Heaps of pieces

(with interactions in mathematics in physics)

Chapter 2a

Ordinary generating functions

(from Ch 1 of Talca course in 2010/2011)

Universidad de Talca, Chile

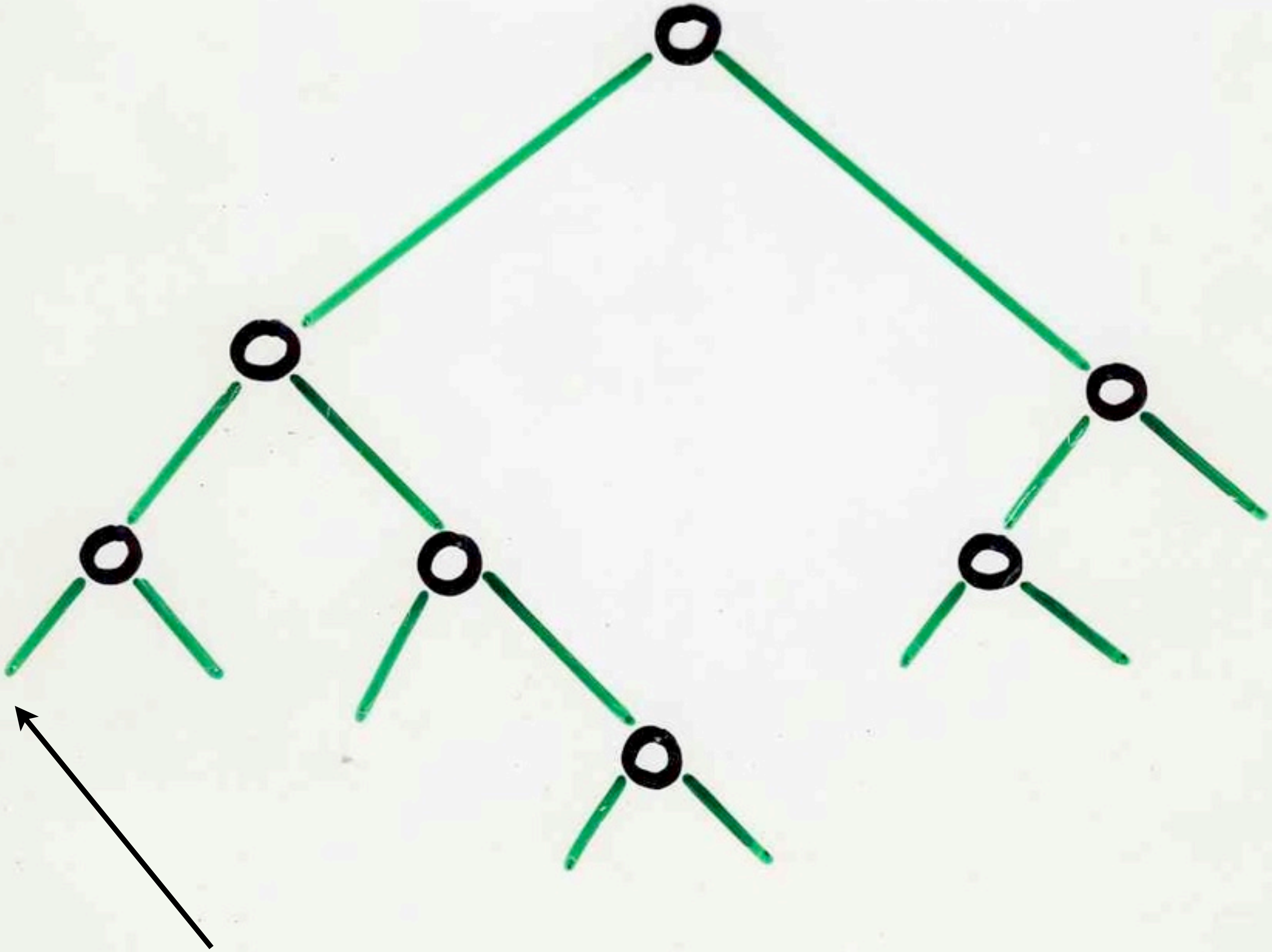
18 December 2013

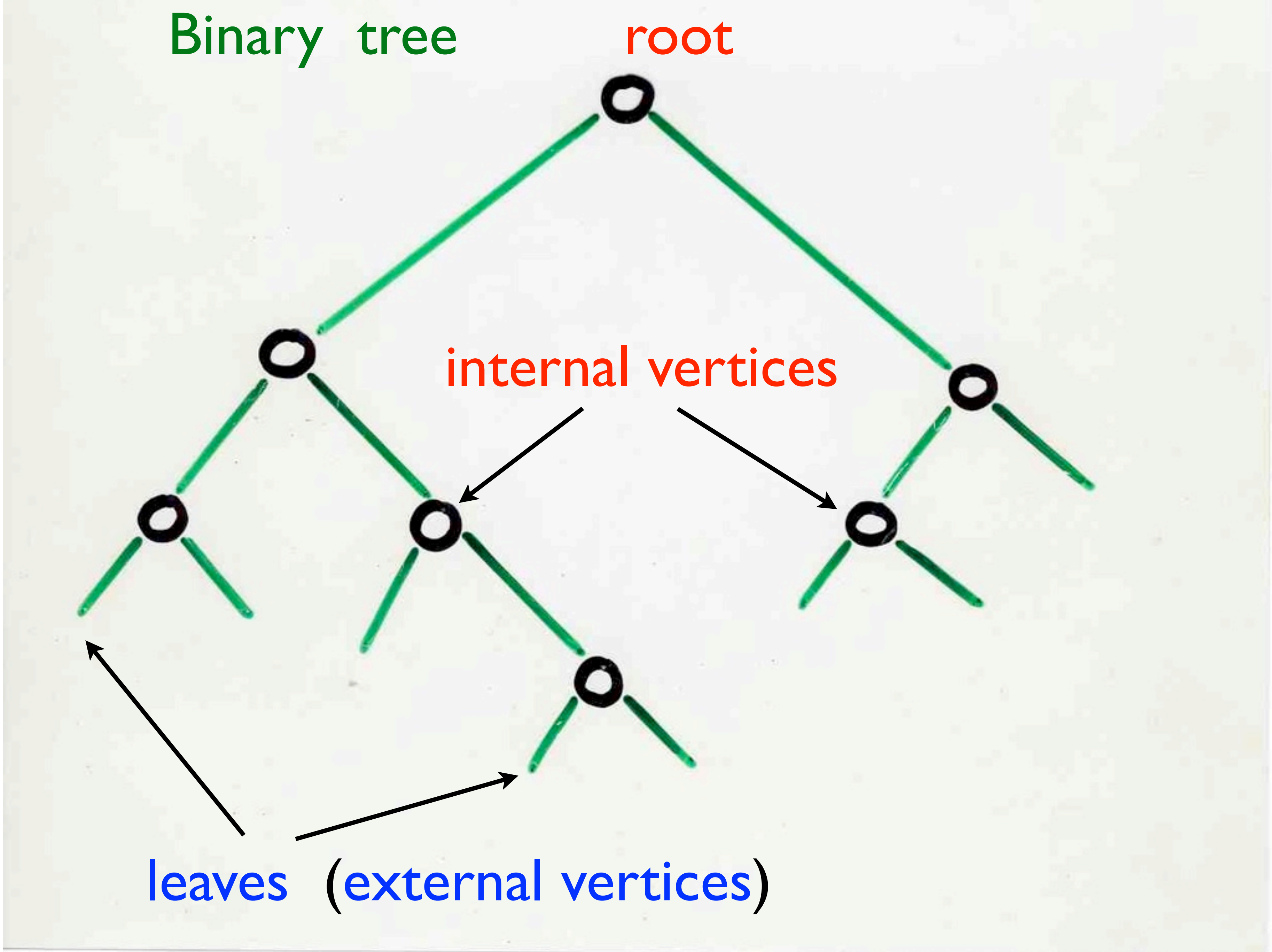
Xavier Viennot

LaBRI, CNRS, Bordeaux

binary tree

Binary tree




Binary tree
root
internal vertices
leaves (external vertices)

C_n = nombre
d'arbres binaires
ayant n
sommets internes

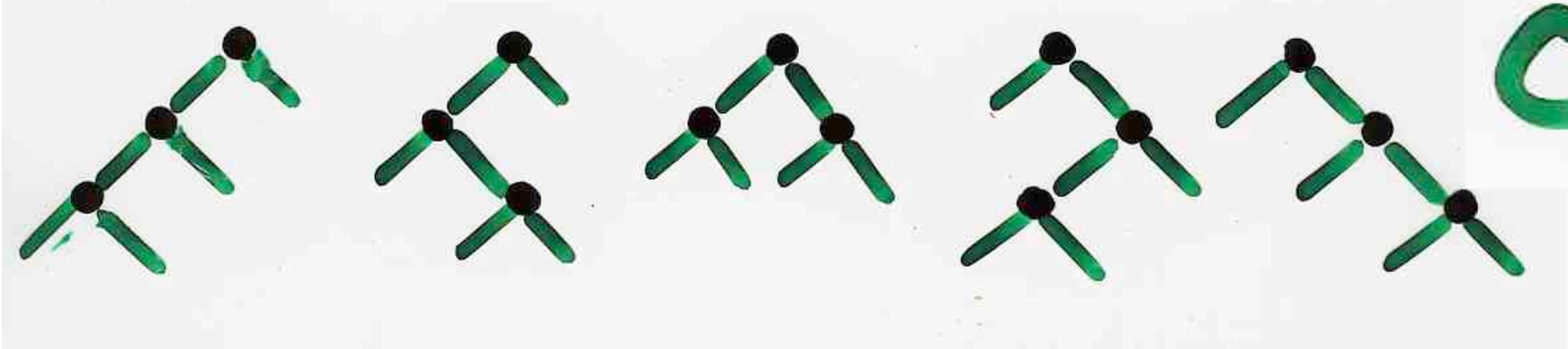
(et donc $n+1$ feuilles)

nombre de Catalan

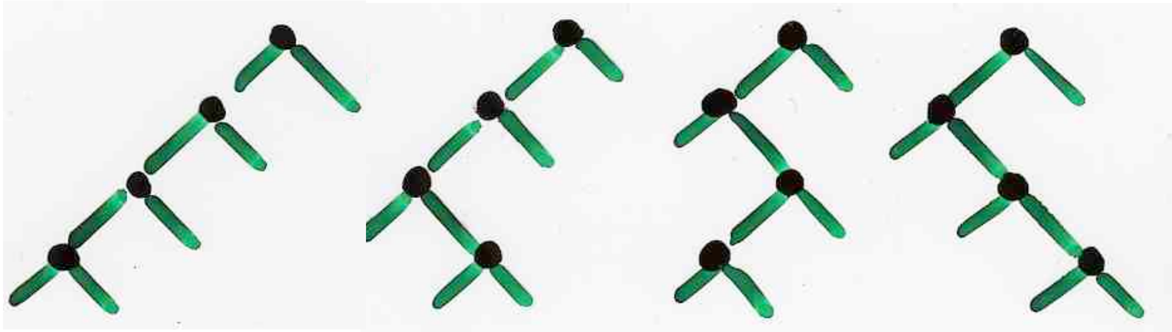
number of binary trees
having n internal vertices
(or n+1) leaves (external vertices)

