Heaps of pieces

(with interactions in mathematics in physics)

Ch6 Heaps and algebraic graph theory

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Basic definitions and theorems:

Ch1 Commutations monoids and heaps of pieces: basic definitions

Ch2 Generating functions for heaps of pieces

Ch3 Heaps and paths, flow monoids, rearrangements

Some applications in classical mathematics:

Ch4 Heaps and linear algebra: bijective proofs of classical theorems

Ch5 Heaps and combinatorial theory of orthogonal polynomials
and continued fractions

Ch6 Heaps and algebraic graph theory

Some applications in theoretical physics:

Ch7 Directed animals and gas model in statistical physics,
Lorentzian triangulations in 2D quantum gravity

Ch8 Polyominoes, q-analogue and SOS model in physics

Applications to more advanced mathematics:

Ch9 Fully commutative class of words in Coxeter groups

[Representation theory in Lie algebras with operators on heaps]

classes at two levels

algebraic graph theory

combinatorial properties of graphs $\longleftrightarrow$ algebraic objects

ex: polynomials

linear algebra

N. Biggs "Algebraic graph Theory" (1974)

Connection with

- Statistical physics
- Knots theory
- Heaps of pieces

some **Polynomials** associated to **Graphs**

graph $G = (V, E) \longrightarrow$ polynomial $P(G\,;x)$

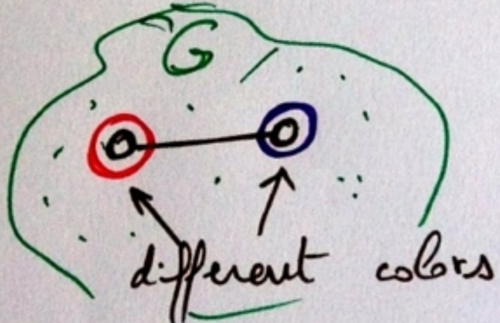
Characteristic polynomial of a graph G

A adjacency matrix

$$A = (a_{ij})_{1 \le i,j \le n} \qquad a_{ij} = \begin{cases} 1 & \overset{\text{edge}}{i \,\text{—}\, j} \\ 0 & i \quad j \end{cases}$$

chromatic polynomial

$\Gamma_G(\lambda)$ = number of ways coloring a graph with λ colors



chromatic number

$\nu(G)$ = smallest number ν such that $\Gamma(G;\nu) \neq 0$

zéros of $\Gamma(G;\lambda)$

graph $G = (V, E) \longrightarrow$ polynomial

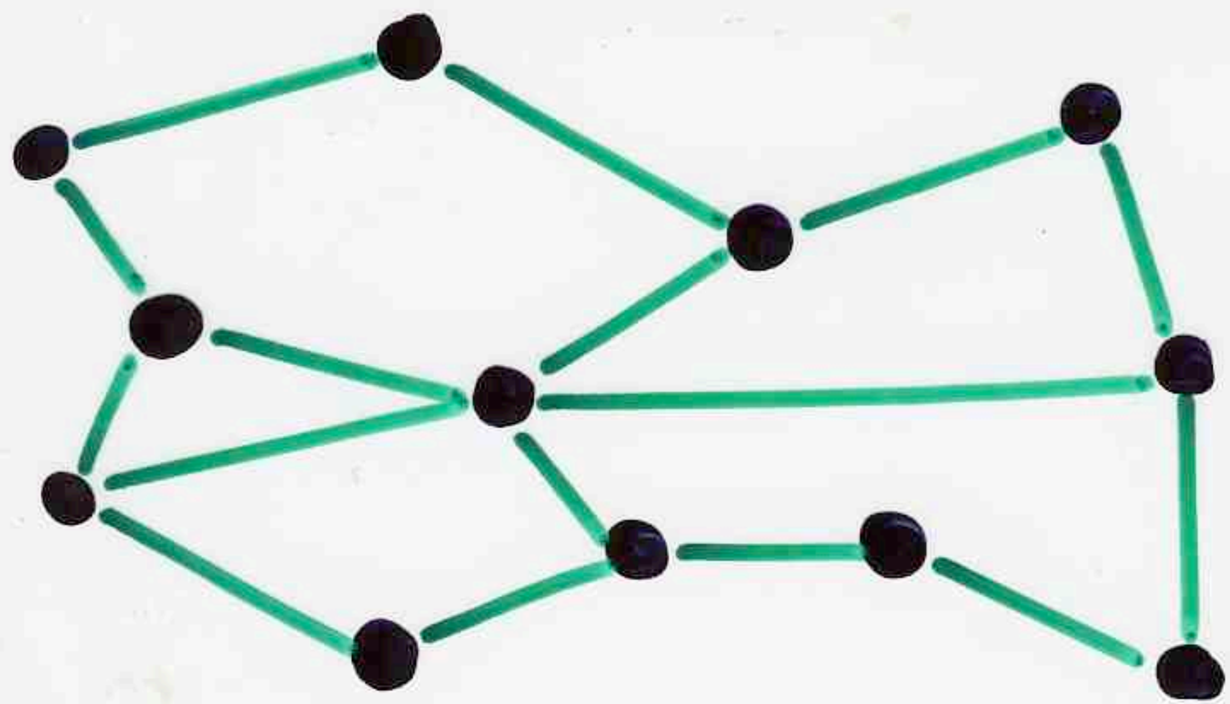
$P(G; x)$

$$\text{Tutte} = \sum_{T \text{ spaning trees}} x^{i(T)} y^{e(T)}$$

$$\text{Whitney} = \sum_{\substack{H \subseteq E \\ \text{spaning subgraph}}} x^{r(H)} y^{cr(H)}$$

Matching polynomial $\qquad G$ graph

$$C_G(x) = \sum_{\substack{\alpha \\ \text{matching}}} (-1)^{|\alpha|} x^{n-2|\alpha|}$$



