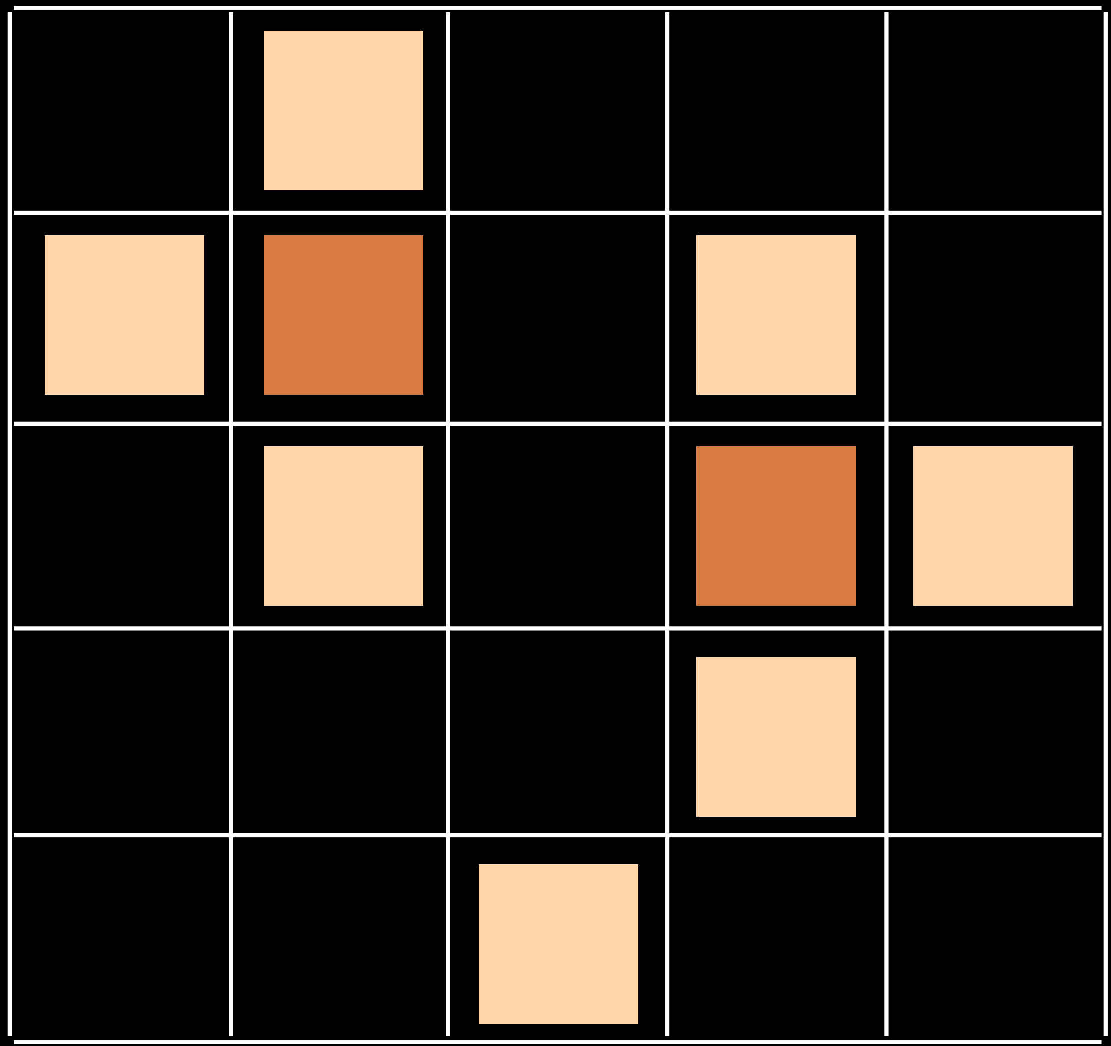
The quadratic algebra Z

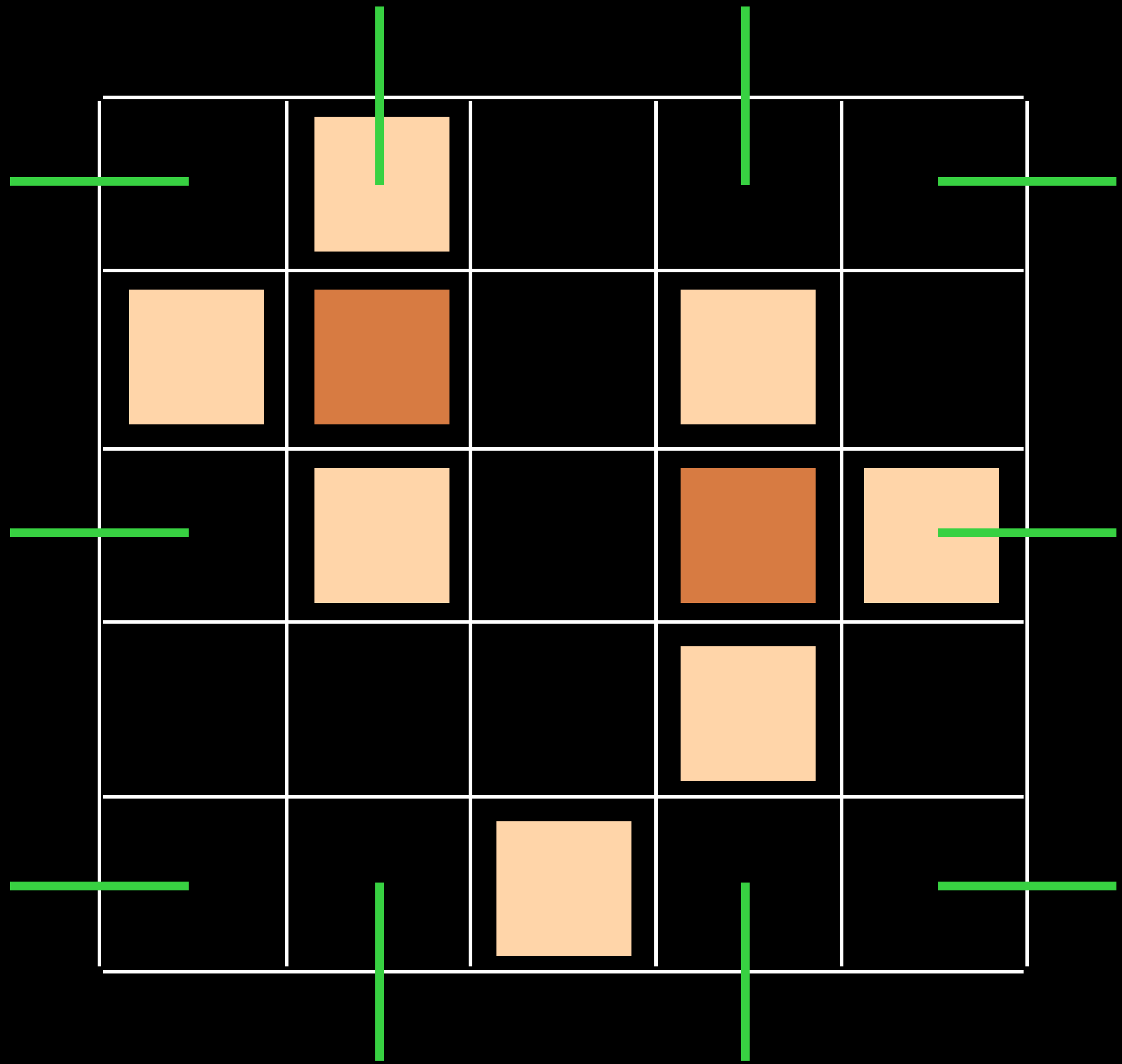
4 generators $B_\bullet \; A_\bullet \; B \; A$