$C_1 = 1$

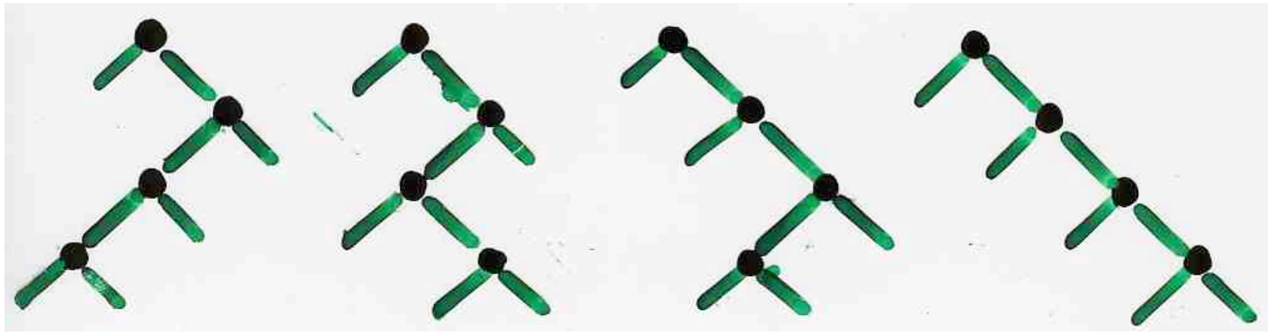
$C_2 = 2$

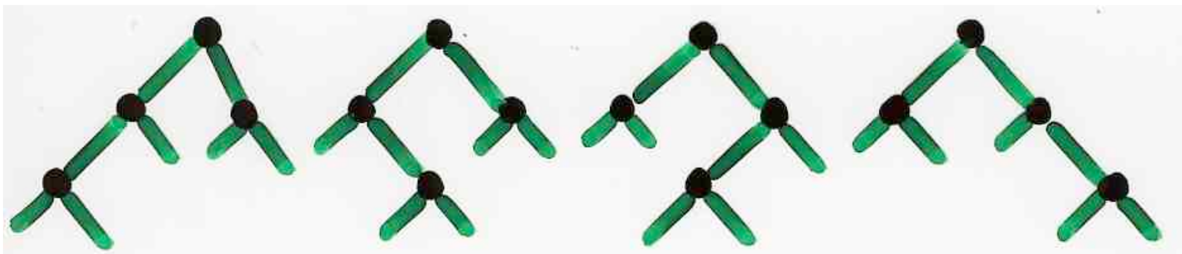


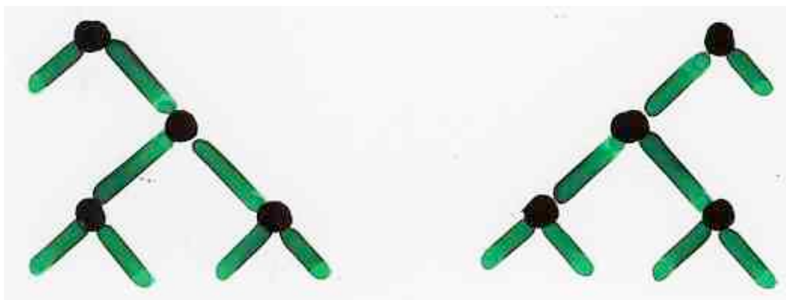
$C_3 = 5$



$C_4 = 14$







recurrence

$$C_{n+1} = \sum_{i+j=n} C_i C_j$$

$$C_0 = 1$$

$$\begin{array}{cccccc} C_0 & C_1 & C_2 & C_3 & C_4 & C_5 \\ 1, & 1, & 2, & 5, & 14, & 42, \ \dots \end{array}$$

$$C_6 = C_0 C_5 + C_1 C_4 + C_2 C_3 + C_3 C_2 + C_4 C_1 + C_5 C_0$$

$$132 \qquad 1 \times 42 + 1 \times 14 + 2 \times 5 + 5 \times 2 + 14 \times 1 + 42 \times 1$$

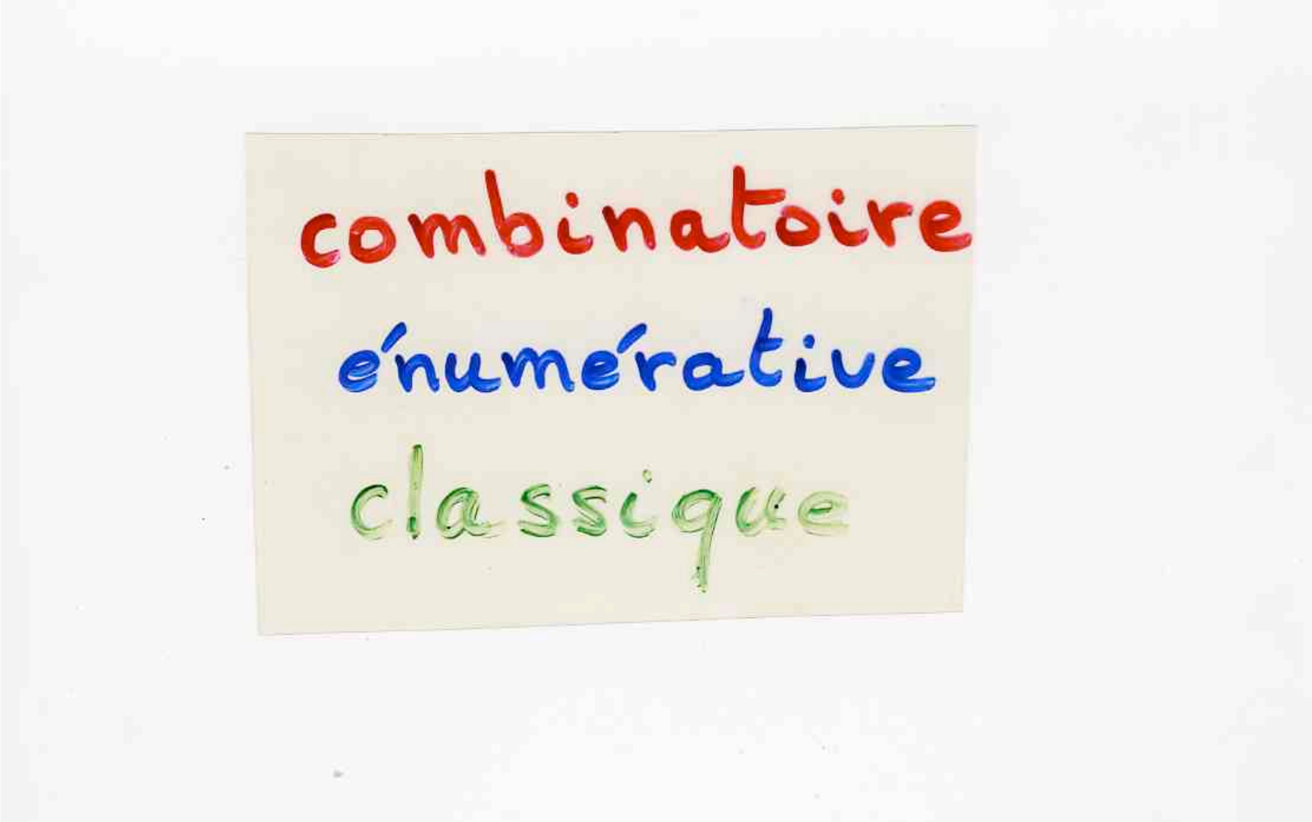
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!\;n!}$$

$$n! = 1 \times 2 \times \dots \times n$$

$$C_4 = \frac{1\times 2\times 3\times 4\times 5\times 6\times 7\times 8}{1\times 2\times 3\times 4 \quad 1\times 2\times 3\times 4\times 5}$$

$$= 14$$



classical

enumerative

combinatorics

(1838)

Note sur une Équation aux différences finies ;

PAR E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_4P_{n-2} + P_3P_{n-1} + P_n, \qquad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+1} = \frac{4n-6}{n} P_n. \qquad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

I.

L'intégrale de l'équation (2) est

$$P_{n+1} = \frac{6}{3}\cdot\frac{10}{4}\cdot\frac{14}{5}\cdots\frac{4n-6}{n}P_3;$$

et comme, dans la question de géométrie qui conduit à ces deux équations, on a $P_3 = 1$, nous prendrons simplement

$$P_{n+1} = \frac{2.6.10.14\ldots(4n-6)}{2.3.\ 4.\ 5\ldots\ n}. \qquad (3)$$

Le numérateur

$$2.6.10.14\ldots(4n-6) = 2^{n-1}.1.3.5.7\ldots(2n-3)$$
$$= \frac{2^{n-1}.1.2.3.4.5\ldots(2n-2)}{2.4.6.8\ldots(2n-2)} = \frac{1.2.3.4\ldots(2n-2)}{1.2.3\ldots(n-1)}.$$

Donc

$$P_{n+1} = \frac{n(n+1)(n+2)\ldots(2n-2)}{2.3.4\ldots n}. \qquad (4)$$

Si l'on désigne généralement par $C_{m,p}$ le nombre des combinaisons de m lettres, prises p à p; et si l'on change n en $n+1$, on aura

$$P_{n+2} = \frac{1}{n+1} C_{2n,n}, \qquad (5)$$

ou bien

$$P_{n+2} = C_{2n,n} - C_{2n,n-1}. \qquad (6)$$

II.

Les équations (1) et (5) donnent ce théorème sur les combinaisons :

$$\left.\begin{aligned}\frac{1}{n+1}C_{2n,n} &= \frac{1}{n}C_{2n-2,n-1} + \frac{1}{n-1}C_{2n-4,n-2}\times\frac{1}{2}C_{2,1}\\ &+ \frac{1}{n-2}C_{2n-6,n-3}\times\frac{1}{3}C_{4,2} + \ldots + \frac{1}{n}C_{2n-2,n-1}.\end{aligned}\right\} \qquad (7)$$

III.

On sait que le $(n+1)^e$ nombre figuré de l'ordre $n+1$, a pour expression, $C_{2n,n}$: si donc, dans la table des nombres figurés, on prend ceux qui occupent la diagonale; savoir :

1, 2, 6, 20, 70, 252, 924...;

qu'on les divise respectivement par

1, 2, 3, 4, 5, 6, 7...;

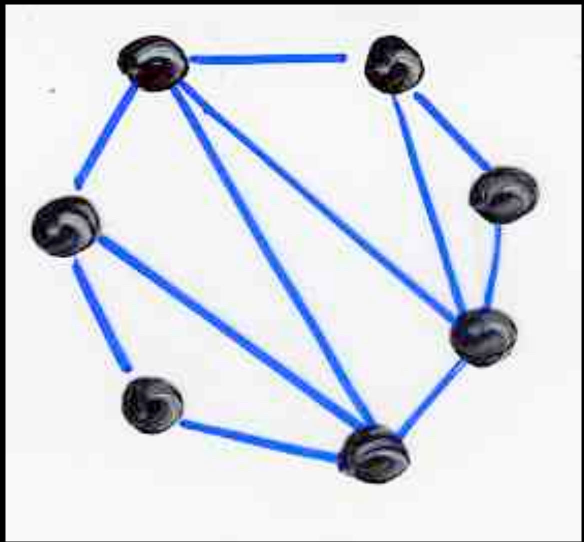
e nombres,

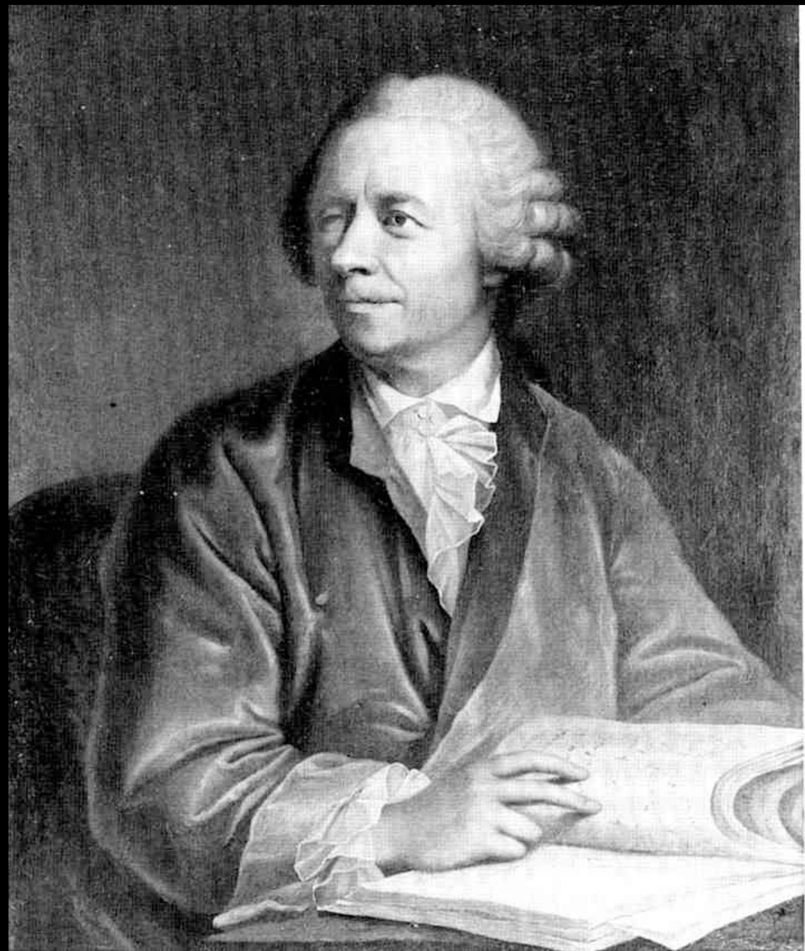
14, 42, 132..., (A)

té :

ite (A) est égal à la somme des
ivant au-dessous d'elle-même, et
s termes précédents, et en multi-
les deux séries.