Matching polynomial G graph

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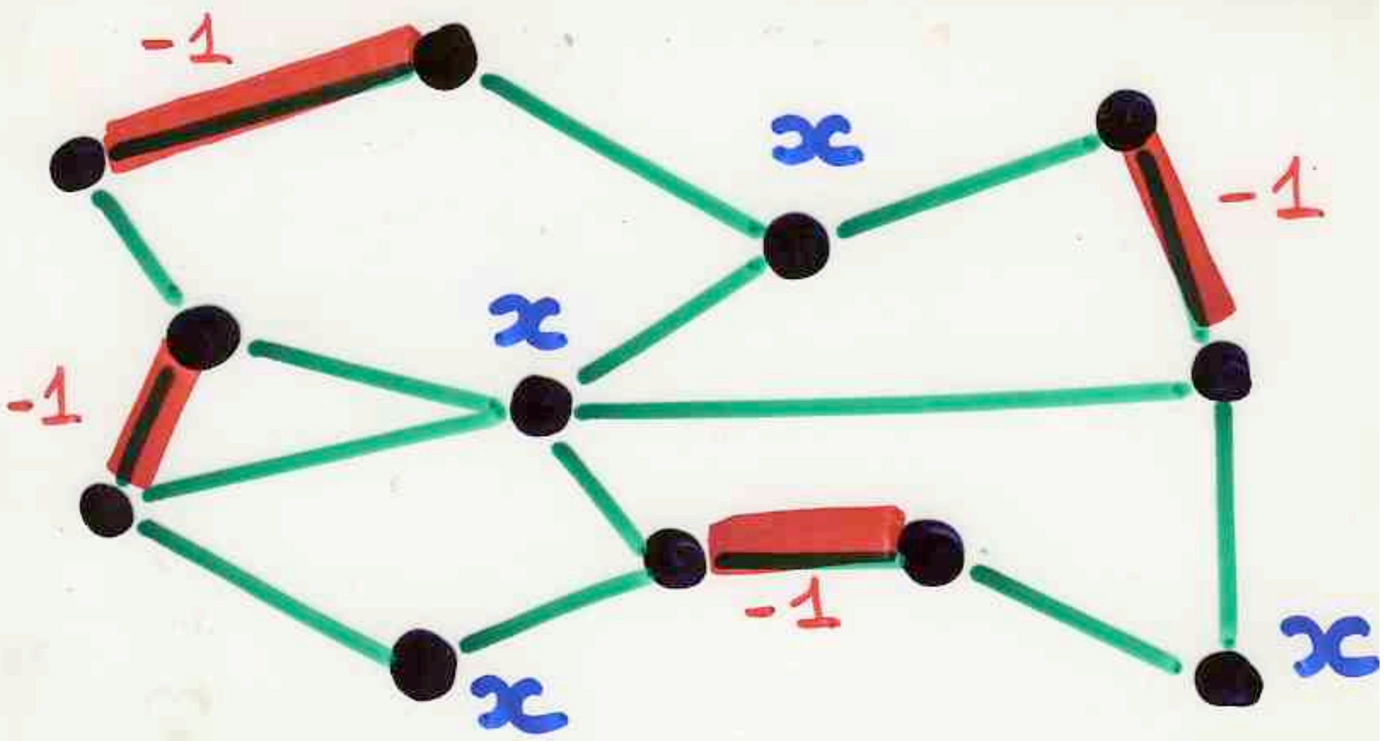
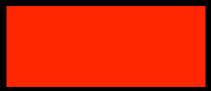
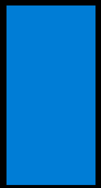


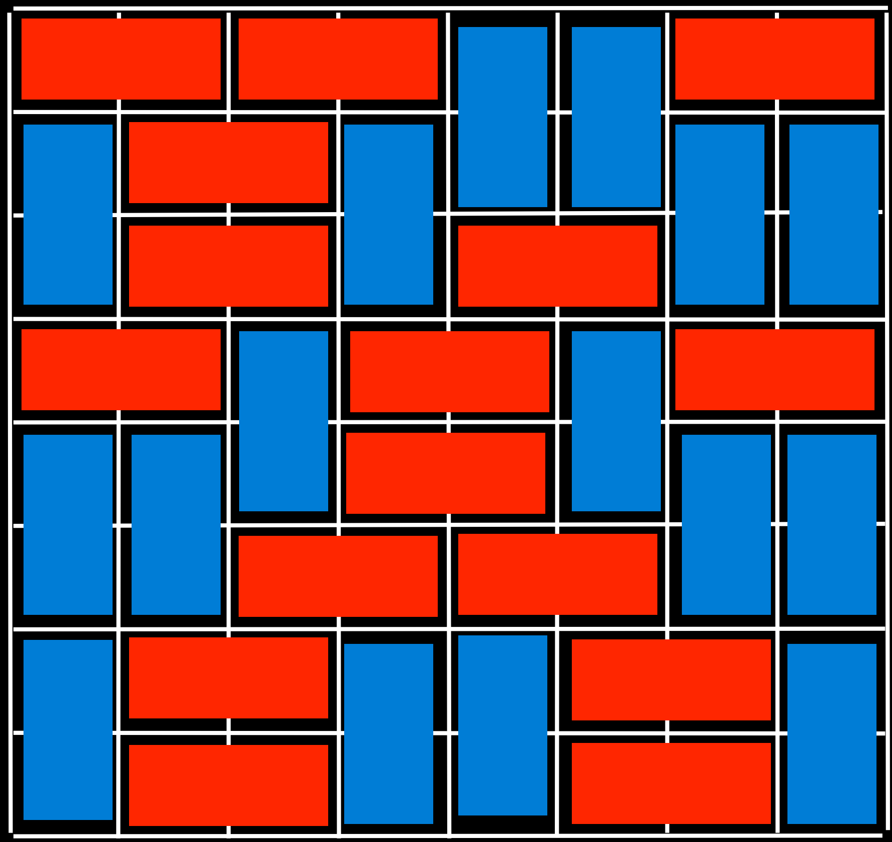
- number of perfect matchings
  = constant term in the matching polynomial

→ • Pfaffian, determinant (for planar graph)
  • Ising model (magnetism ...)