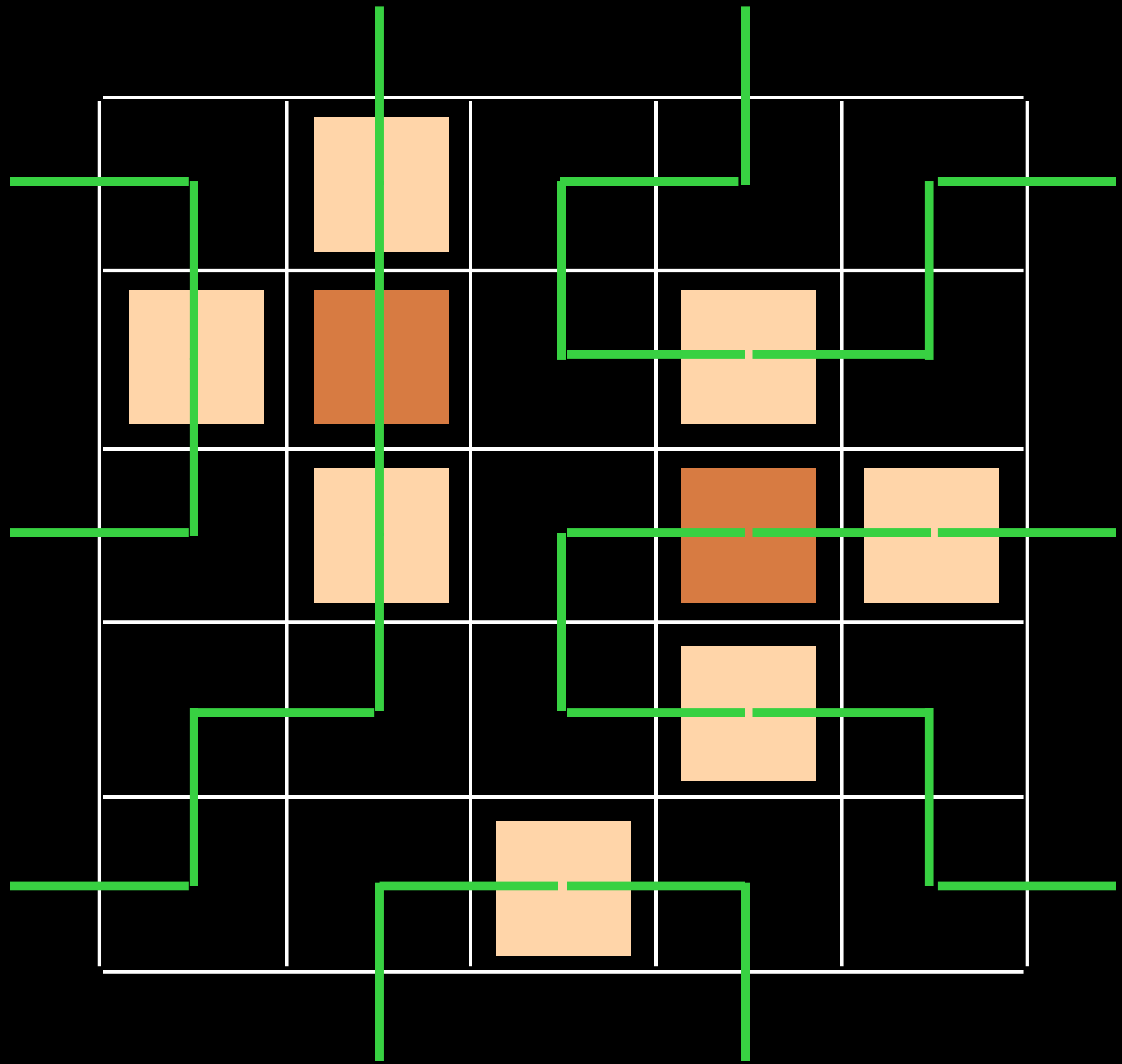
8 parameters $q_{\dots}$, $t_{\dots}$

$$
\begin{cases}
B\,A = q_{\circ\circ}\, A\,B + t_{\circ\circ}\, A_\bullet B_\bullet \\
B_\bullet A_\bullet = q_{\bullet\bullet}\, A_\bullet B_\bullet + \bigcirc\; A\,B \\
B_\bullet A = q_{\bullet\circ}\, A\,B_\bullet + t_{\bullet\circ}\, A_\bullet B \\
B\,A_\bullet = q_{\circ\bullet}\, A_\bullet B + \bigcirc\; A\,B_\bullet
\end{cases}
$$