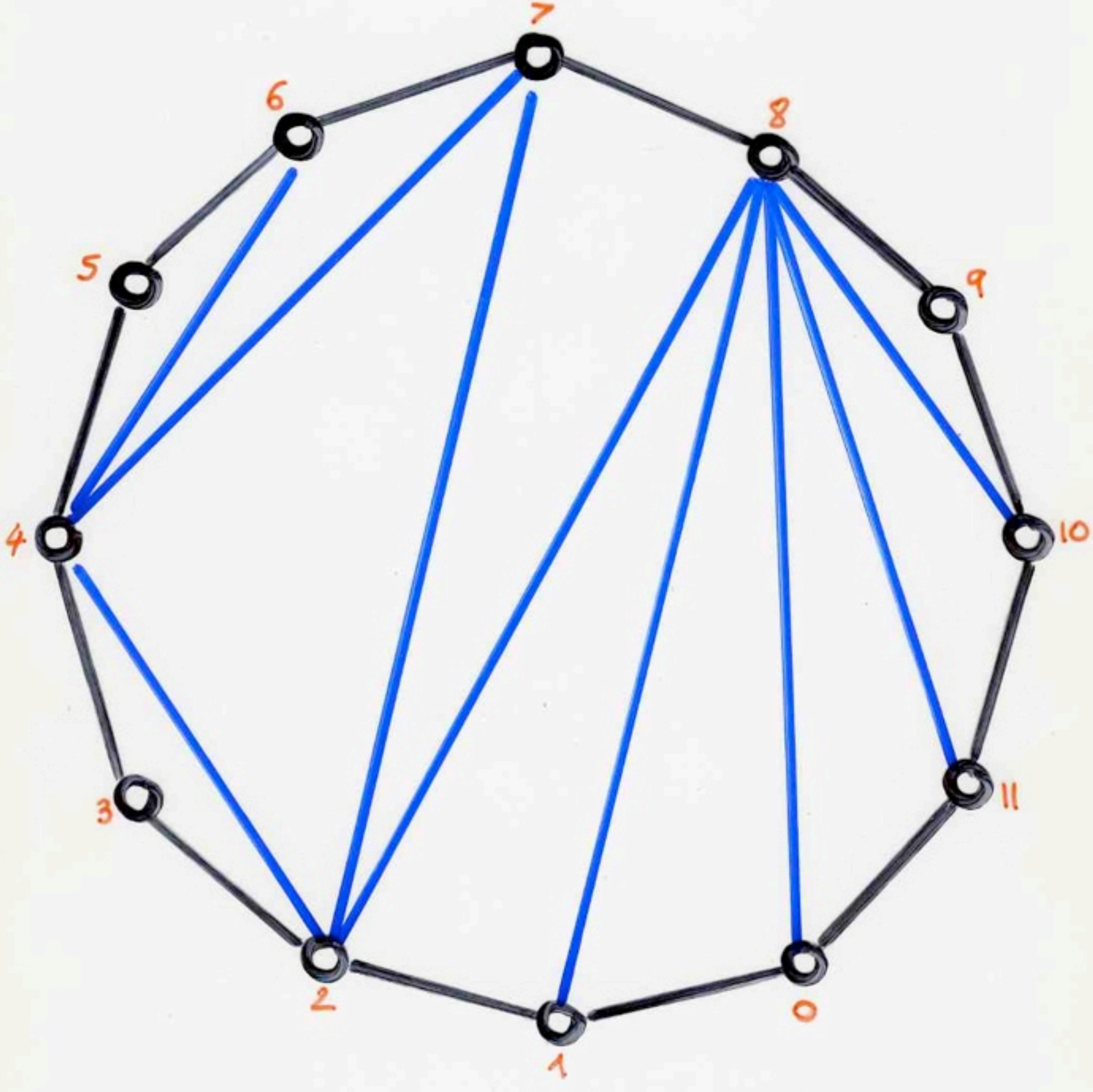
5+5.2+14.1+42.1.





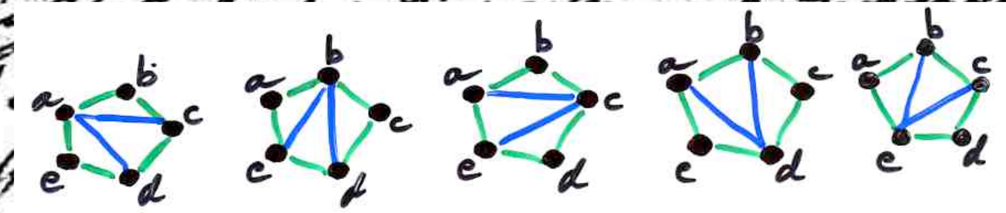
$$2(2n+1)C_n = (n+2)C_{n+1}$$

$$\frac{1}{n+1}\binom{2n}{n}$$



7
6
8
5
9
4
10
3
11
2
0
1

durch die Diagonales I. ac ad; II. bd be; III ca ce; IV db da; V ec eb



durch $n-3$ Diagonales in $n-2$ Triangula zerschnitten werden ...

... $= x$

wann $n =$ 1, 2, 5, 14, 42, 132, 429, 1430, ...

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430$ ... generaliter

$$x = \frac{2.\ 6.\ 10.\ 14.\ 18.\ 22.\ \ldots\ (4n-10)}{2.\ 3.\ 4.\ 5.\ 6.\ 7.\ \ldots\ (n-1)} = \frac{(2n)!}{(n+1)!\,n!}$$

$5 = 2 \cdot \frac{10}{4}$; $14 = 5 \cdot \frac{14}{5}$; $42 = 14 \cdot \frac{18}{6}$;

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

$$\frac{1-2a-\sqrt{1-4a}}{2a^2}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc}$$

$$1+2a+5a^2+14a^3+42a^4+132a^5+\text{etc} = \frac{1-2a-\sqrt{(1-4a)}}{2aa}$$

also wenn $a=\frac{1}{4}$ so ist $1+\frac{2}{4}+\frac{5}{4^2}+\frac{14}{4^3}+\frac{42}{4^4}+\text{etc} = 4.$

Berlin d. 4. Sept 1751.

4 Sept 1751
Berlin

L. Euler

ordinary generating functions

formal power series

Catalan numbers

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

polynomial

$$1 + 1t + 2t^2 + 5t^3 + 14t^4 + 42t^5$$

$$+ \cdots$$

formal power series

$$y = 1 + 2t + 5t^2 + 14t^3$$

$$+ 42t^4 + \dots + C_n t^n$$

$$+ \dots$$

Wilf

corde à linge

série génératrice

$$f(t) = a_0 + a_1 t + a_2 t^2 + \dots\dots$$

$$\dots + a_n t^n + \dots\dots$$

generating function

formal power series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots$$

a little exercise

$$\frac{1}{1-(t+t^2)} = \; ?$$

$$\frac{1}{1-(t+t^2)} = ?$$

$$= 1 + t + 2t^2 + 3t^3 + 5t^4$$
$$+ 8t^5 + 13t^6 + 21t^7$$
$$+ 34t^8 + 55t^9 + \dots$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$

$$(t^2 + 2t^3 + t^4)$$

$$(t^3 + 3t^4 + 3t^5 + t^6)$$

$$(t^4 + 4t^5 + 6t^6 + \dots$$

$$+ (t^5 - - - -$$

$$\sum_{i \geq 0} (t + t^2)^i =$$

$$1 + (t + t^2)$$
$$(t^2 + 2t^3 + t^4)$$
$$(t^3 + 3t^4 + 3t^5 + t^6)$$
$$(t^4 + 4t^5 + 6t^6 + \ldots$$
$$+ (t^5 \text{ --- --- ---}$$

1 2 3 5 8

$$F_{n+1} = F_n + F_{n-1}$$

$$F_0 = F_1 = 1$$

Fibonacci

$$f(t) = \sum_{n \geq 0} a_n t^n$$

$t + t + t + \ldots\ldots t + \ldots$

$1 + 1 + 1 + \ldots$

$$t + t + t + \ldots\ t + \ldots$$

$$1 + 1 + 1 + \ldots$$

formal power series algebra

formalisation

Formal power series

Formal power series algebra in one variable

$\mathbb{K}$ commutative ring

$\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \quad \mathbb{Z}[\alpha, \beta, \ldots]$

$\mathbb{K}[t]$ polynomials algebra

$\deg(P)$ degree

$\mathbb{K}[[t]]$ formal power series algebra

(in one variable t and coefficients in $\mathbb{K}$)

$$a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$$a_0 + a_1 t + a_2 t + \dots + a_n t^n + \dots$$

generating power series

of the coefficients (numbers) a_n

$$\sum_{n \geq 0} a_n t^n = f(t)$$

s.g. ordinaire

exponential $\sum a_n \frac{t^n}{n!}$

convergence

- formal
- real (complex)

ultrametric topology

algebra

- sum $\qquad f+g=h, \quad a_n+b_n=c_n$
- product $\qquad fg=h, \quad c_n=\sum_{\substack{p+q=n \\ p,q\geq 0}} a_p b_q$
- product (by a scalar) $\qquad \lambda f=h, \quad c_n=\lambda a_n$

$$f=\sum_{n\geq 0} a_n t^n, \quad g=\sum_{n\geq 0} b_n t^n, \quad h=\sum_{n\geq 0} c_n t^n$$

summable family

$$\sum_{i \in I} f_i(t)$$

infinite product

$$\prod_{i \in I} (1 + g_i(t))$$

other operations

- substitution

$$f(t) = \sum_{n\geq 0} a_n t^n \quad , \quad g(t) = \sum_{n\geq 0} b_n t^n$$

$$b_0 = 0$$

$$f \circ g(t) \; ; \; f(g(t)) = \sum_{n\geq 0} a_n (g(t))^n$$

- Inverse

$$\frac{1}{1-f} = 1 + f + f^2 + \dots + f^n + \dots$$

(si $\mathrm{ord}(f) \geq 1$)

- derivative

$$f' \qquad \frac{df}{dt} = \sum_{n\geq 1} n\, a_n t^{n-1}$$

exponential

logarithm

binomial power series

$$\exp(t) = \sum_{n \geq 0} \frac{t^n}{n!}$$

$$\log(1-t)^{-1} = \sum_{n \geq 1} \frac{t^n}{n}$$

$$(1+t)^{\alpha} = \sum_{n \geq 0} \binom{\alpha}{n} t^n$$

$$= \sum_{n \geq 0} \alpha(\alpha-1)\ldots(\alpha-n+1)\frac{t^n}{n!}$$

$\mathrm{ord}(f) \geq 1$ $\exp(f)$ $\log(1+f)$ $(1+f)^{\alpha}$

formal power series in several variables

$$f(t_1, t_2, \dots, t_p) = \sum_{n_1, \dots, n_p} a_{n_1, \dots, n_p} t_1^{n_1} t_2^{n_2} \cdots t_p^{n_p}$$

$\mathbb{K}[t_1, \dots, t_p]$

$\mathbb{K}[[t_1, \dots, t_p]]$

algebras

operations

$\partial / \partial t_i$

rational power series $\qquad \sum_{n\geq 0} a_n t^n = \frac{N(t)}{D(t)}$

$\cap$

algebraic power series $\qquad P(y,t) = 0$

$\cap$

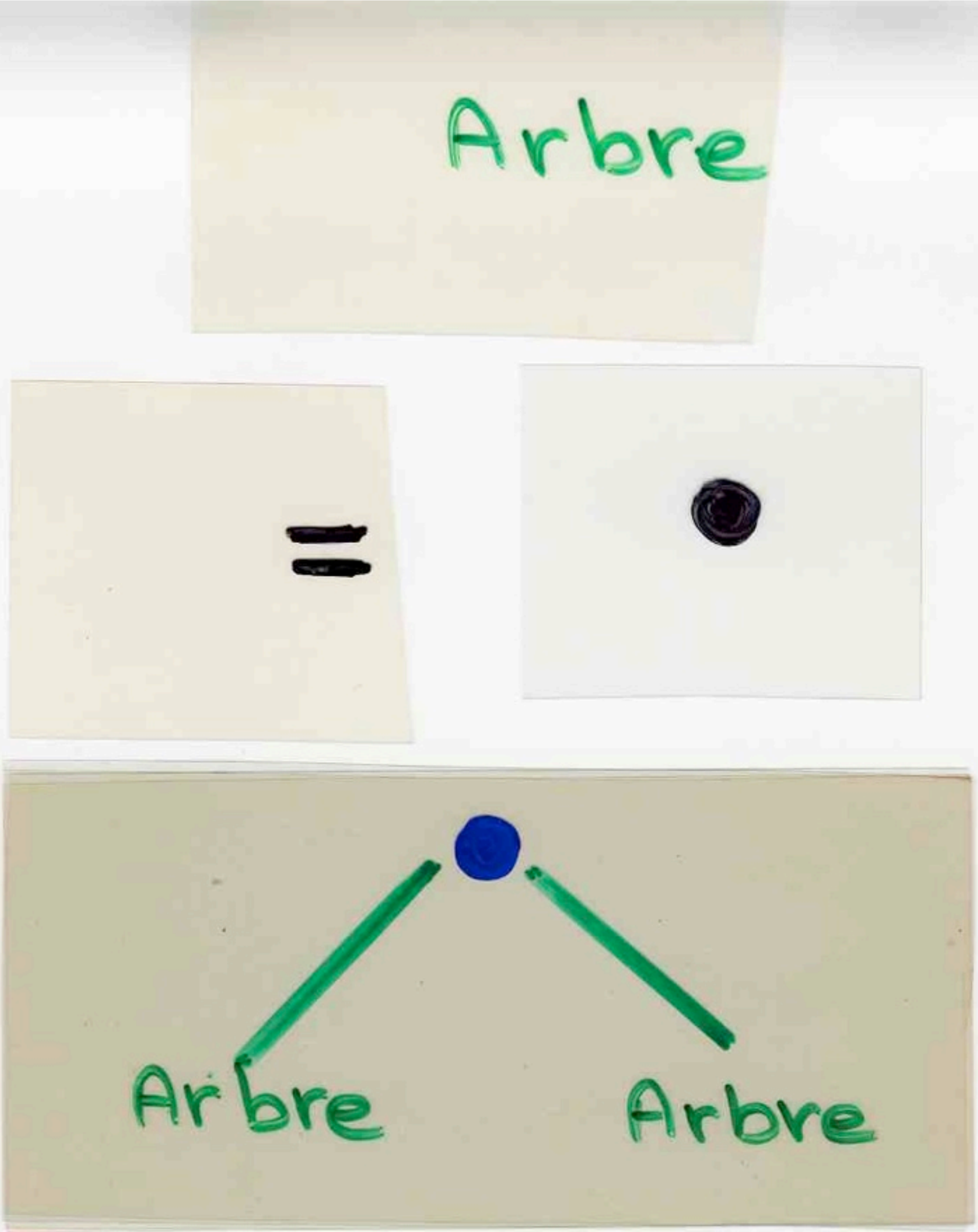
P-recursive (D-finite) power series

$$P_k(n)a_{n+k} + P_{k-1}(n)a_{n+k-1} + \cdots + P_0(n)\,a_n = 0$$

operations on combinatorial objects

example: binary trees

Binary Tree



$$y = 1 + \begin{tikzcd} & t \arrow[dl, dash] \arrow[dr, dash] & \\ y & & y \end{tikzcd}$$