Tilings of a chessboard with dimers

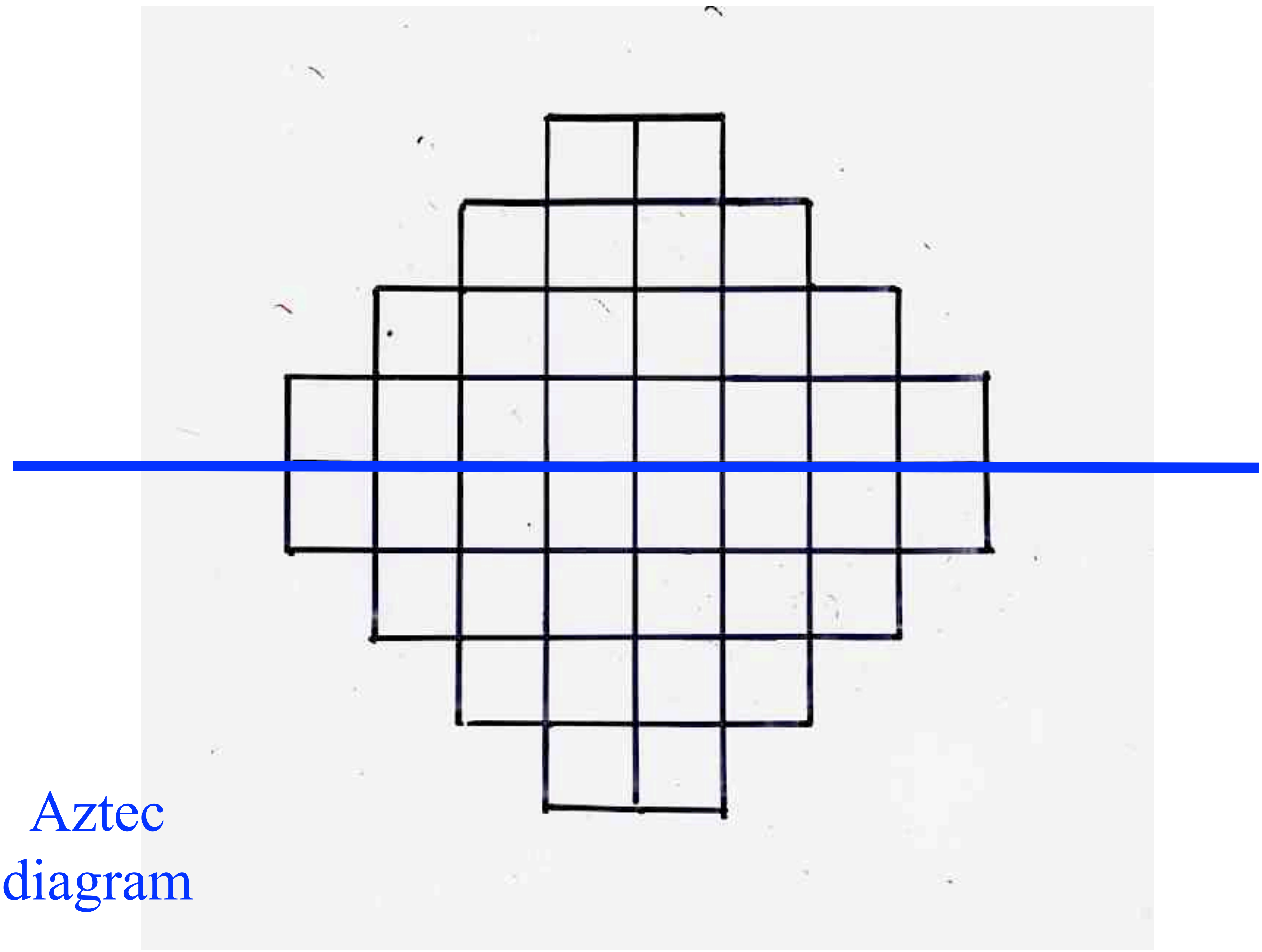






the number of tilings for the 8 x 8 chessboard

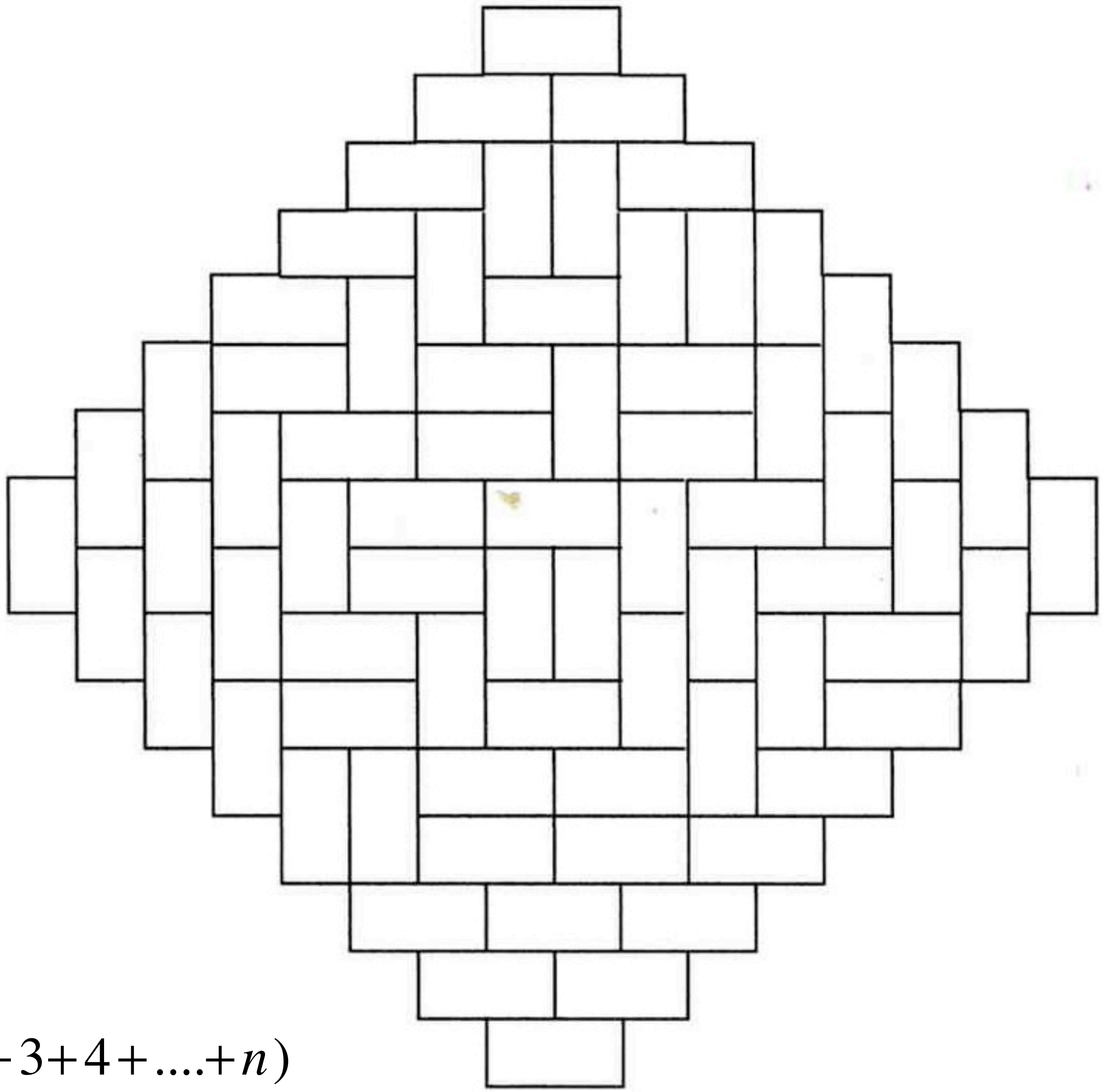
= 12 988 816



Aztec
diagram

the number
of
tilings of the
Aztec diagram
with dimers
is

$$2^{\frac{n(n+1)}{2}}$$



$$2^{(1+2+3+4+....+n)}$$

the number of tilings with dimers a $m \times n$ rectangle is

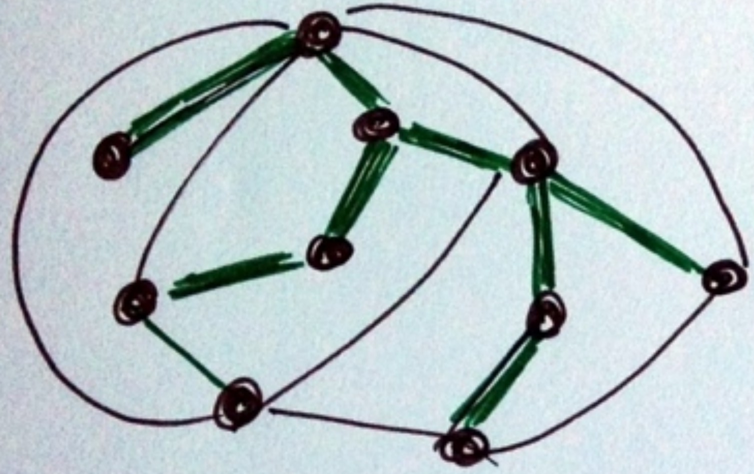
$$\prod_{i=1}^{m/2} \prod_{j=1}^{n/2} \left(4 \cos^2 \frac{i\pi}{m+1} + 4 \cos^2 \frac{j\pi}{n+1} \right)$$

Kasteleyn (1961)