FPL
fully packed loops



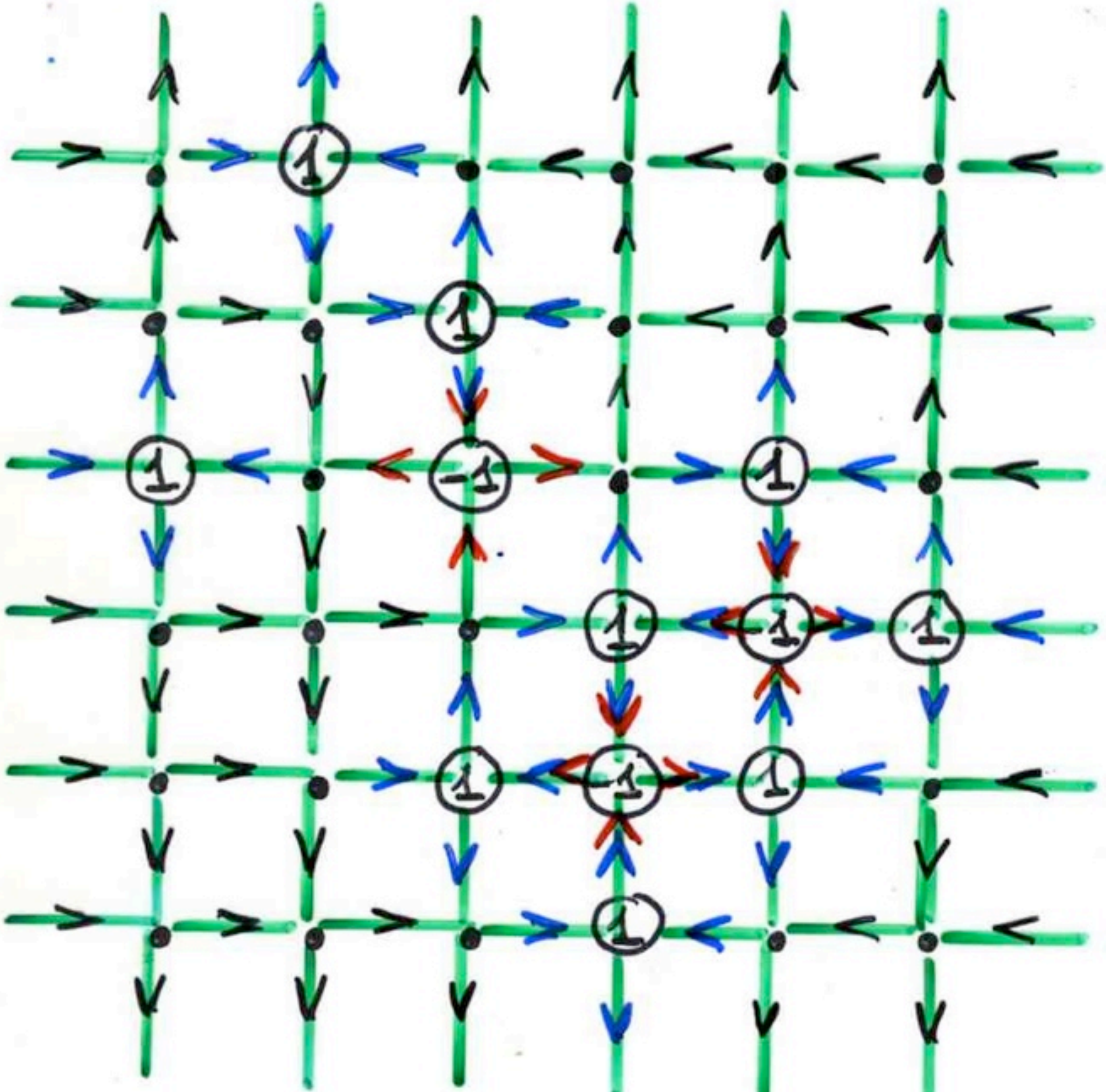


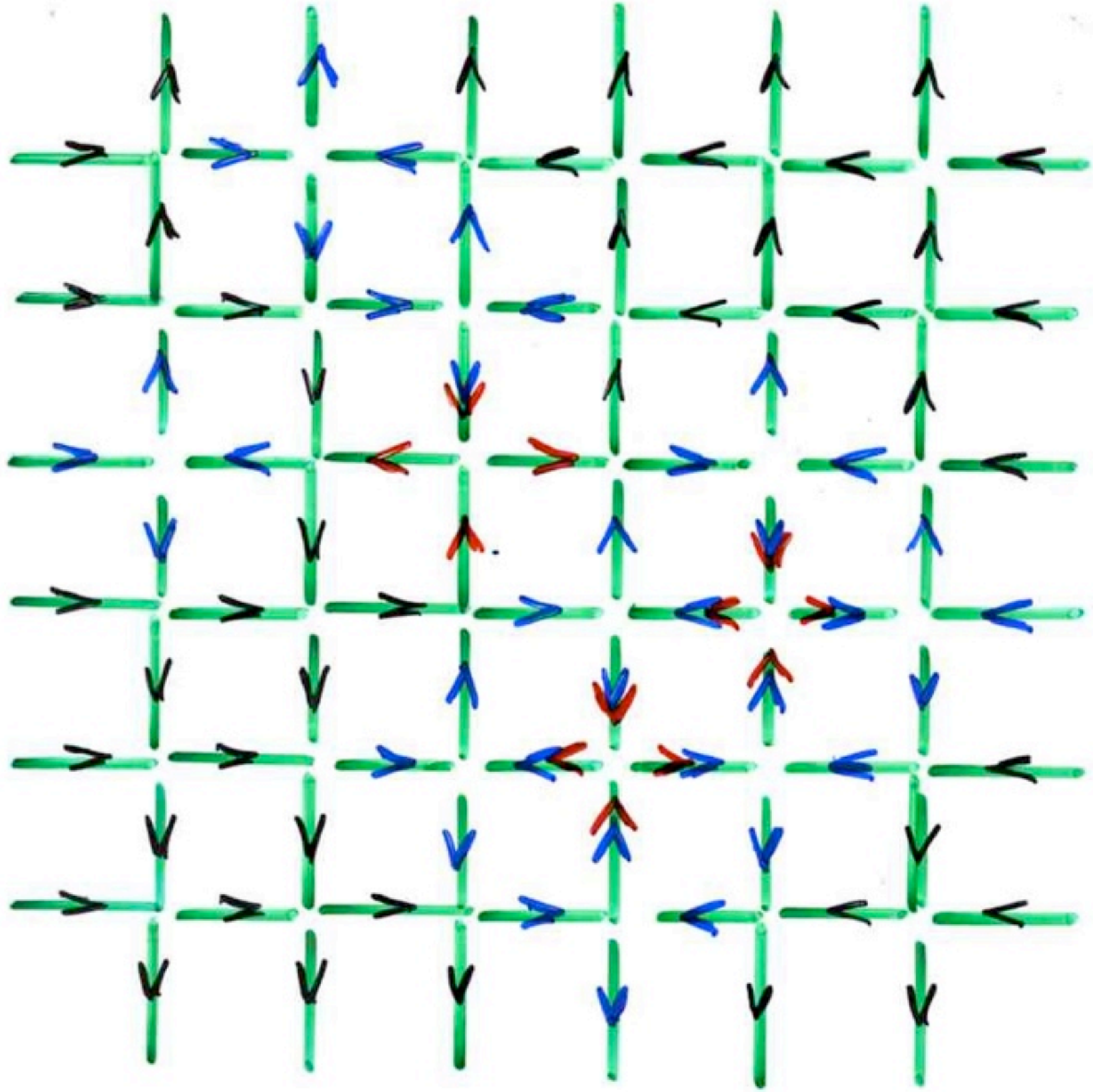


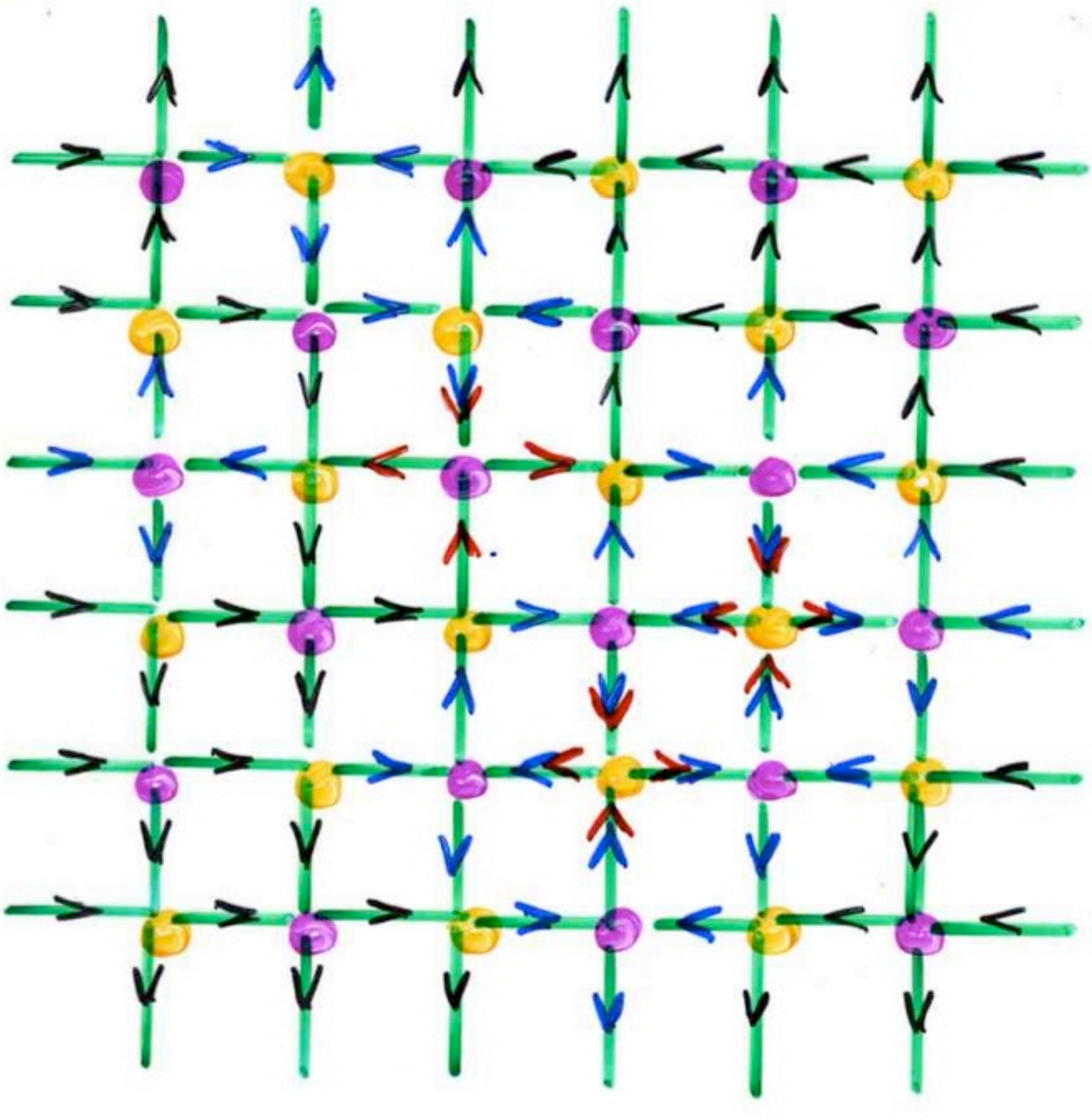