$$y = 1 + t(y)^2$$

algebraic equation

$$y = 1 + t y^2$$

équation algébrique

$$y = \frac{1-(1-4t)^{1/2}}{2t}$$

$$(1+u)^{m} =$$

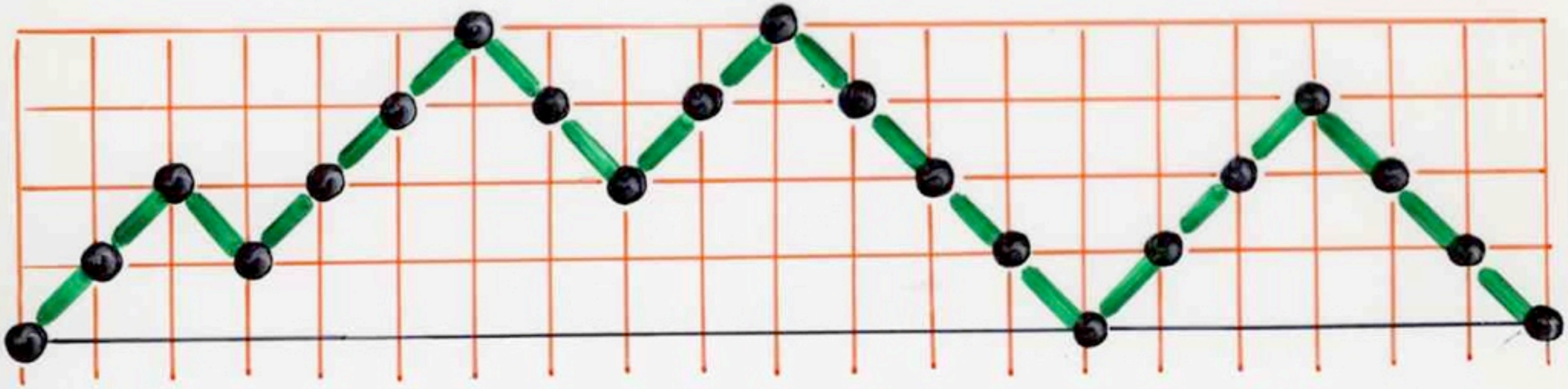
$$1 + \frac{m}{1!}u + \frac{m(m-1)}{2!}u^{2} + \frac{m(m-1)(m-2)}{3!}u^{3} +$$