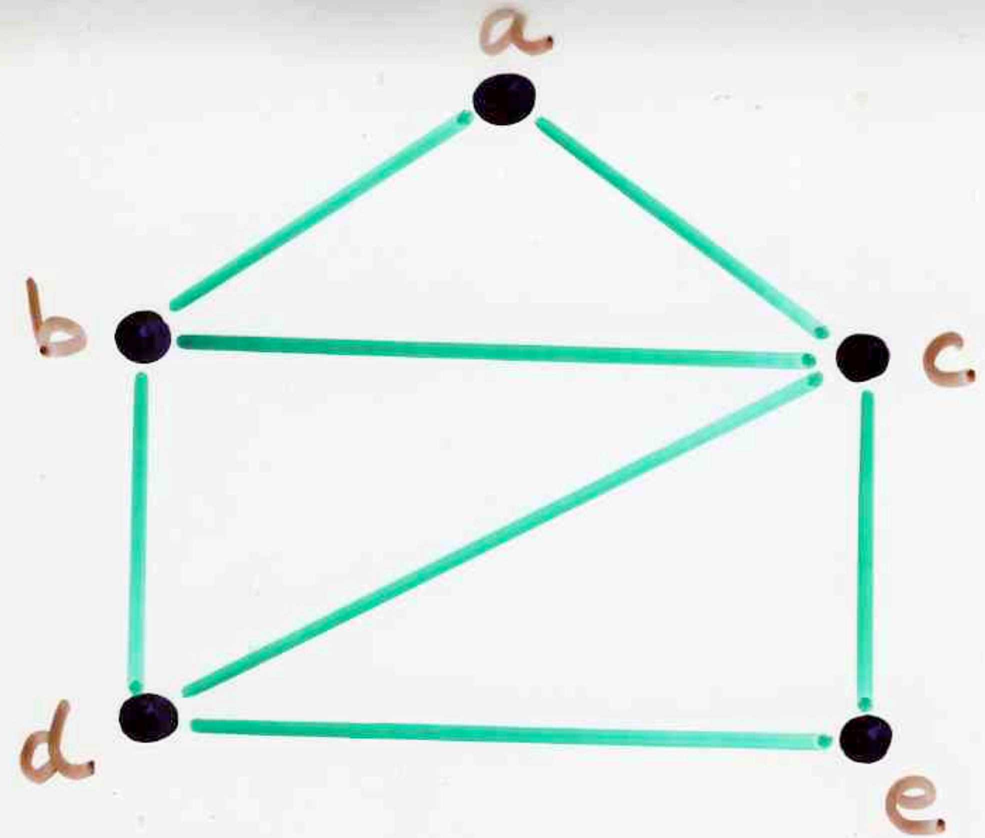
it is an integer !

for a chessboard m=8, n=8: 12 988 816

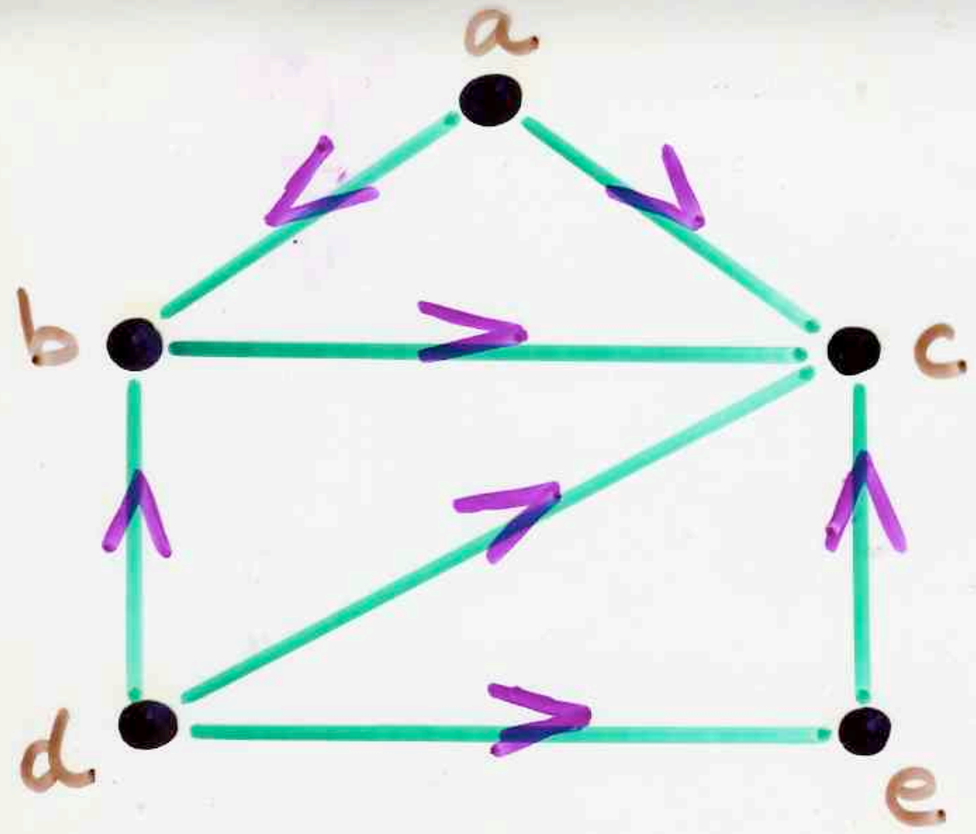
- number of spanning trees



- number of acyclique orientations of G


a
b
c
d
e

Orientation acyclique



Characteristic polynomial

$\left\{\begin{array}{l} \text{- eigenvalues } \lambda \\ \text{- eigenvectors } V \end{array}\right.$ of A