The
bijection
AMS
FPL

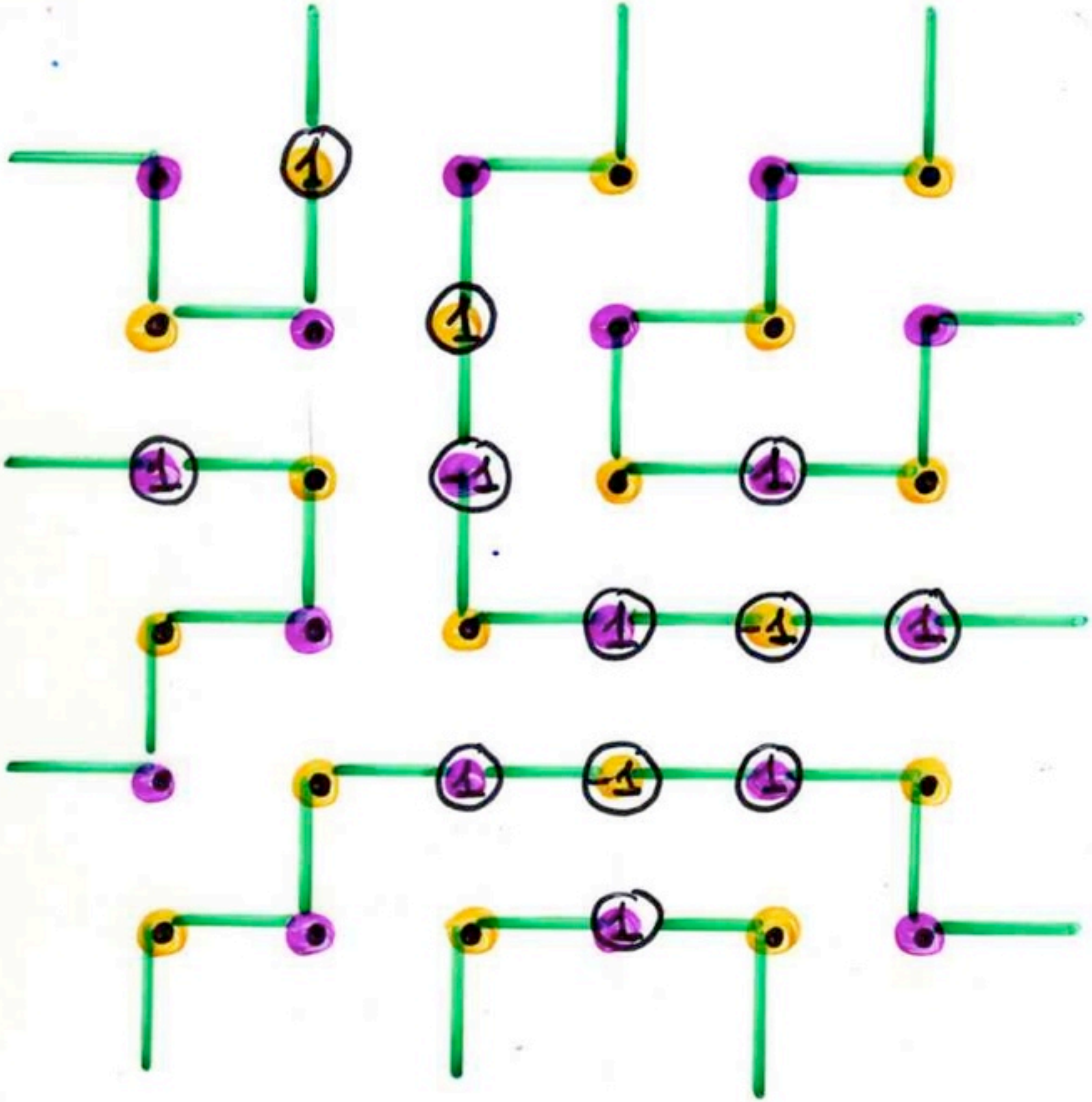
•	1	•	•	•	•
•	•	1	•	•	•
1	•	-1	•	1	•
•	•	•	1	-1	1
•	•	1	-1	1	•
•	•	•	1	•	•