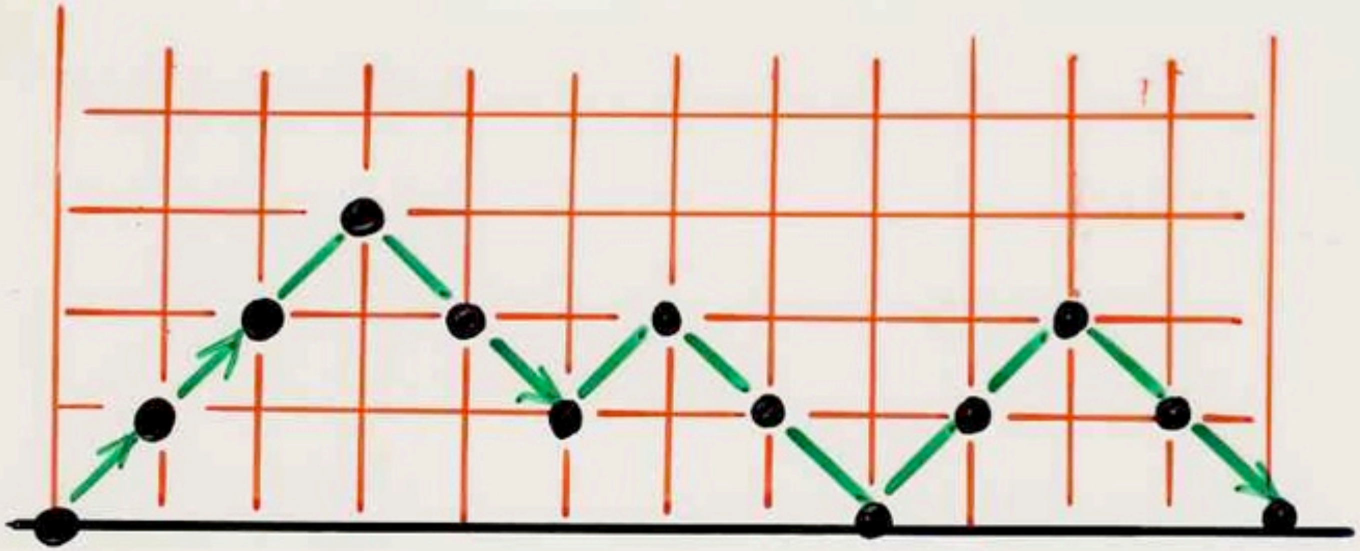
$$+ \cdots$$

$$m = \frac{1}{2}$$

$$u = -4t$$

Dyck paths



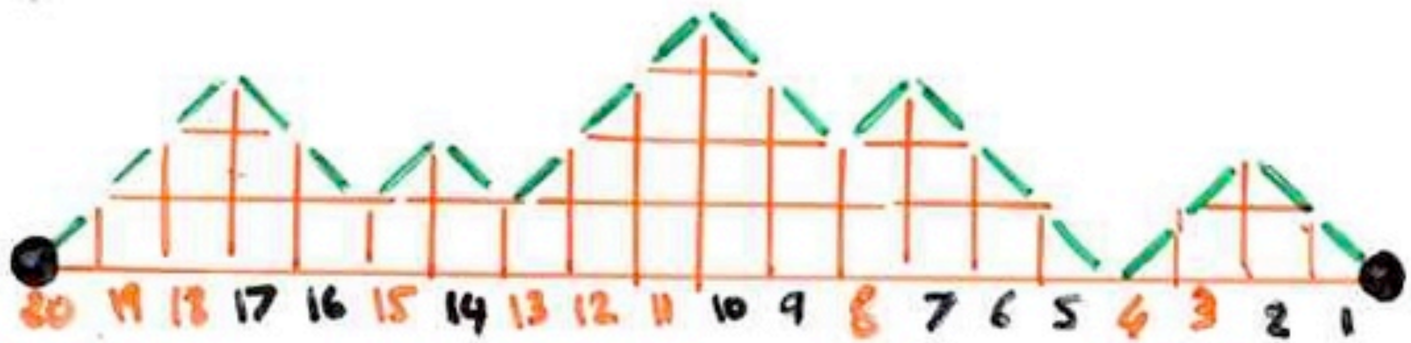


length $2n = 12$

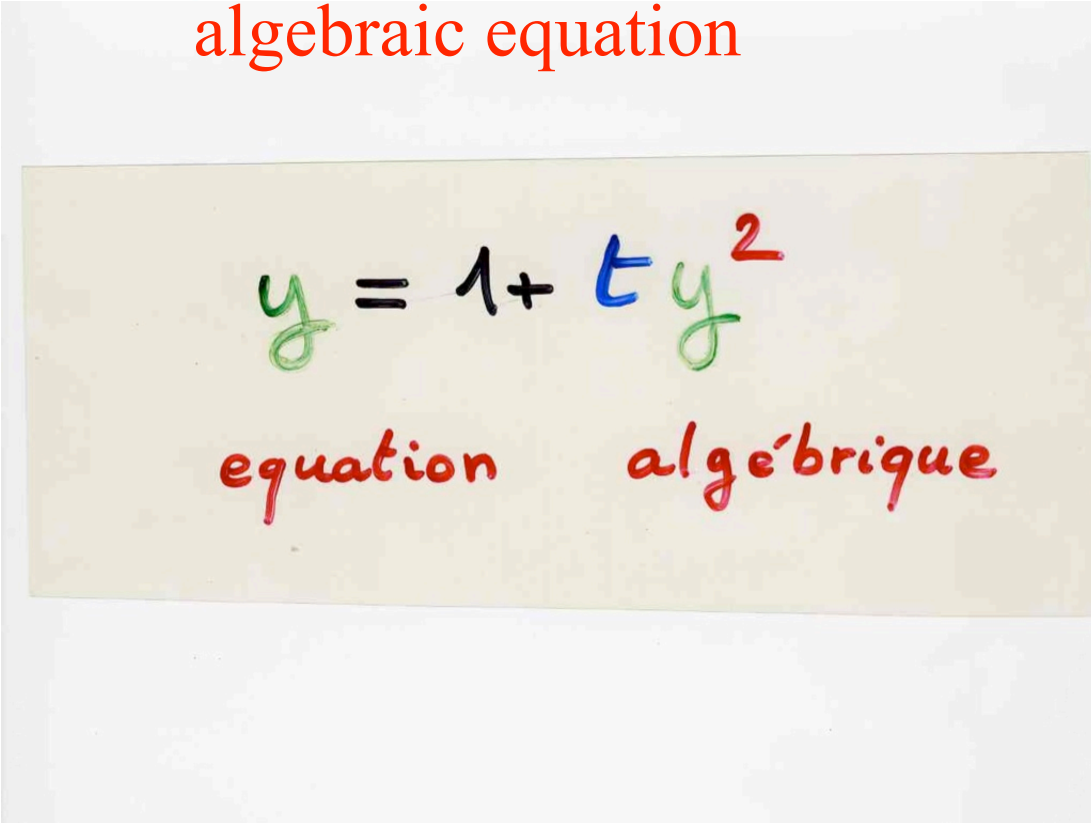
Dyck path

C_n = number of Dyck path of length $2n$

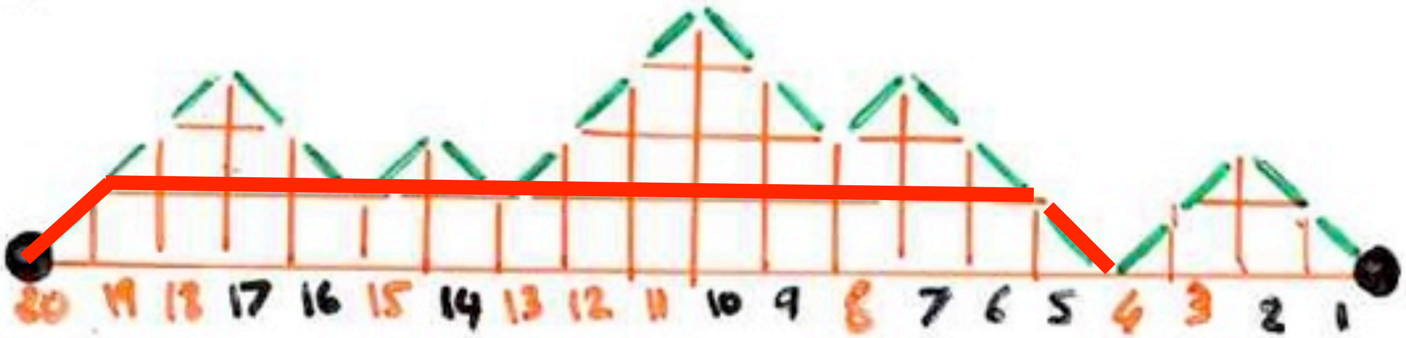
Dyck path

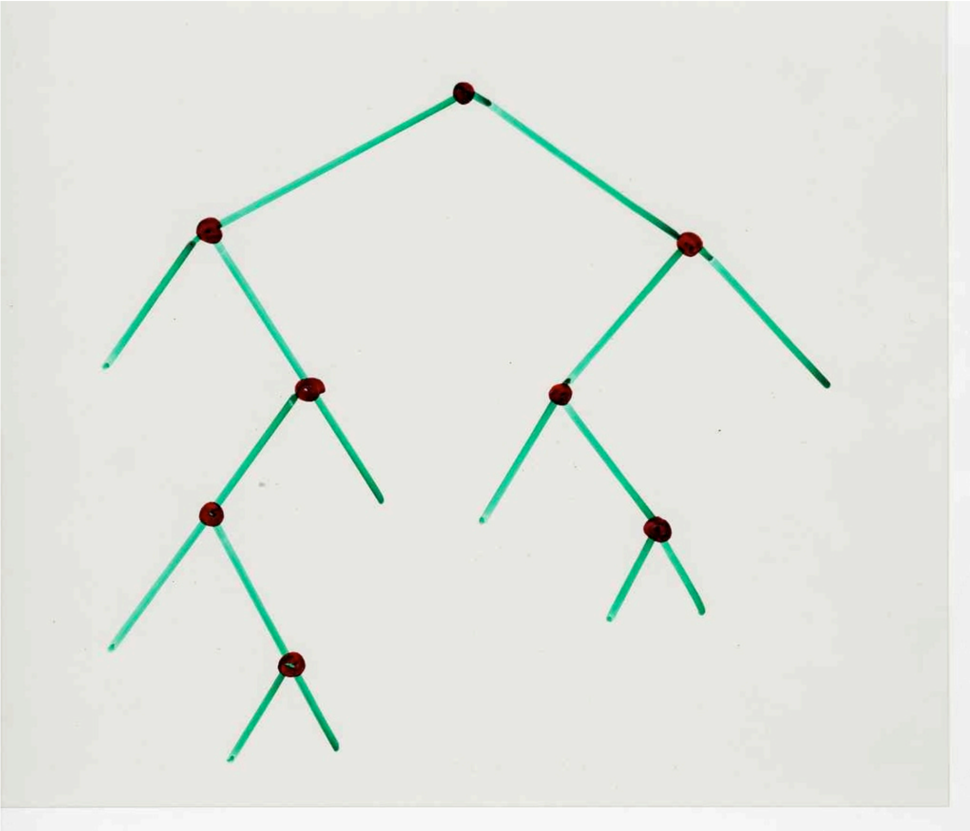


algebraic equation

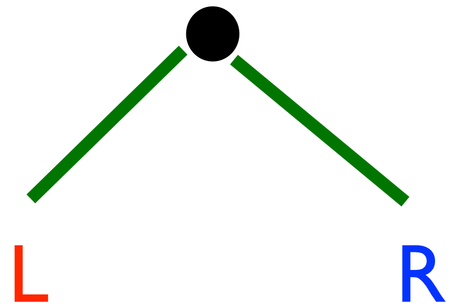


Dyck path

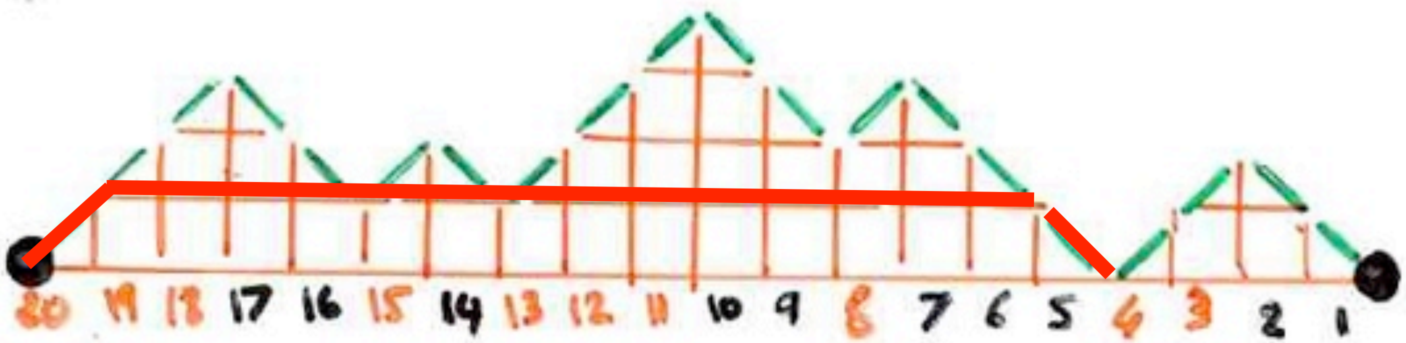


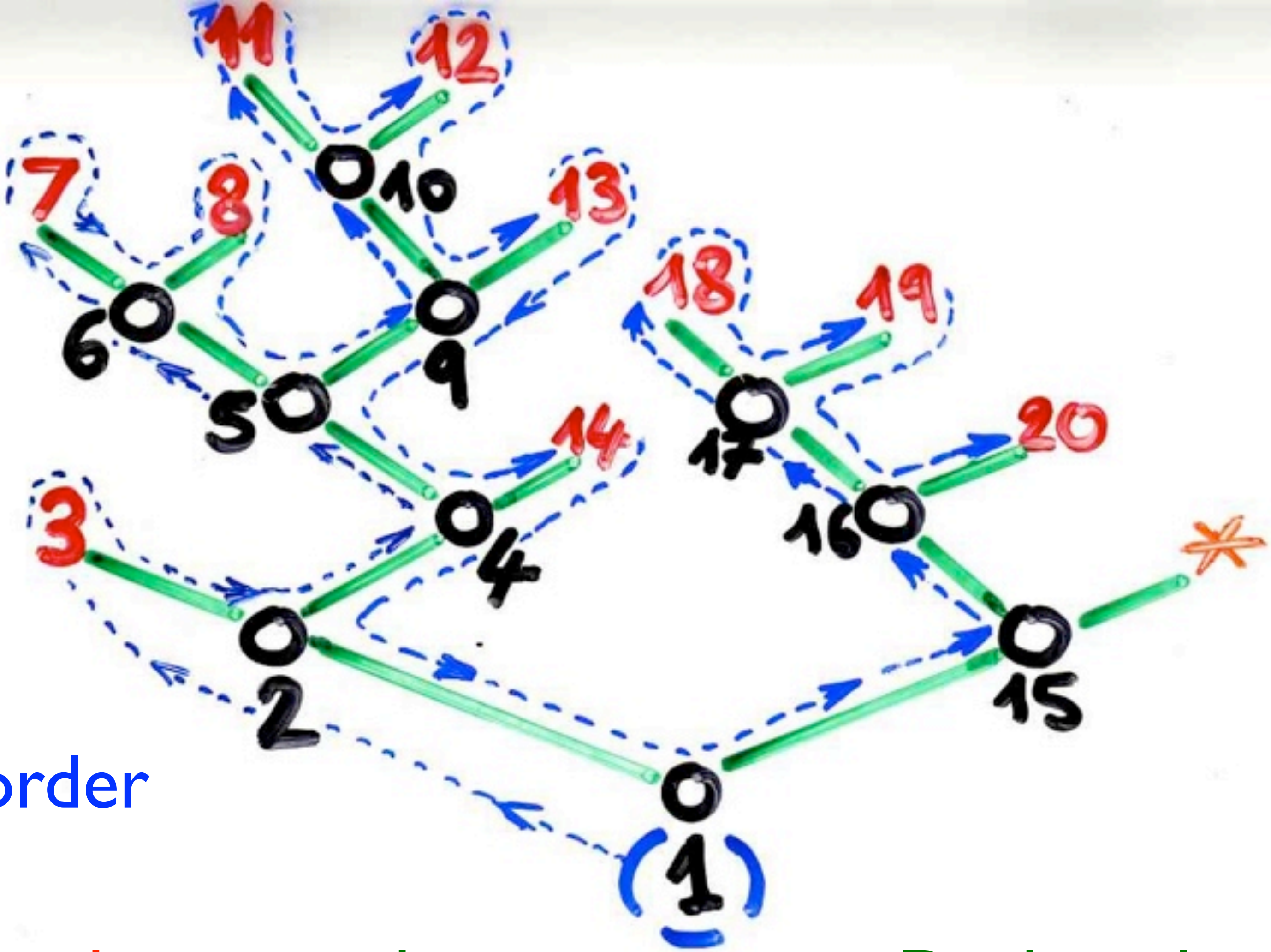


binary tree



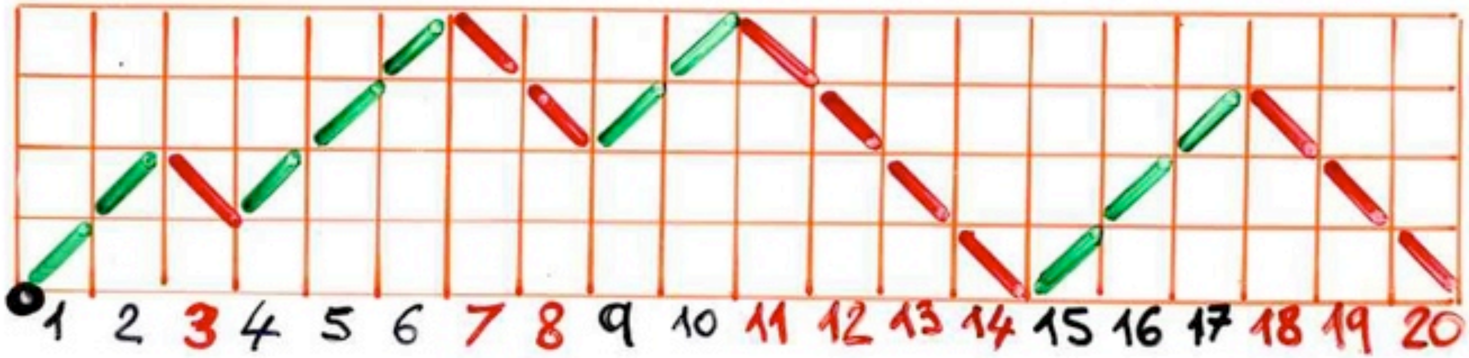
Dyck path





prefix order

bijection binary trees -- Dyck paths

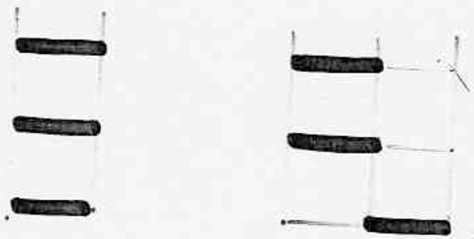


equation for semi-pyramids

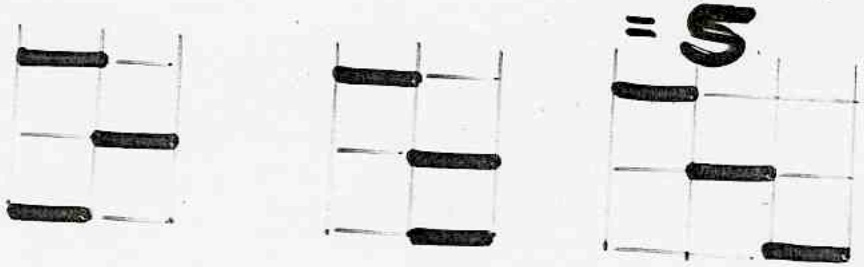
ex:

b_n = nb of pyramids of dimers on $\mathbb{N}$, having n dimers, and such that

$$\pi\left(\begin{array}{c}\text{maximal}\\\text{piece}\end{array}\right) = \{0, 1\}$$