$$AV = \lambda V$$

λ zero of the polynomial

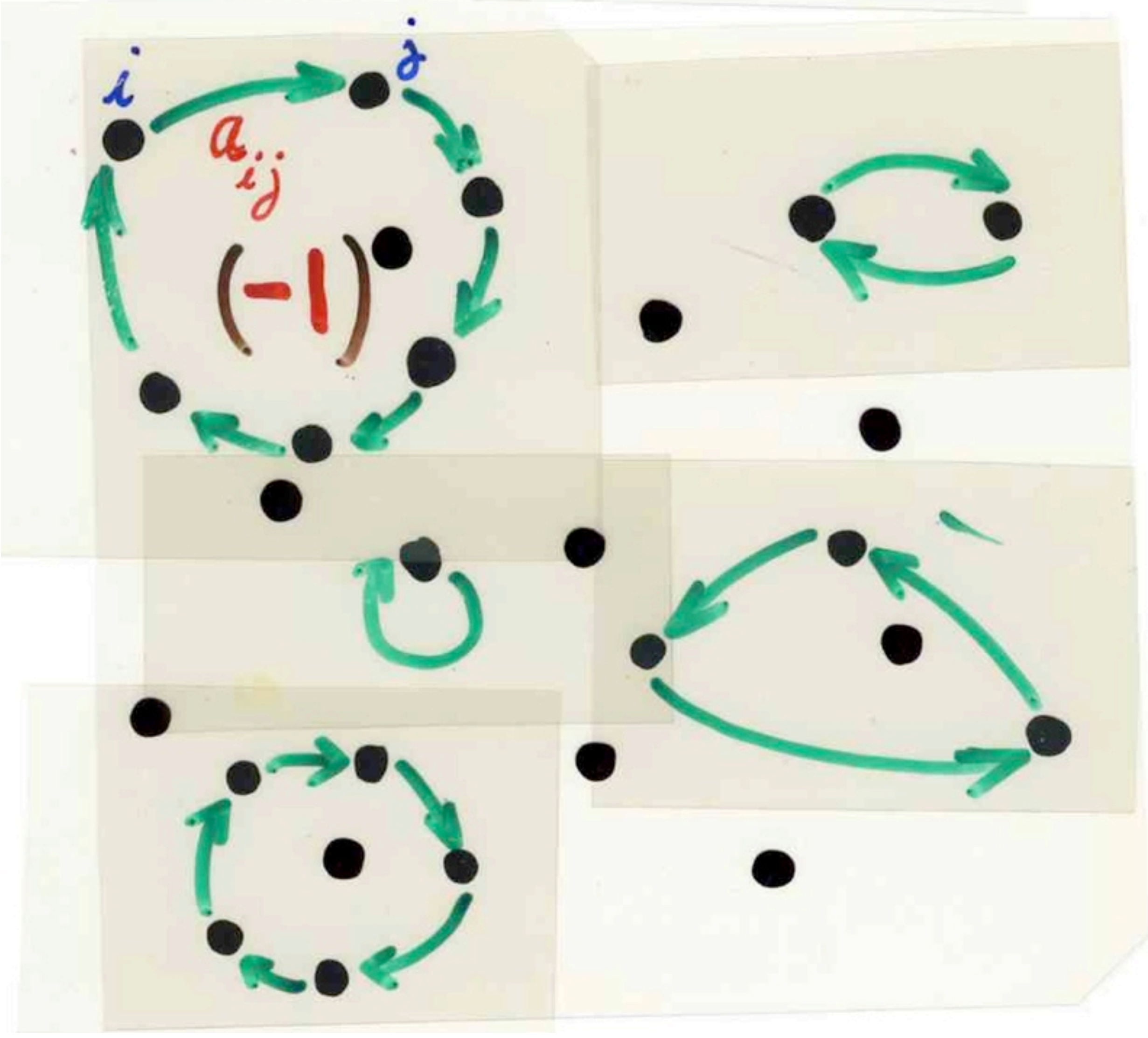
$$\chi(x) = \det(\lambda I - A)$$

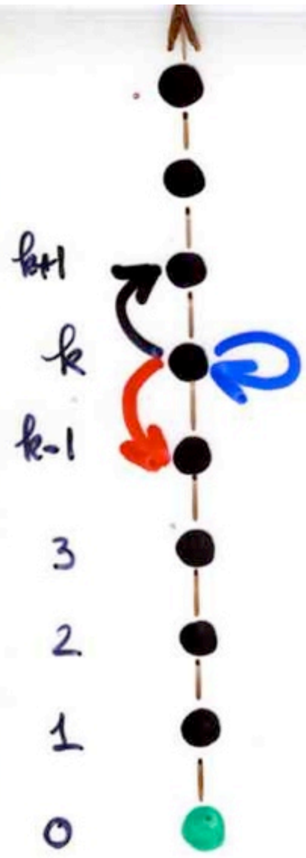
characteristic polynomial of G

Prop - The zeros of the characteristic polynomial of a graph G are real numbers

- the eigenvalues of a symmetric matrix A are real numbers

$$\det(A_n - xI)$$



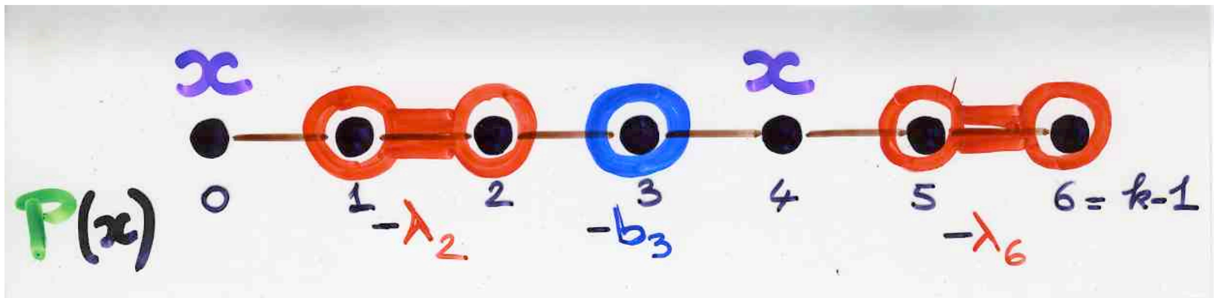


Tridiagonal matrix

$$
A = \begin{bmatrix}
b_0 & 1 & & & & \\
\lambda_1 & b_1 & 1 & & & \\
& \lambda_2 & b_2 & 1 & & \\
& & \lambda_3 & b_3 & 1 & \\
& & & \lambda_4 & \ddots & \ddots \\
& & & & \ddots & \ddots
\end{bmatrix}
$$

$$P_k(x) = \sum_{T} (-1)^{|T|}\, v(T)\, x^{pi(T)}$$

trivial heap
monomers – dimers
over $[0, k-1]$



<u>Prop.</u>

If $\{\lambda_k\}_{k\geq 1}$ $\quad \lambda_k \geq 0$

then the zeros of the orthogonal polynomial related to $\{b_k\}_{k\geq 0}$, $\{\lambda_k\}_{k\geq 1}$ are real numbers

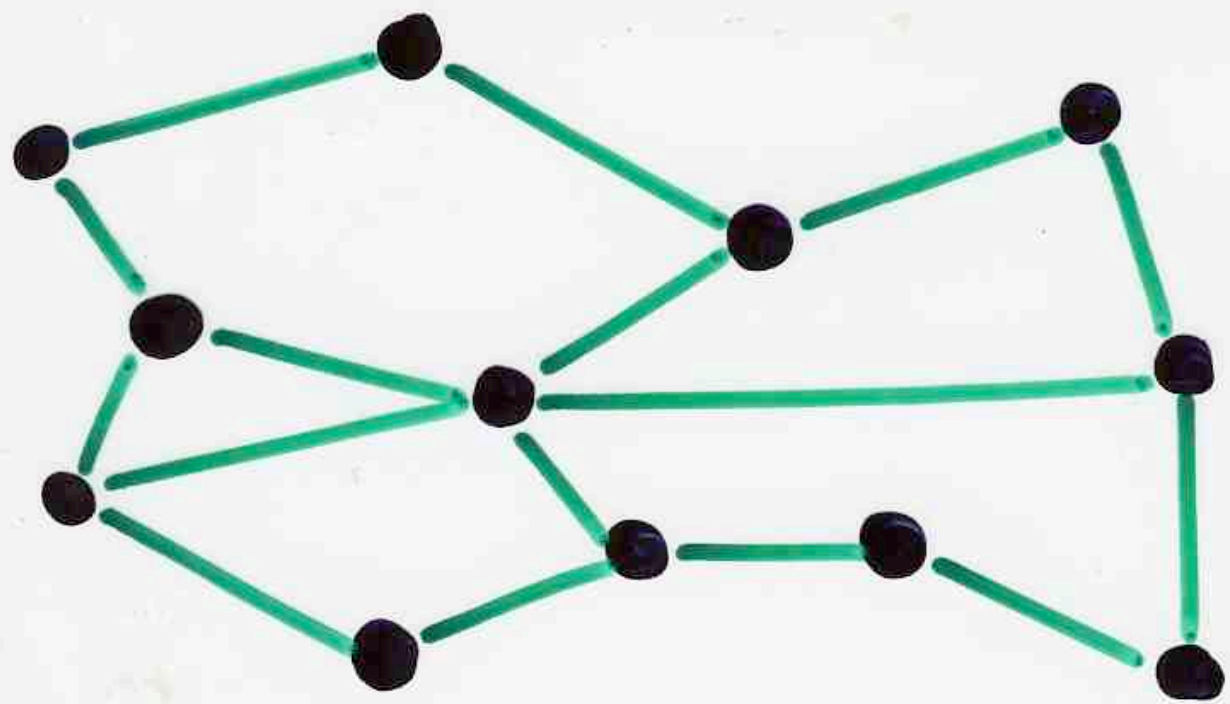
$$P_{k+1}(x) = (x - b_k) - \lambda_k P_{k-1}(x)$$