The 6-vertex model

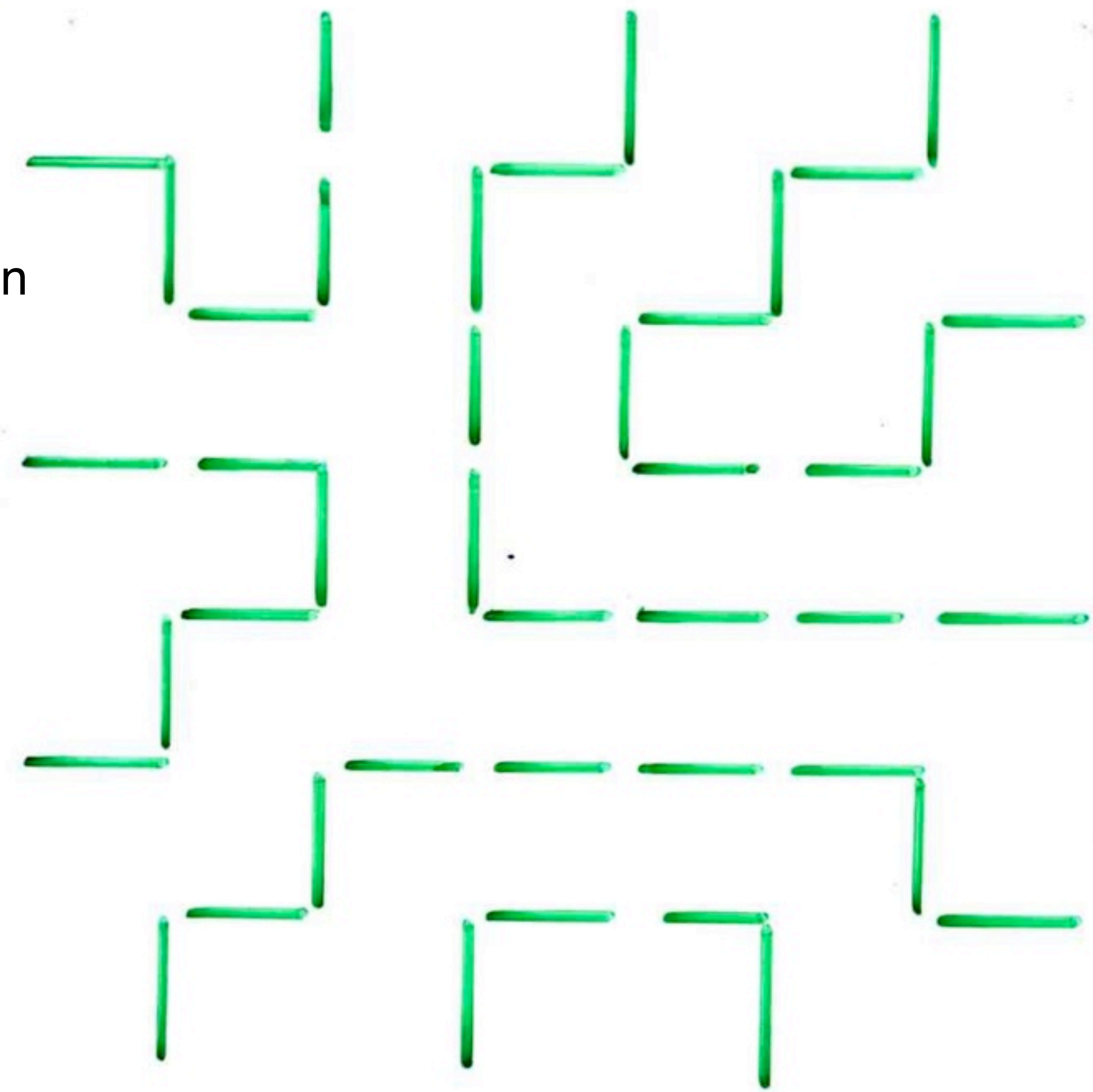




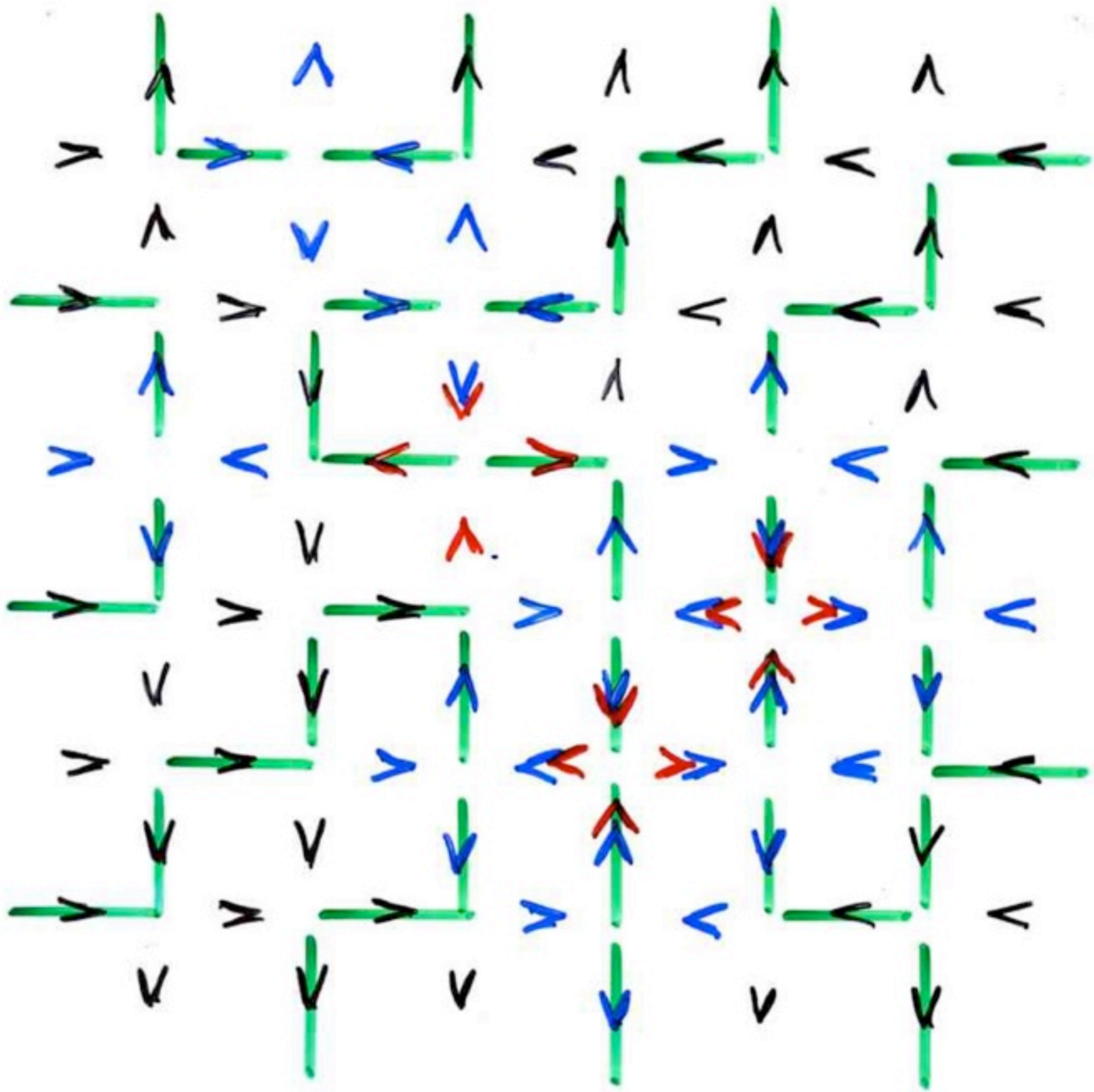




FPL
“Fully
Packed
Loop”
configuration



dual
FPL



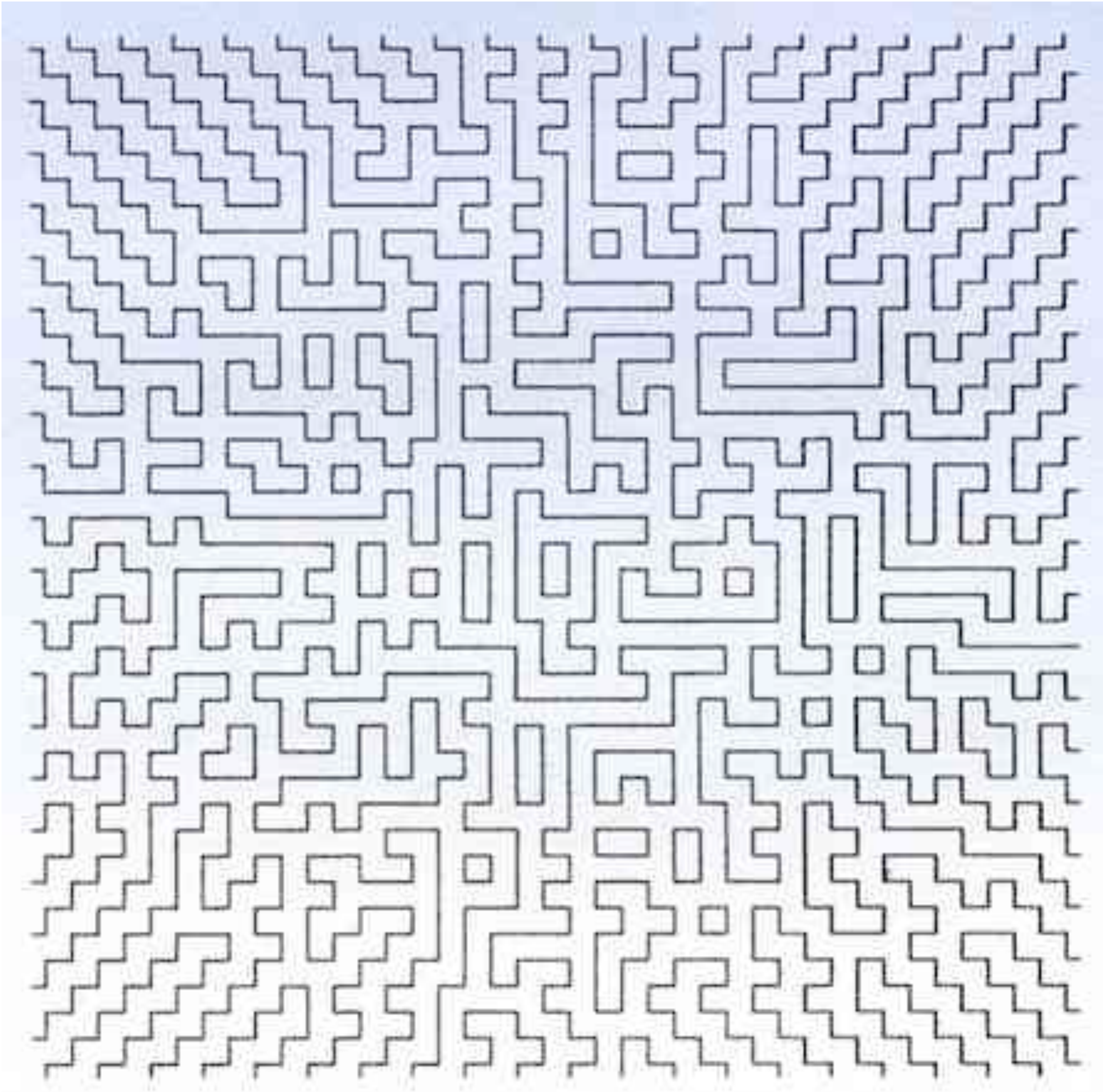
The quadratic algebra $\mathcal{Z}$

4 generators $B_\bullet \; A_\bullet \; B \; A$

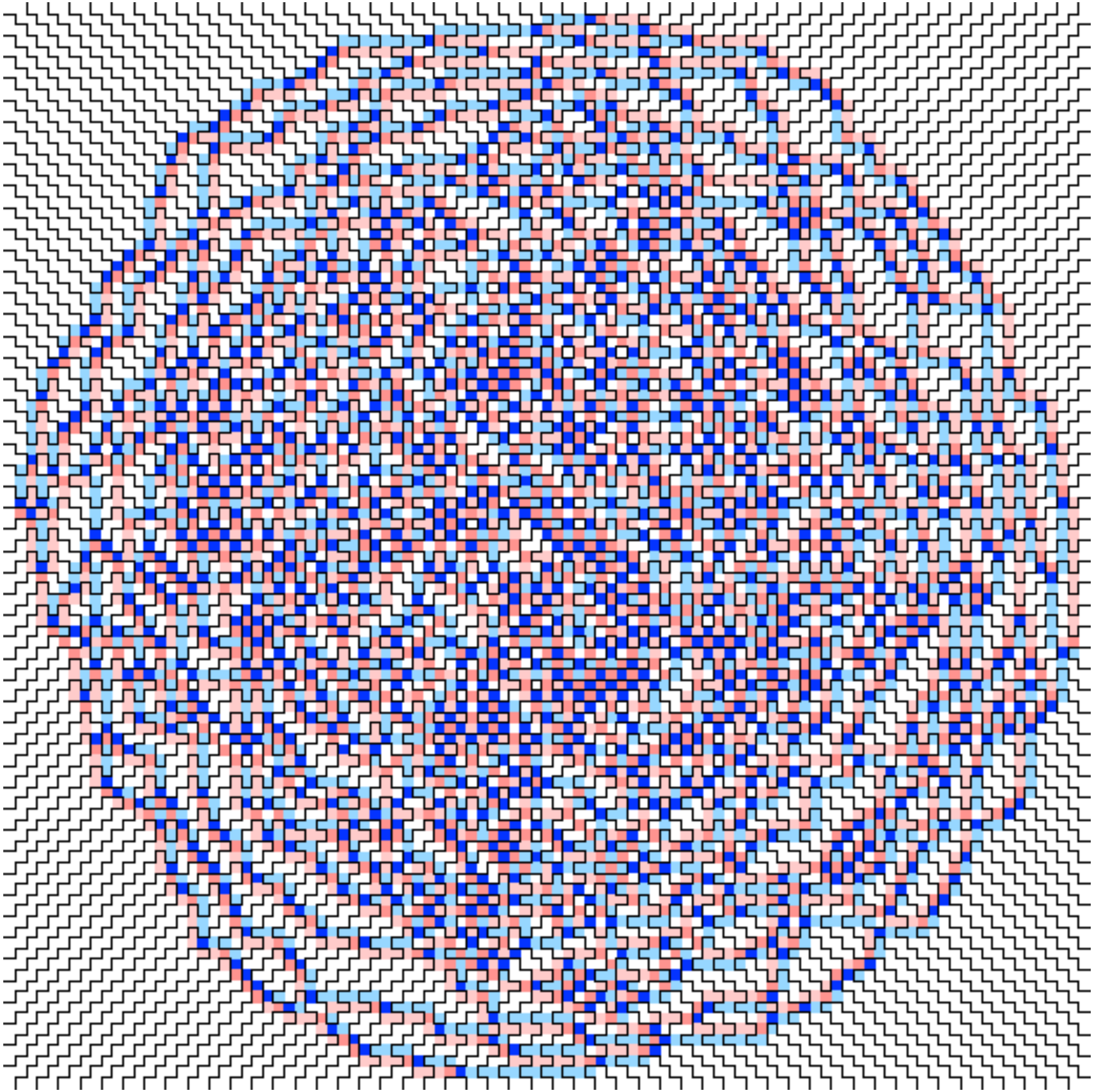
8 parameters $q_{\dots}, \; t_{\dots}$

$$
\begin{cases}
B A = \bigcirc \, A B + t_{\circ\circ} \, A_\bullet B_\bullet \\
B_\bullet A_\bullet = \bigcirc \, A_\bullet B_\bullet + t_{\bullet\bullet} \, A B \\
B_\bullet A = q_{\bullet\circ} \, A B_\bullet + t_{\bullet\circ} \, A_\bullet B \\
B A_\bullet = q_{\circ\bullet} \, A_\bullet B + t_{\circ\bullet} \, A B_\bullet
\end{cases}
$$