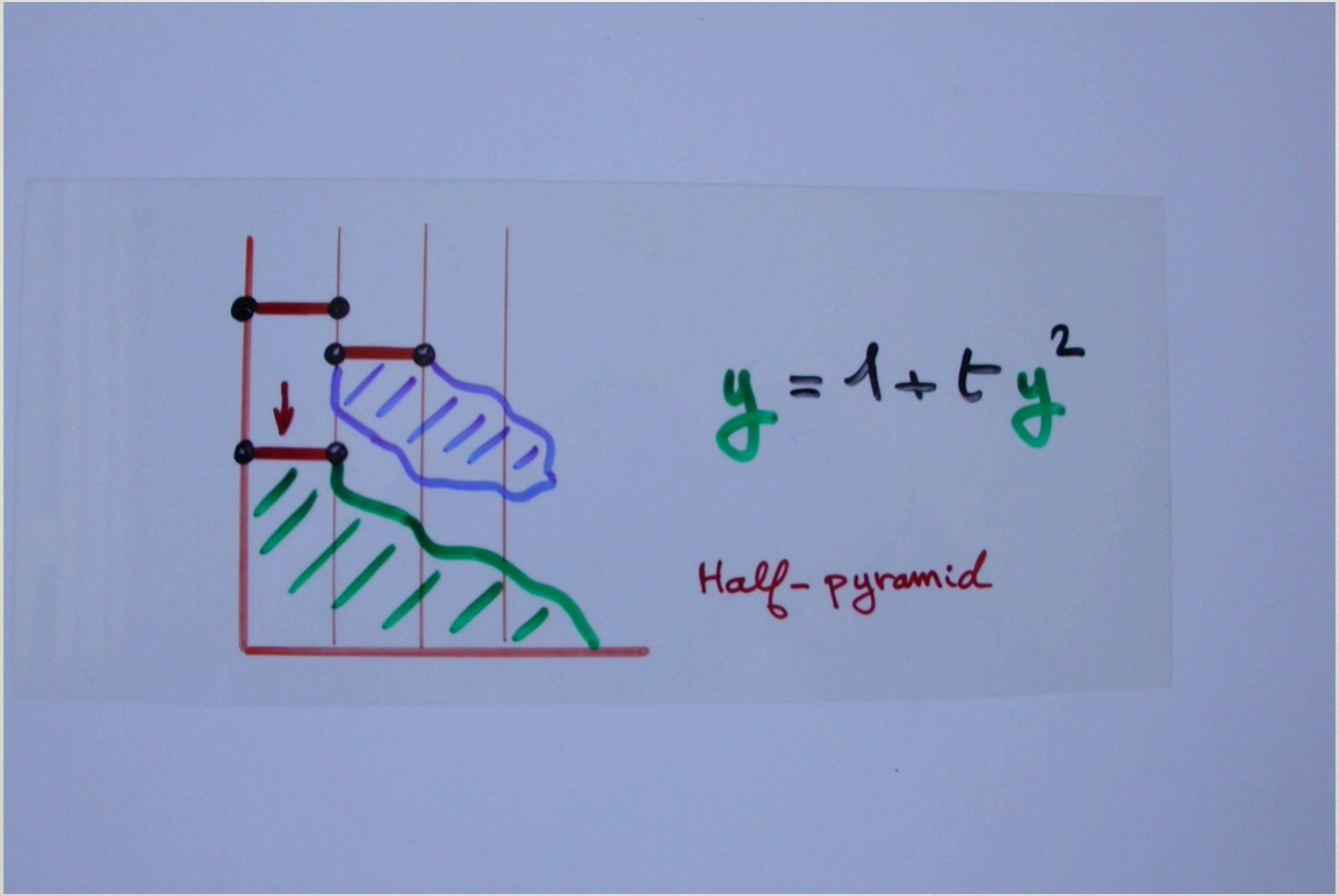


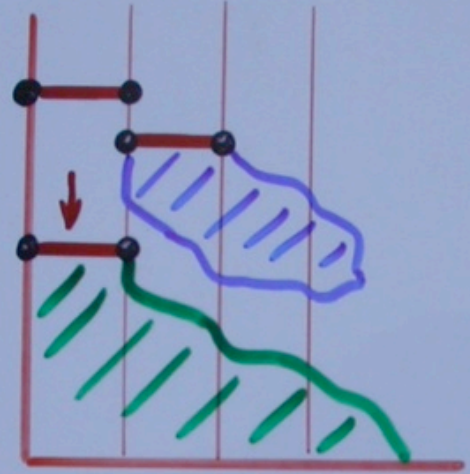
$$b_3 = \frac{6!}{3!\,4!} = \frac{1}{4}\binom{6}{3} = 5$$