$$A^{\#} = \begin{bmatrix} b_0 & \sqrt{\lambda_1} & & & \\ \sqrt{\lambda_1} & b_1 & \sqrt{\lambda_2} & & \\ & \sqrt{\lambda_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\lambda_k} \\ & & & \sqrt{\lambda_k} & b_{k+1} \end{bmatrix}$$

matching polynomial

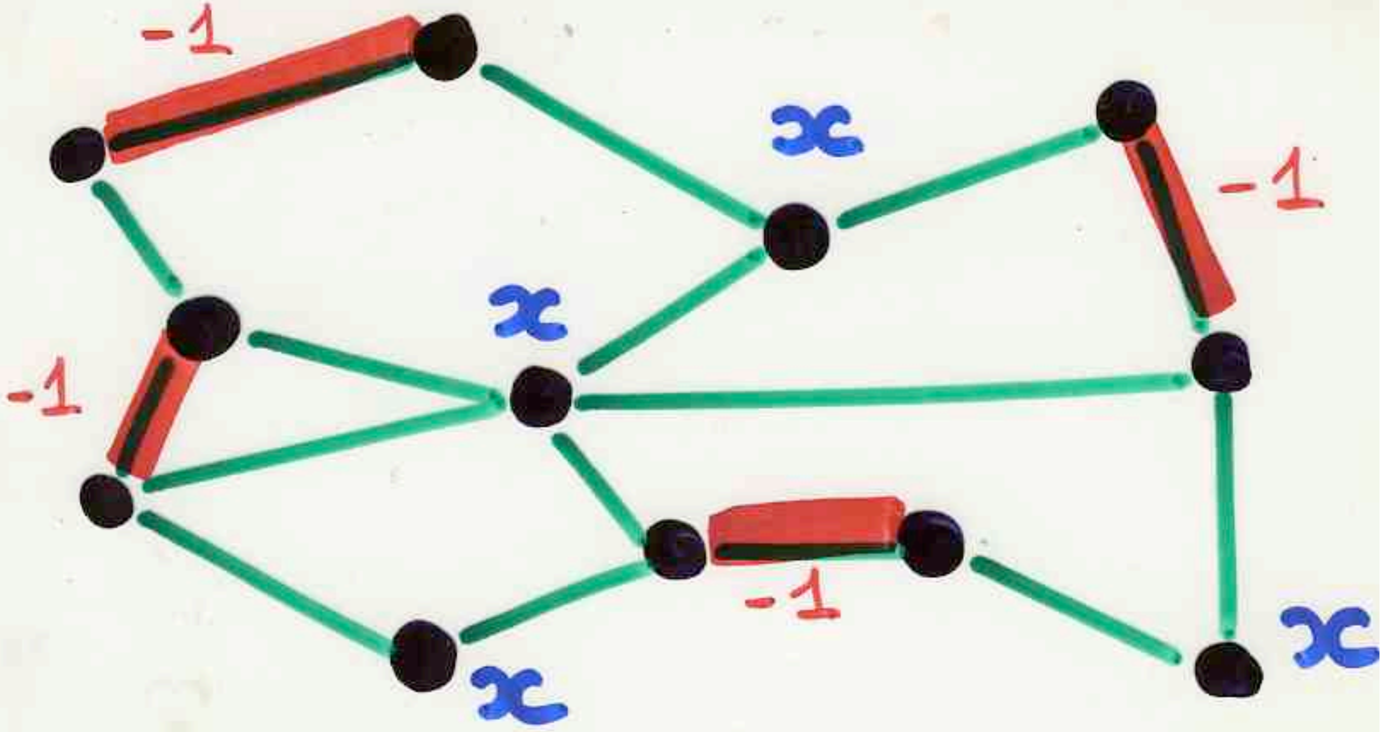
Matching polynomial $\quad G$ graph

$$C_G(x) = \sum_{\substack{\alpha \\ \text{matching}}} (-1)^{|\alpha|} x^{n-2|\alpha|}$$



Matching polynomial G graph

$$C_G(x) = \sum_{\substack{\alpha \\ \text{matching}}} (-1)^{|\alpha|} x^{n-2|\alpha|}$$

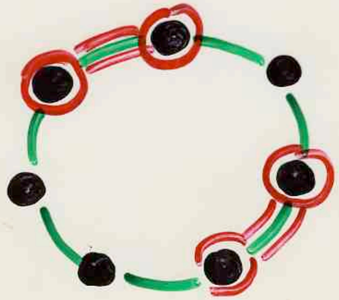


$$F_n(x) = \sum_{k=1}^{n} (-1)^k a_{n,k}\, x^{n-2k}$$

$$U_n(x) = F_n(2x)$$

$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

$$\cos(n\theta) = T_n(\cos\theta)$$



Tchebycheff

1st $\binom{2n}{n}$

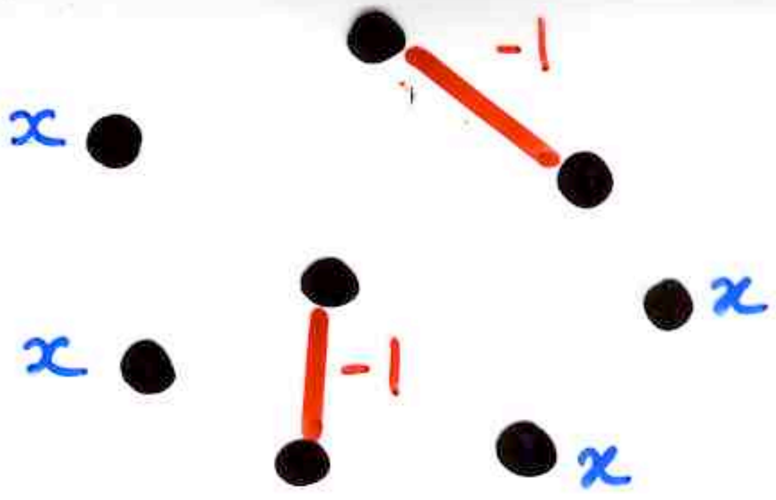
2nd C_n Catalan

ex: Hermite

$$H_n(x) = \sum_{\substack{\text{matching } \gamma \\ \text{of } K_n}} (-1)^{|\gamma|} x^{\text{fix}(\gamma)}$$

$\mu_{2n+1} = 0$ $\qquad$ $\mu_{2n} = 1 \times 3 \times \dots \times (2n-1)$

number of perfect matchings of K_{2n}



matching

Prop. For every graph G the zeros of the matching polynomial $C(G;x)$ are real numbers

$$\frac{1}{C^{*}_{G}(x)} = \sum_{E} x^{|E|}$$

heap of dimers over G

Prop

If G is a tree, then

$$C(G;x) = \chi(x)$$

$$\det(xI - A)$$

matching polynomial

characteristic polynomial

$$T_u(G)$$

Tree-like path

on a graph $G = (P, \mathcal{E})$

Def- $\omega \longrightarrow (\eta, F)$ (bijection ch III)

η: self-avoiding path

F: cycles heap on G

tree-like ("arborescent") iff all cycles of F have length 2

Bijection

$$\text{Paths}_{u \rightsquigarrow v}\ \omega \longrightarrow (\eta, E)$$

- η self-avoiding path going from u to v
- E heap of cycles, $\pi(\alpha)$, $\alpha \in \max(E)$ intersects η