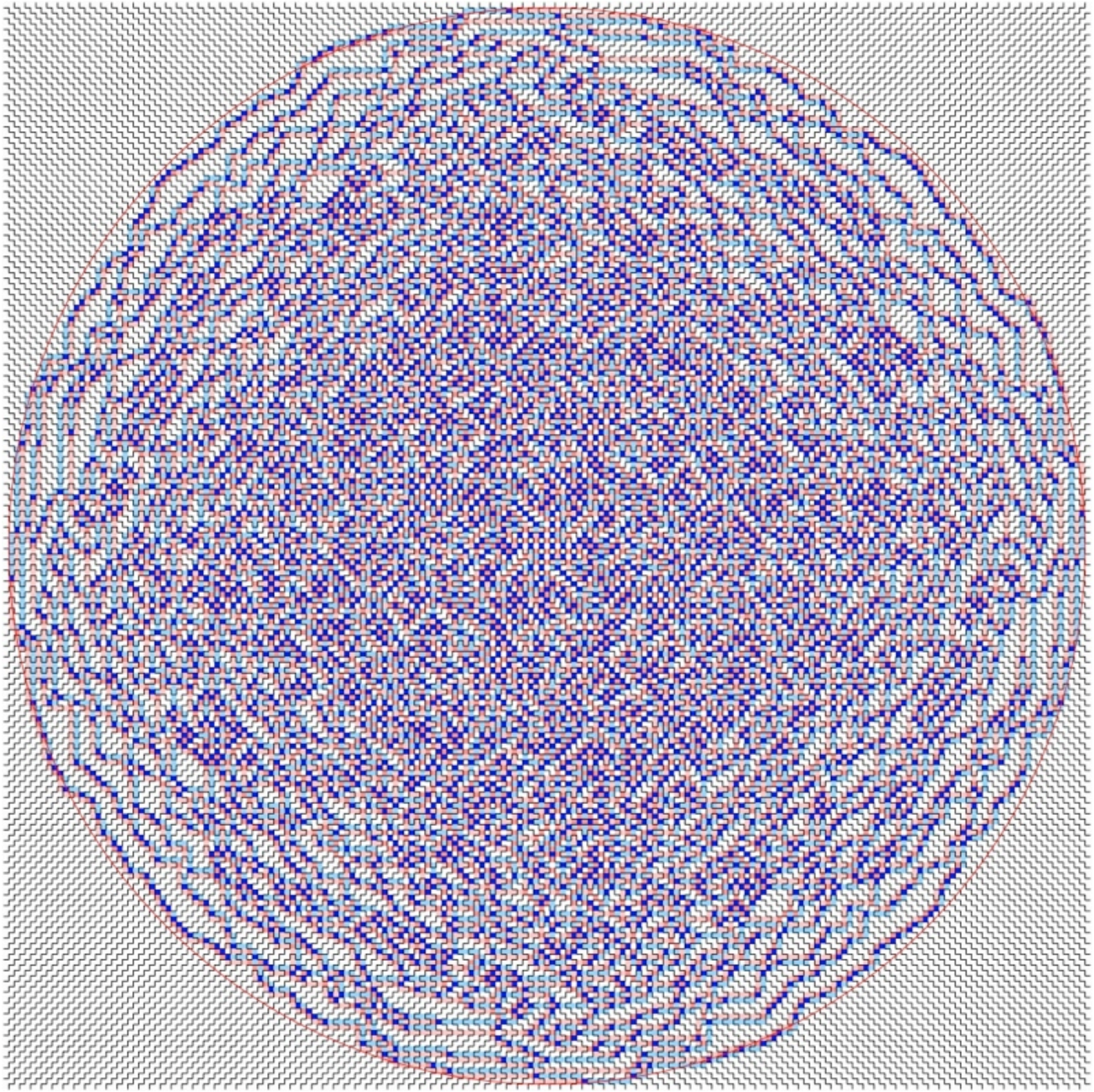
random
FPL



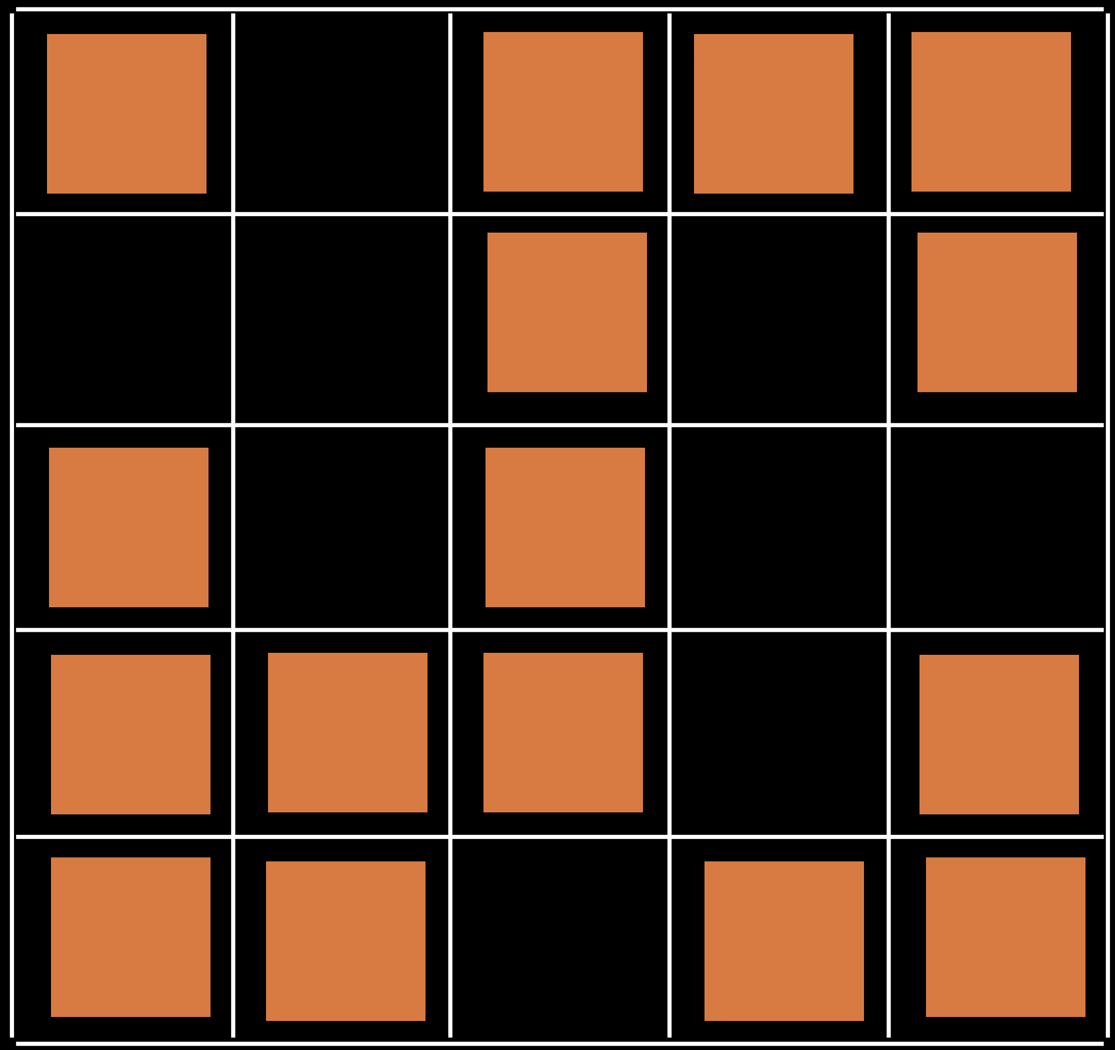
random
FPL



random
FPL

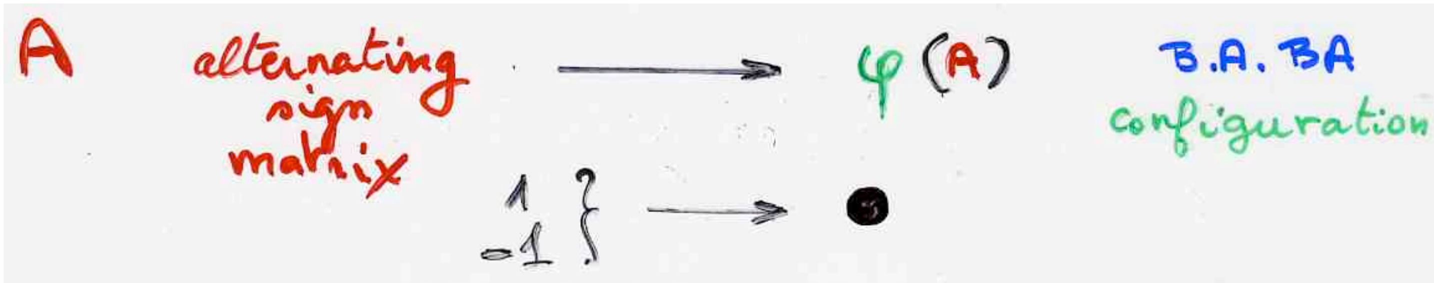


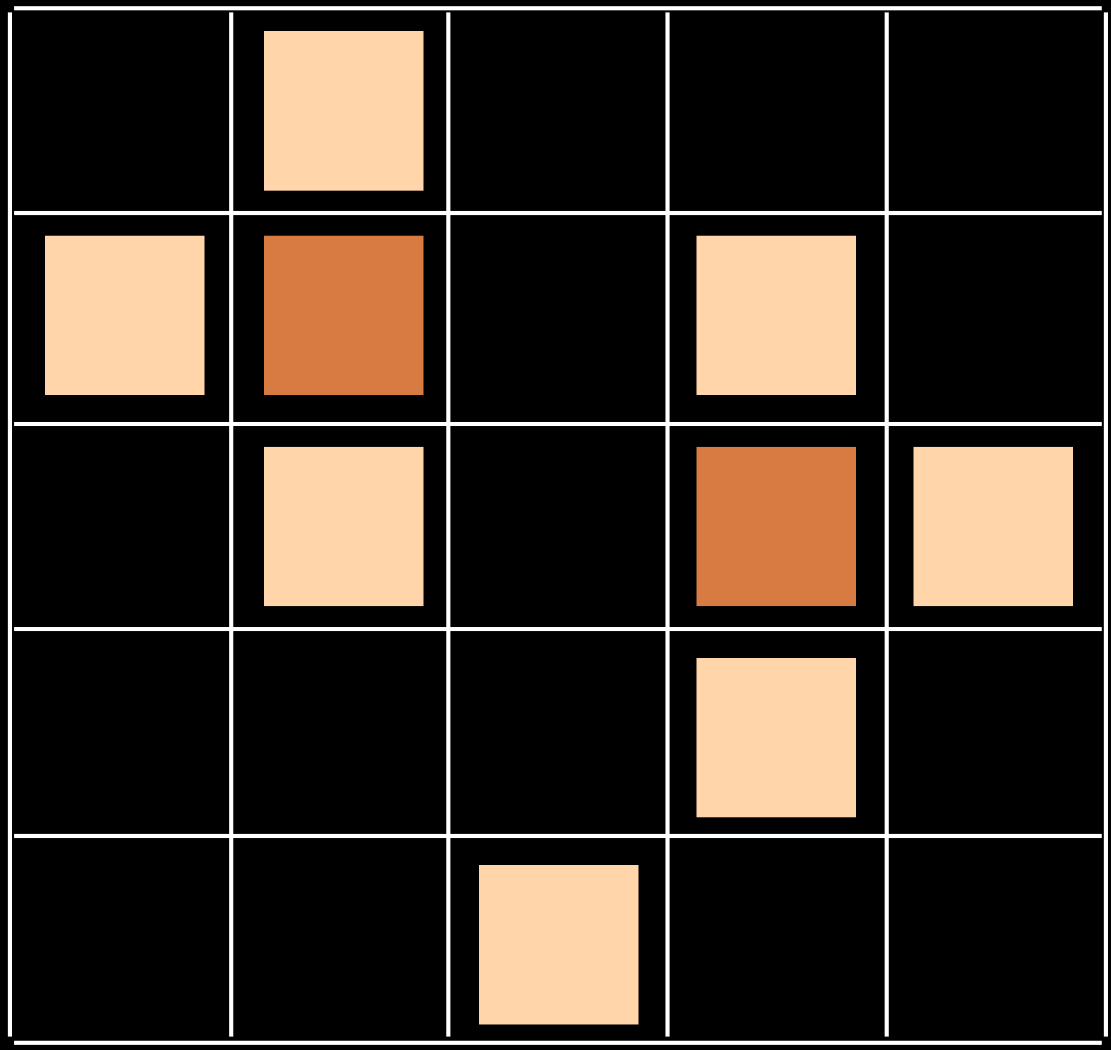
configurations B.A.BA

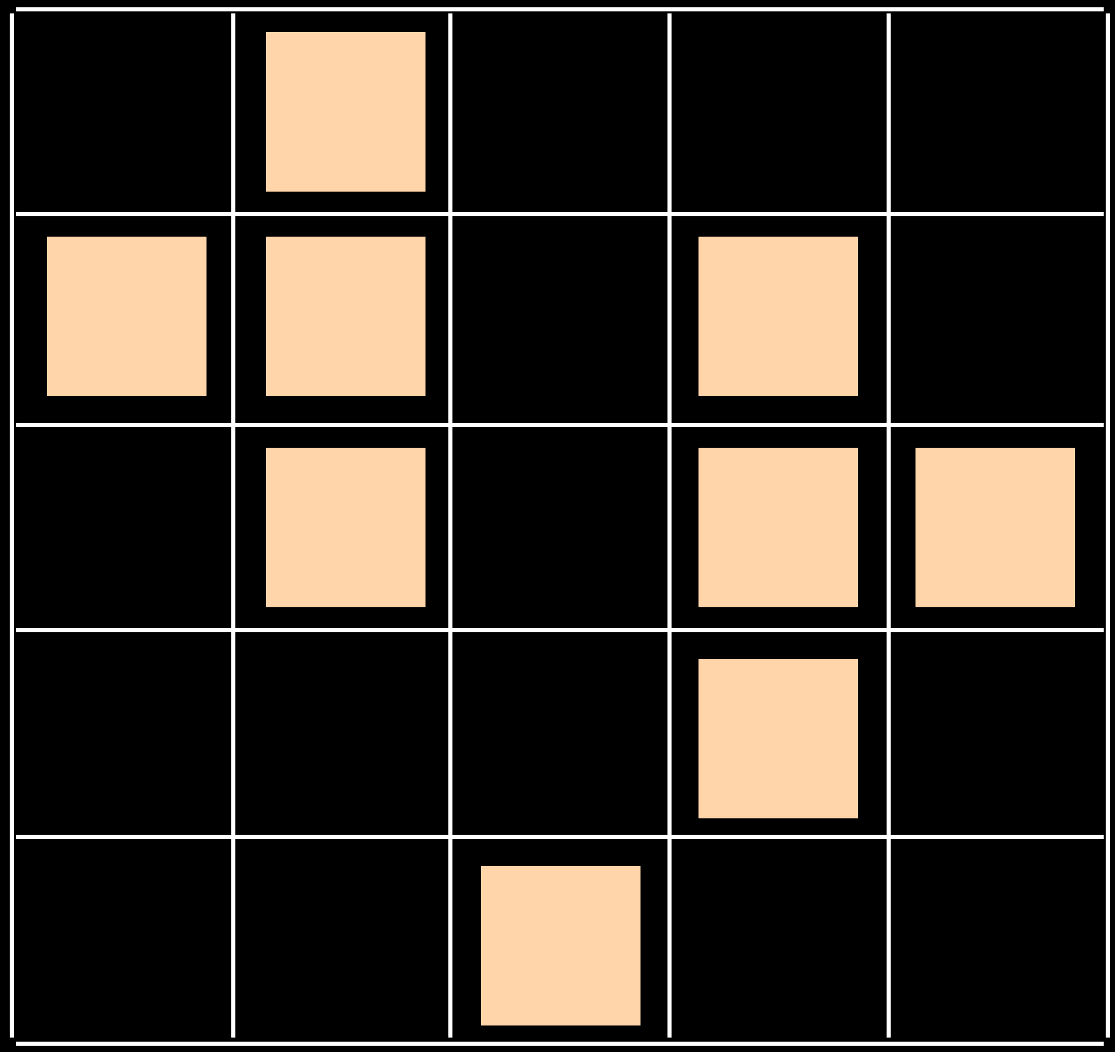


configuration
B. A. BA

Prop. The number of configurations B. A. BA on $n \times n$ is $2^{(n^2)}$


A
alternating sign matrix
φ (A)
B.A.BA
configuration
1
-1

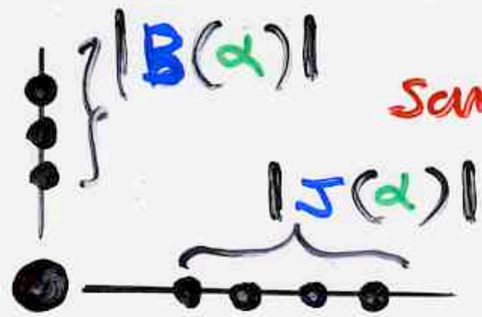




A alternating sign matrix $\longrightarrow$ $\varphi(A)$ B.A.BA configuration

$\left.\begin{matrix} 1 \\ -1 \end{matrix}\right\} \longrightarrow \bullet$

characterisation :



same parity

cell

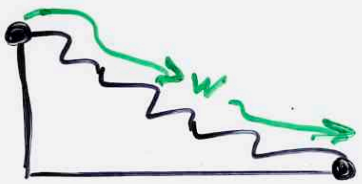
$\alpha \in C$

\+ in each row and column odd number of cells in C

Z-tableaux
and
B.A.BA configurations

Configurations B.A.BA
on a Ferrers diagram F