$$b_n = C_n$$

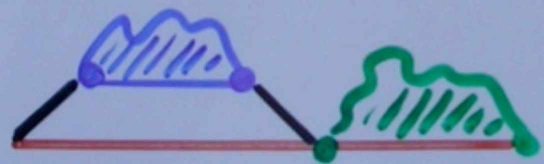
Catalan number


y = 1 + t y^2
Half-pyramid



$$y = 1 + t\,y^2$$

Half-pyramid



$$y = 1 + t\,y^2$$

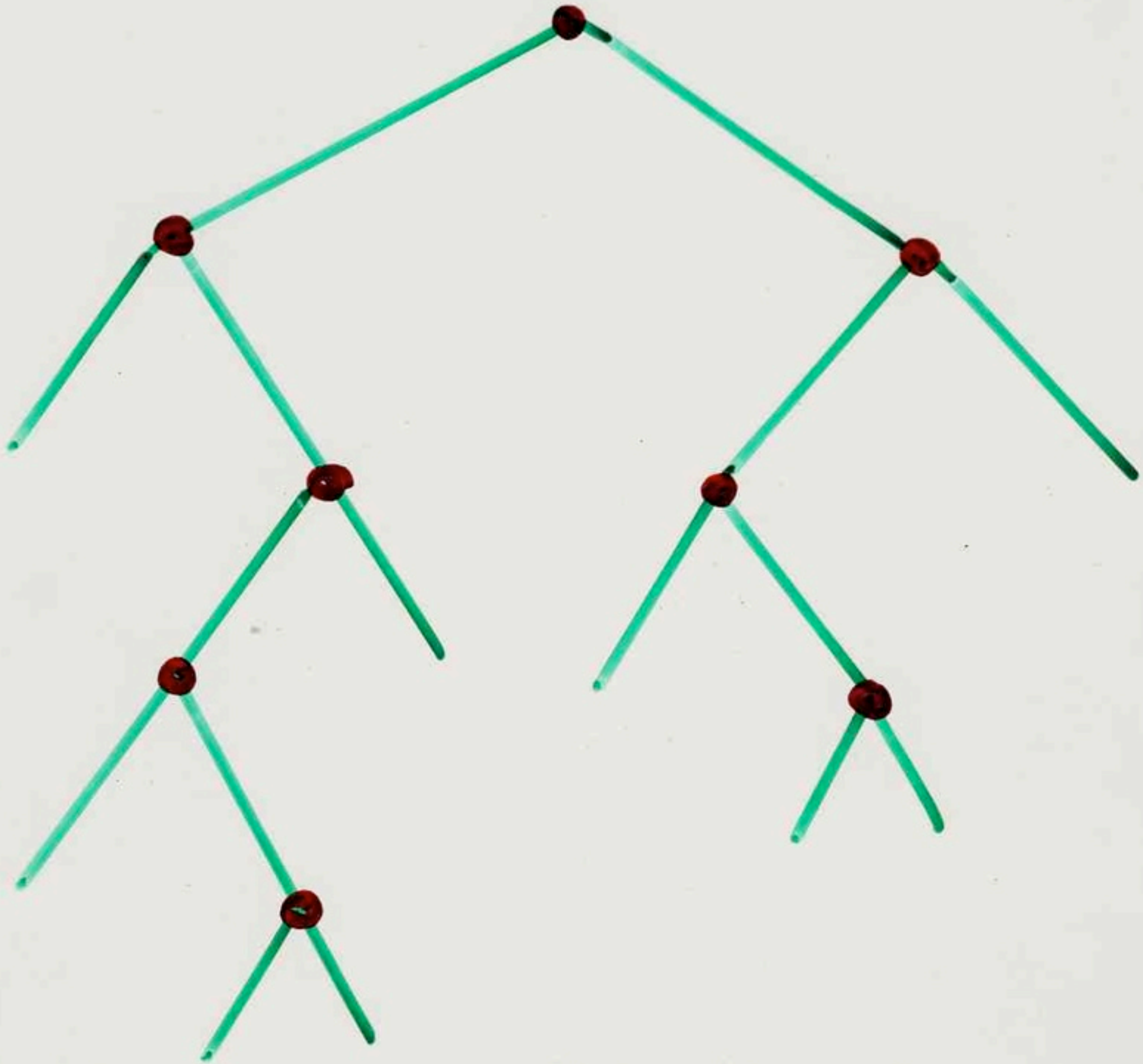
Dyck path

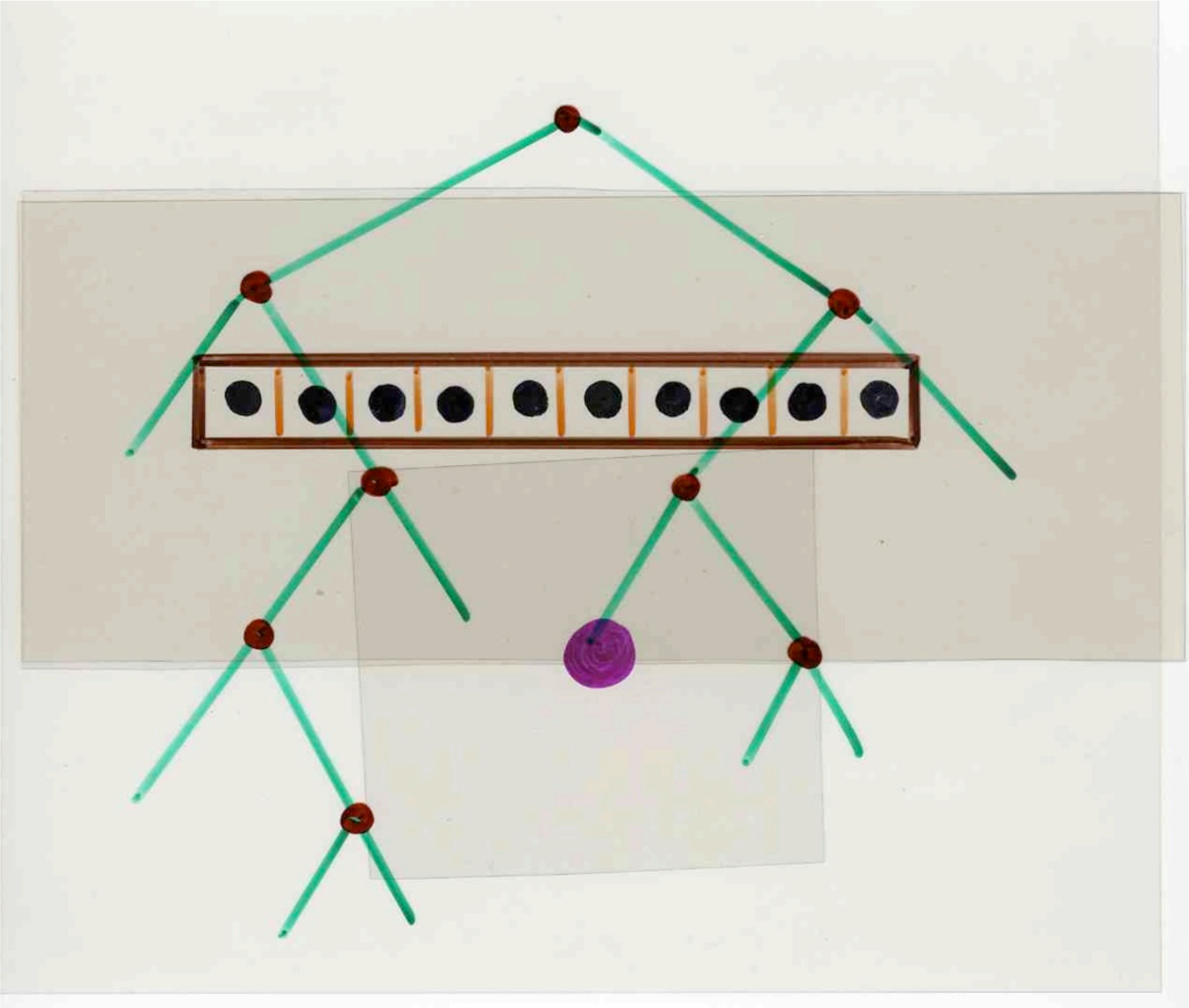
bijective combinatorics

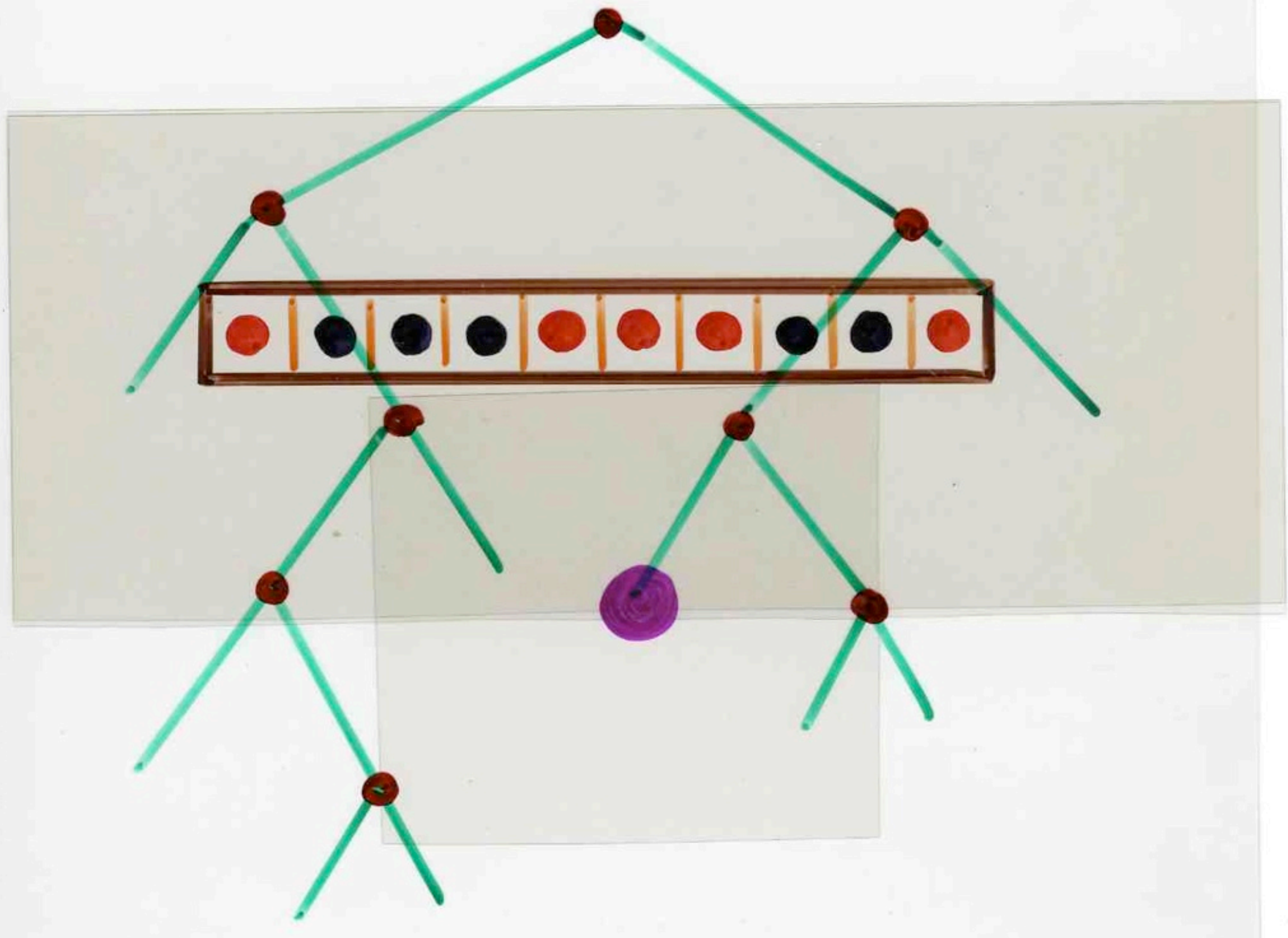
example: Catalan numbers

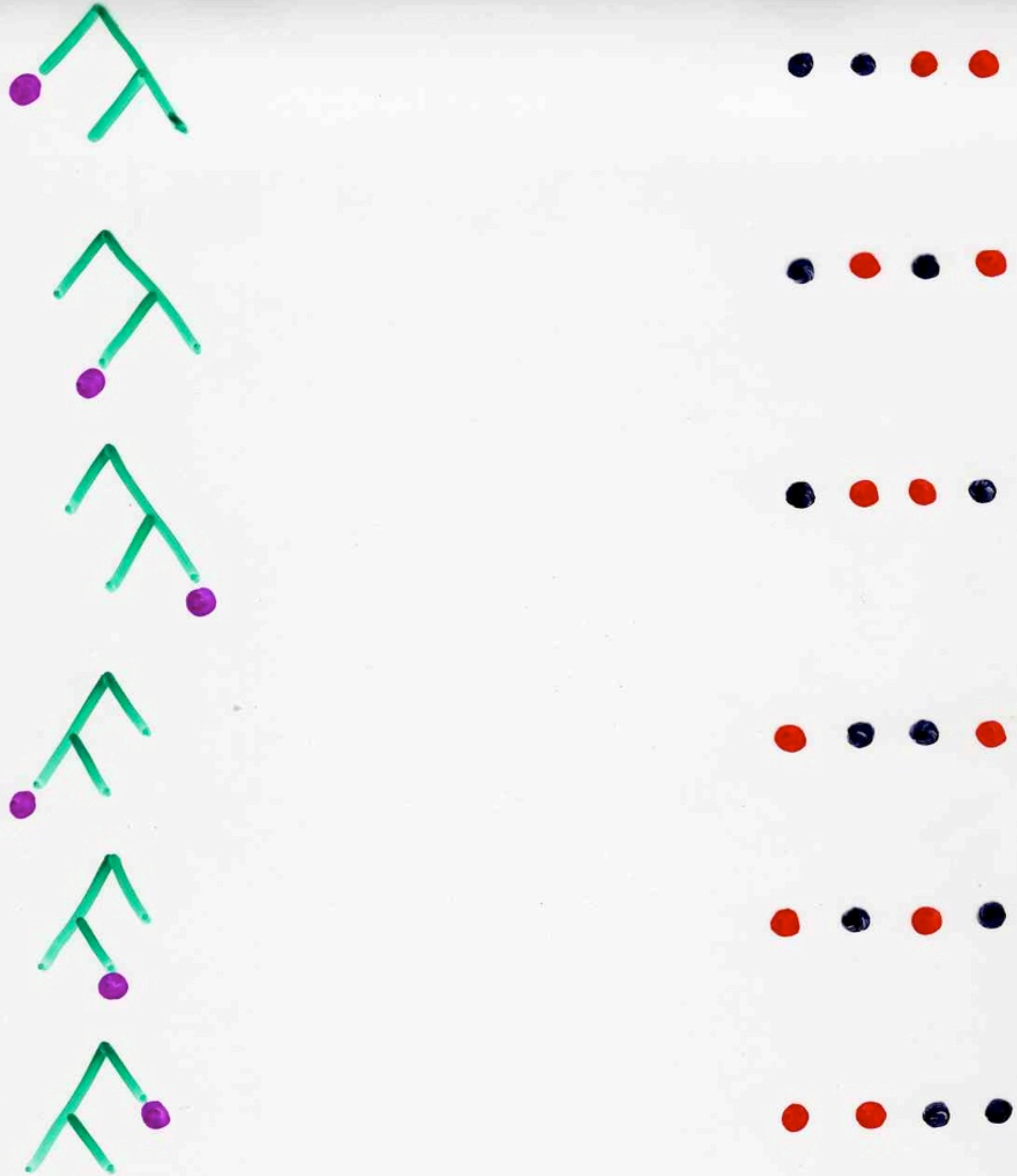
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

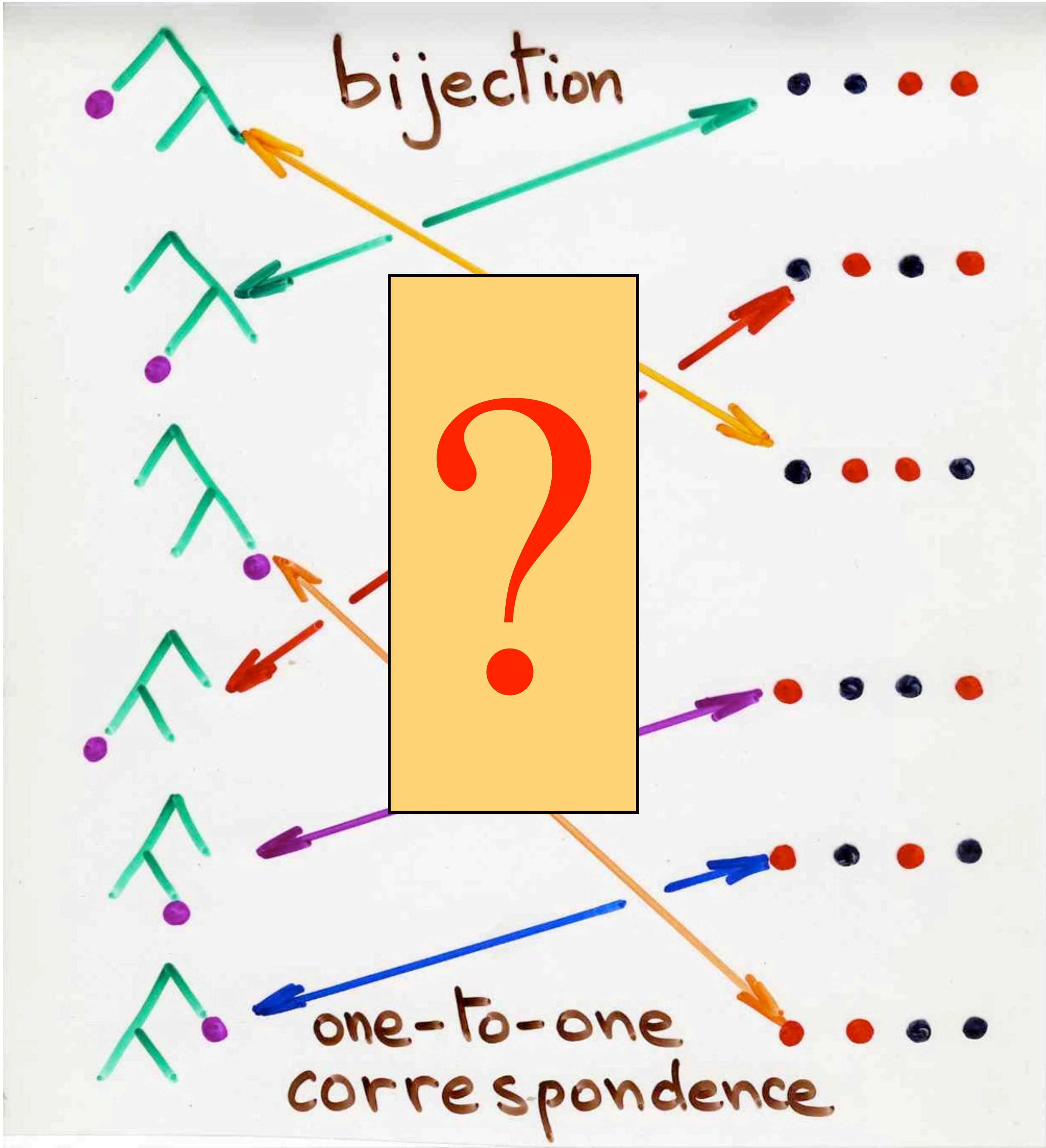
$$(n+1)\,C_n = \binom{2n}{n}$$










bijection
?
one-to-one
correspondence

operations on combinatorial objects

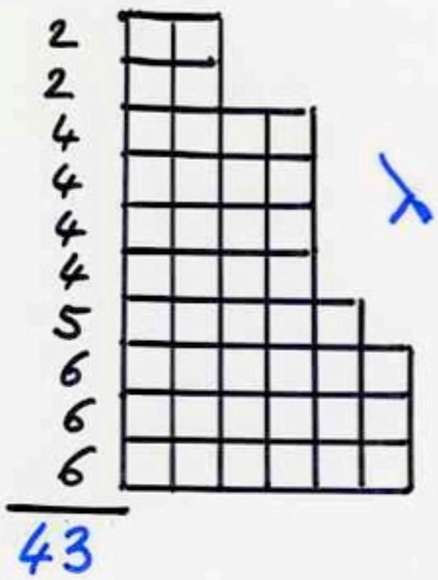
example: integers partitions

q-series

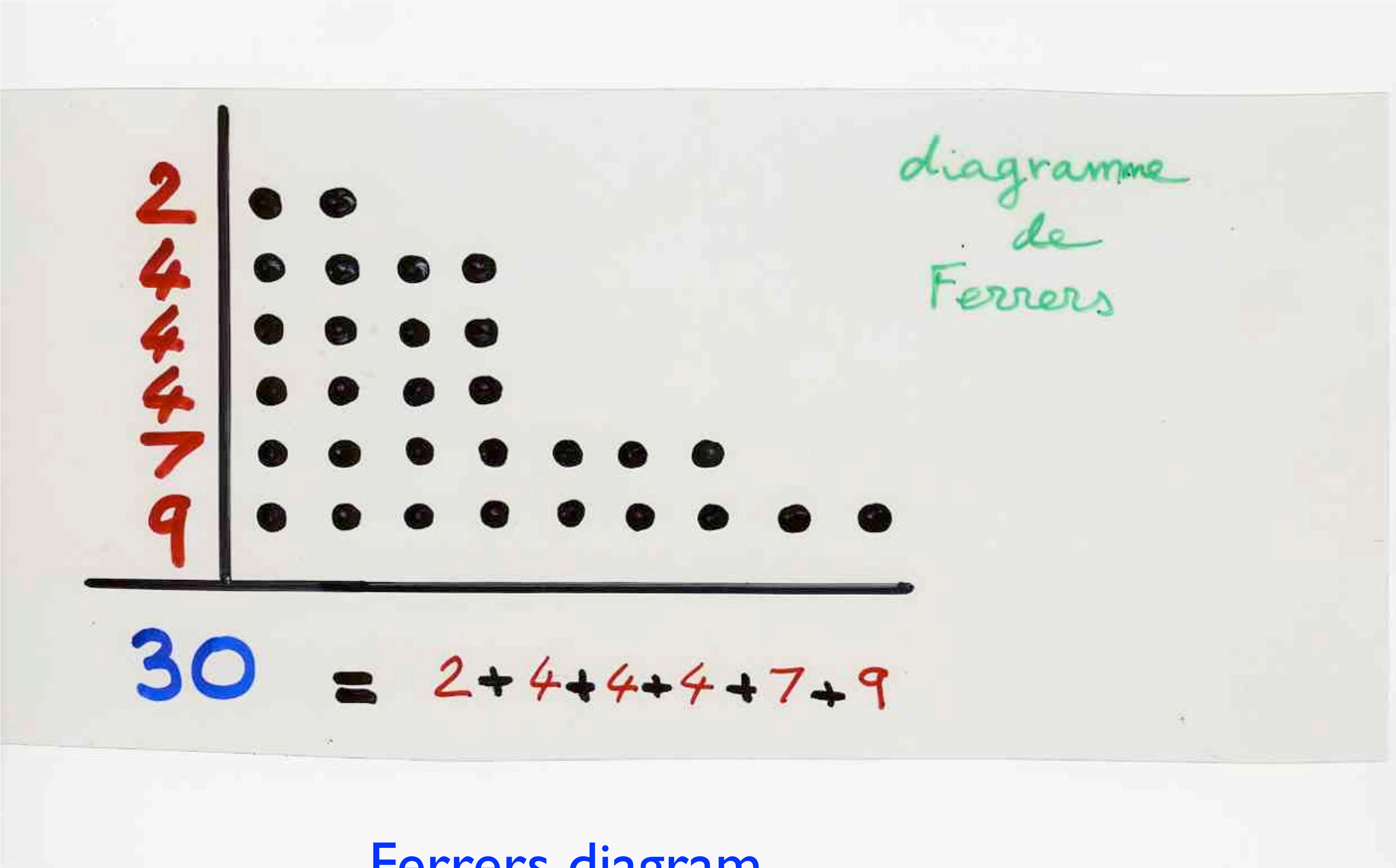
partition of an integer n

$$\lambda = (6, 6, 6, 5, 4, 4, 4, 4, 2, 2)$$

$$n = 43 = 6+6+6+5+4+4+4+4+2+2$$



Ferrers diagram



Ferrers diagram

$$1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

série génératrice

des partitions (d'entiers)