$\omega = (s_0 = u, \dots, s_n = v)$ path on B $u \rightsquigarrow v$

$$\omega \longrightarrow (\eta\,;\ \{\gamma_1, \dots, \gamma_{r_n}\})$$

self-avoiding path $u \rightsquigarrow v$ ("coupe")

sequence of cycles

for $T = 0, 1, \dots, n$, $\begin{cases} \mathrm{Coupe}_T(\omega) : \text{self-avoiding path} \\ \mathrm{Suite}_T(\omega) : \text{cycles sequence} \end{cases}$

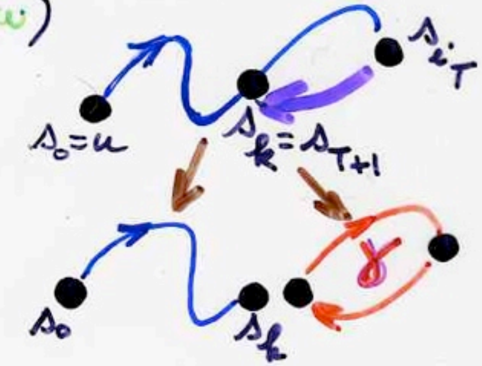
- $\mathrm{Coupe}_0(\omega) = (s_0)$ $\qquad$ $\mathrm{Suite}_0(\omega) = \emptyset$

- $\begin{cases} \mathrm{Coupe}_T(\omega) = (s_0, \dots, s_{i_T}) \\ \mathrm{Suite}_T(\omega) = (\gamma_1, \dots, \gamma_{r_T}) \end{cases}$

(i) if $s_{T+1} \notin \mathrm{Coupe}_T(\omega)$

$\mathrm{Coupe}_{T+1}(\omega) = (s_0, \dots, s_{i_T}, s_{T+1})$ $\qquad$ $\mathrm{Suite}_{T+1}(\omega) = \mathrm{Suite}_T(\omega)$

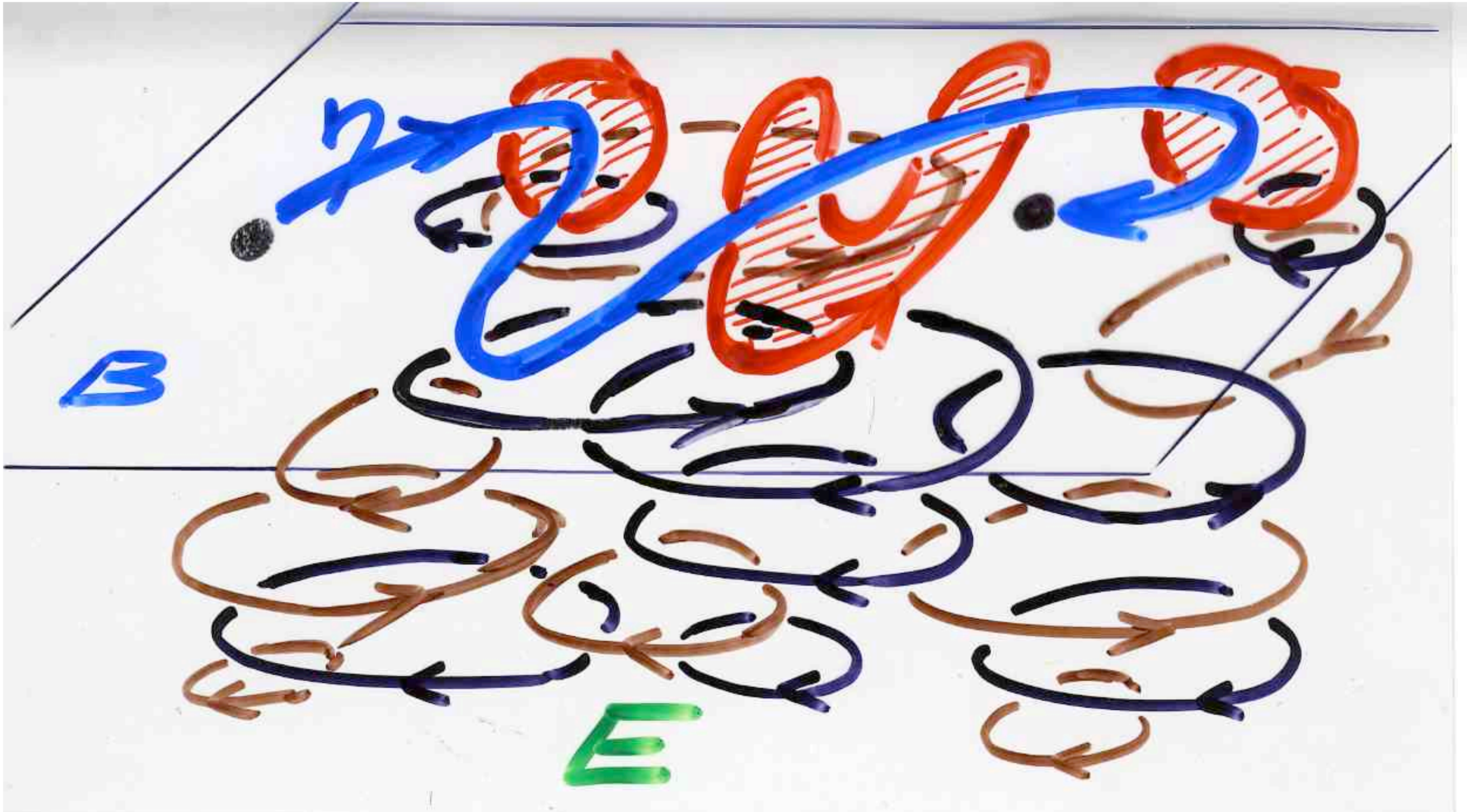
(ii) if $s_{T+1} = s_k \in \mathrm{Coupe}_T(\omega)$



$\begin{cases} \mathrm{Coupe}_{T+1}(\omega) = (s_0, \dots, s_k) \\ \mathrm{Suite}_{T+1}(\omega) = (\gamma_1, \dots, \gamma_r, \gamma) \end{cases}$