word $w \in \{B_\bullet, A_\bullet, B, A\}^*$ $\longrightarrow$ diagram $F(w)$



$w = w_1 \dots w_m$

i^{th} step of  is $=$ if $w_i = B_\bullet, B$

$\|$ if $w_i = A_\bullet, A$

Bijection(s)

(word w, C) $\longleftrightarrow$ T

C: B.A.BA configuration on the diagram $F(w)$

T: Z-tableau (with diagram $F(w)$)

F Ferrers diagram

For each cell α of F,
for each of the pair $B\,A$, $B_\bullet A_\circ$, $B_\bullet A$, $B\,A_\circ$,
we fix a rule for the labeling
of F by q_{xy} or t_{xy} $\left(\begin{matrix} x = \bullet \text{ or } \circ \\ y = \bullet \text{ or } \circ \end{matrix} \right.$
according to $\alpha \in C$ or not.

Bijection(s)

(word w, C) $\longleftrightarrow$ T

C ↑ B.A.BA configuration on the diagram $F(w)$

T: Z-tableau (with diagram $F(w)$)

T_0 Z-tableau on diagram F

$\downarrow$ ϕ

T

$2^{\ell(F) + |F|}$ such bijections

square lattice $n \times n$ $\rightarrow 2^{2n+n^2}$