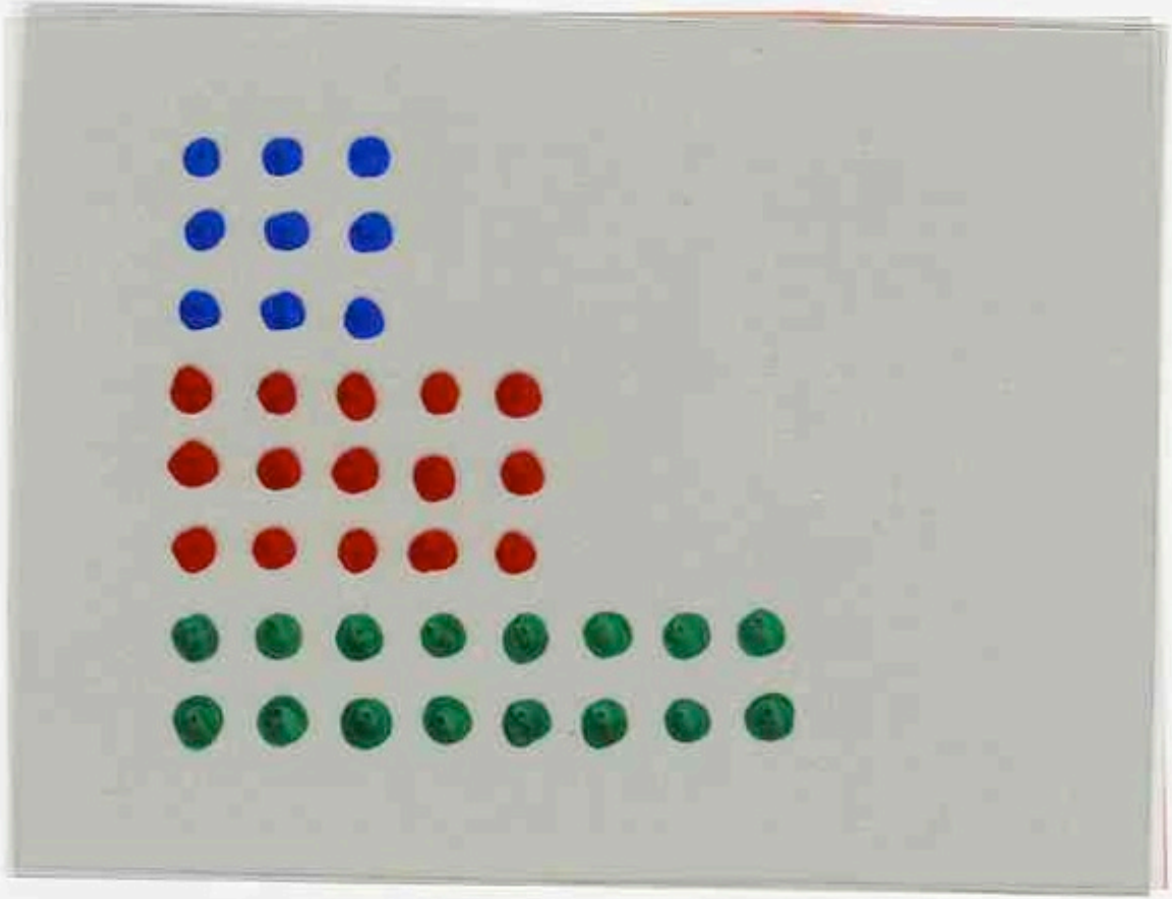
$$\sum_{n \geqslant 0} a_n q^n$$

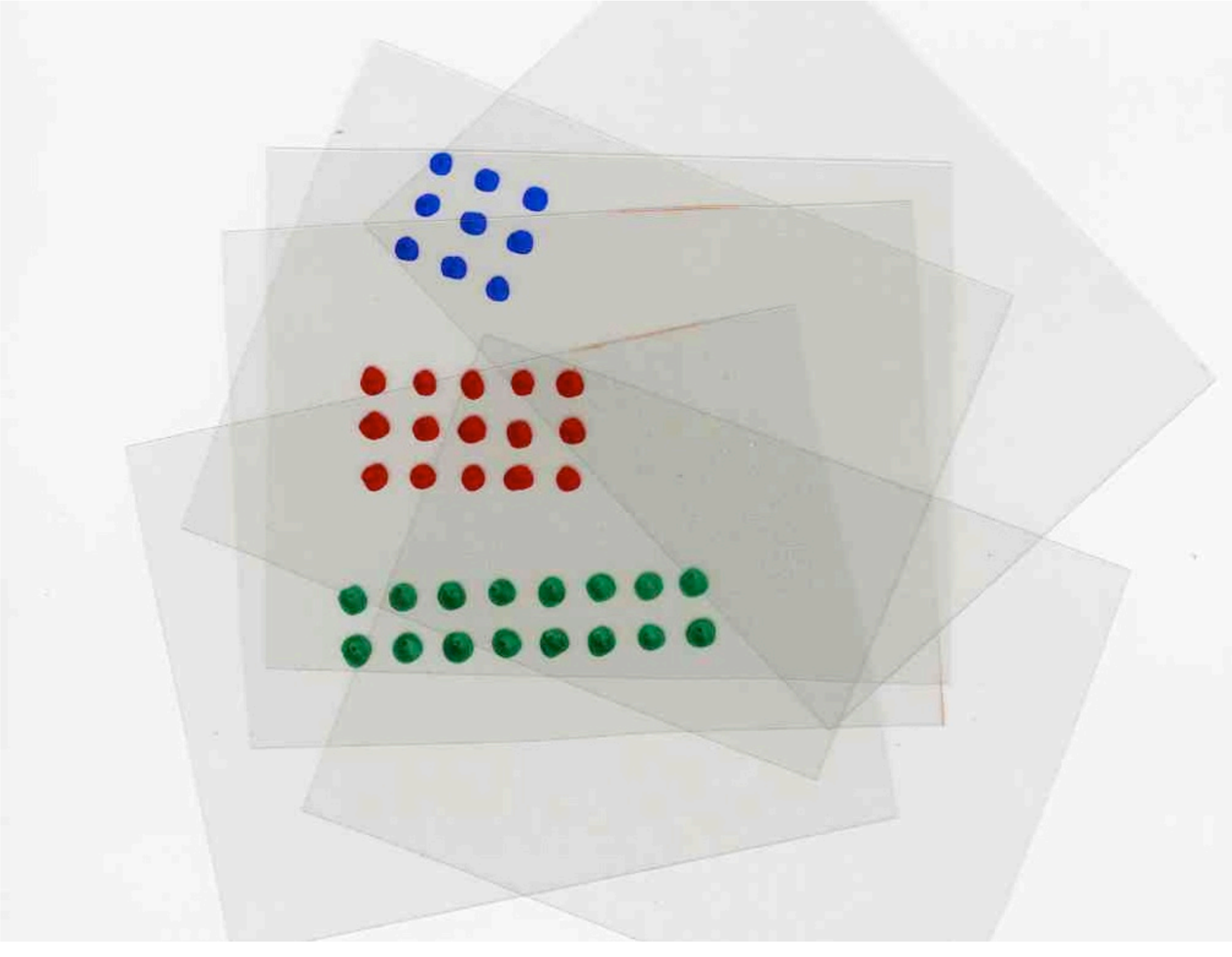
generating function

for (integer) partitions

$\underbrace{\bullet\bullet\bullet\bullet\bullet}_{i}$ $\qquad$ q^{i}

$$\frac{1}{1-q^{i}}$$





$$\frac{1}{(1-q)(1-q^2)\cdot\ldots\ (1-q^m)}$$

$$\frac{1}{(1-q)(1-q^2)\cdot\dots\,(1-q^m)}$$

$$\prod_{i\geq 1}\frac{1}{(1-q^i)}$$

exercise

$$\sum_{n\geq 0} p(n,I)\, q^n = \prod_{i\in I} \frac{1}{1-q^i}$$

partitions
parts $\lambda_j \in I$

D- partition $\qquad \lambda_i - \lambda_{i+1} \geq 2$
$\lambda = (\lambda_1, \ldots, \lambda_k) \qquad (1 \leq i < k)$

série génératrice
des D- partitions

$$\sum_{m\geq 0} \frac{q^{(m^2)}}{(1-q)(1-q^2)\cdots(1-q^m)}$$

Rogers - Ramanujan identities

$$R_I \quad \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \bmod 5}} \frac{1}{(1-q^i)}$$

D-partitions

$$\begin{cases} 9 \\ 8+1 \\ 7+2 \\ 6+3 \\ 5+3+1 \end{cases}$$

partitions

parts $\equiv 1,4 \bmod 5$

$$\begin{cases} 9 \\ 4+4+1 \\ 6+1+1+1 \\ 4+1+1+1+1+1 \end{cases} \begin{cases} 1+\dots+1 \end{cases}$$

Rogers - Ramanujan identities

$$R_{I} \quad \sum_{n\geqslant 0} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 1,4 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

$$R_{II} \quad \sum_{n\geqslant 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \mathrm{mod}\ 5}} \frac{1}{(1-q^i)}$$

R_{II}

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{\substack{i\equiv 2,3 \\ \text{mod } 5}} \frac{1}{(1-q^i)}$$

D-partitions

parts $\neq 1$

Partitions

parts $\equiv 2, 3$ mod 5

$$\begin{cases} 7+2 \\ 6+3 \\ 9 \end{cases} \qquad \begin{cases} 2+2+2+3 \\ 3+3+3 \\ 7+2 \end{cases}$$

operations on combinatorial objects

formalisation

Operations on combinatorial objects

Def- class of valued combinatorial objects

$\alpha = (A, v)$ $\quad A$ finite or enumerable set

$v : A \to \mathbb{K}[x]$
valuation

(*) for w monomial of $\mathbb{K}[x]$,

let $A_w = \{\alpha \in A$, coeff of w in $v(\alpha)$ is $\neq 0\}$.

then for every monomial w,

A_w is finite

$v(\alpha)$ weight or valuation of α

$\{v(\alpha), \alpha \in A\}$ is summable

Def

$$f_A = \sum_{\alpha \in A} v(\alpha)$$

generating power series of objects $\alpha \in A$

weighted by v

ex: objects of size n $\mathcal{A}$

$X = \{t\}$ $v(\alpha) = t^n$

n is the size of α, $|\alpha|$

$a_n = \mathcal{A}_{t^n}$ (finite)

$=$ number of objects $\alpha \in \mathcal{A}$ of size n

$$f_{\mathcal{A}} = \sum a_n t^n$$

ex: more generally

$X = \{t\} \cup Y$ $v(\alpha) = w(\alpha)\, t^n$

in general $a_0 = 1$, only one *empty* object

ε with weight $v(\varepsilon) = 1$

$\alpha = (A, v_A)$ $\beta = (B, v_B)$

- **sum** $\alpha + \beta = \gamma$

$A \cap B = \emptyset$ $= (C, v_C)$

- $C = A \cup B$ (disjoint union)
- $v_C/A = v_A$ $v_C/B = v_B$

Lemma $f_\gamma = f_\alpha + f_\beta$

- product $\mathcal{A} \cdot \mathcal{B} = \mathcal{C}$

$$= (C, v_C)$$

- $C = A \times B$
- $(\alpha, \beta) \in C$

$$v_C(\alpha, \beta) = v_A(\alpha) v_B(\beta)$$

ex: size $|(\alpha, \beta)| = |\alpha| + |\beta|$

ex: binary tree

Lemma $f_{\mathcal{C}} = f_{\mathcal{A}} \cdot f_{\mathcal{B}}$

- sequence

$\mathcal{A} = (A, v_A)$ $\quad\quad$ $\mathcal{C} = (C, v_C)$

$$\mathcal{C} = \{\epsilon\} + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^n + \ldots$$

$$= \mathcal{A}^*$$

Lemma

$$f_{\mathcal{A}^*} = \frac{1}{1 - f_{\mathcal{A}}}$$

ex: Ferrers diagrams

binary trees

generating power series

power series algebra

operations on combinatorial objects

bijective combinatorics Dyck paths

operations on combinatorial objects

example: integers partitions

formalisation