- $\mathrm{Coupe}(\omega) = \mathrm{Coupe}_n(\omega)$

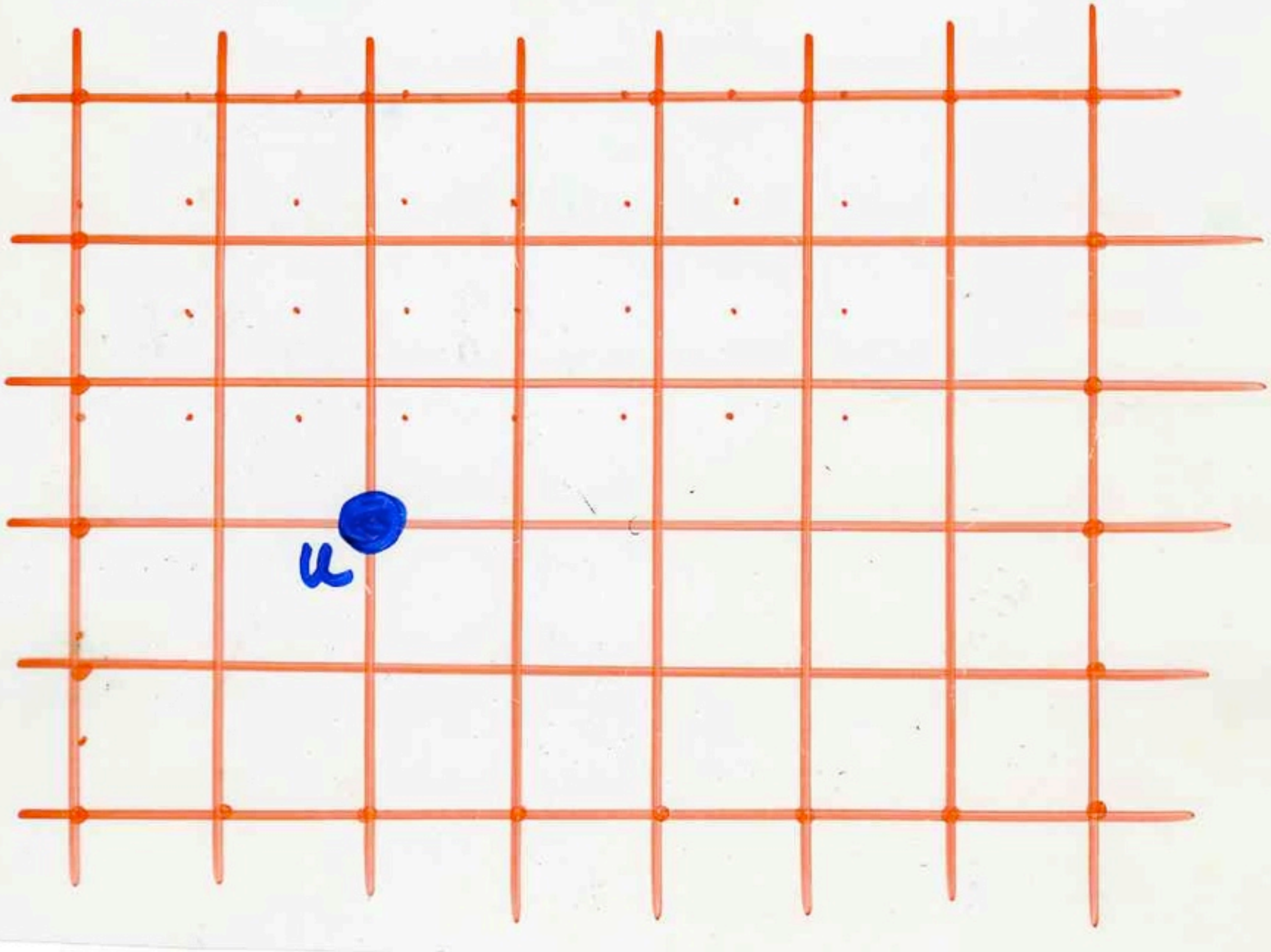
$\{\gamma_1, \dots, \gamma_{r_n}\} = \mathrm{Suite}(\omega) = \mathrm{Suite}_n(\omega)$

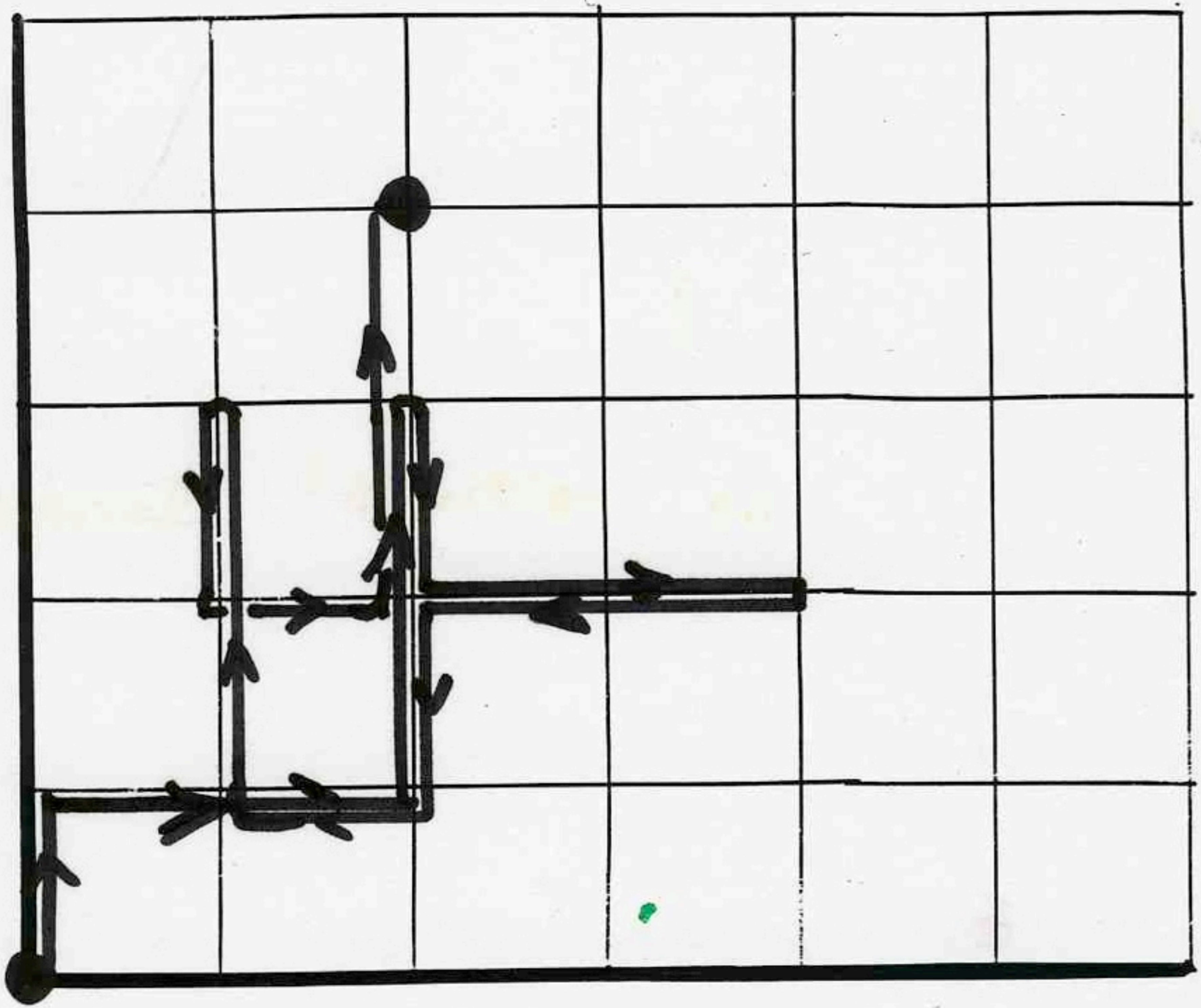


B
E

Tree-like paths on a graph G

Godsil (1981)

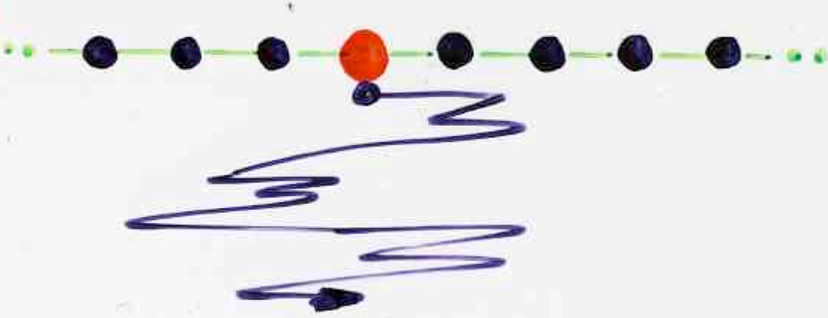




Particular cases.

- **Dyck** path

- bilateral **Dyck** paths

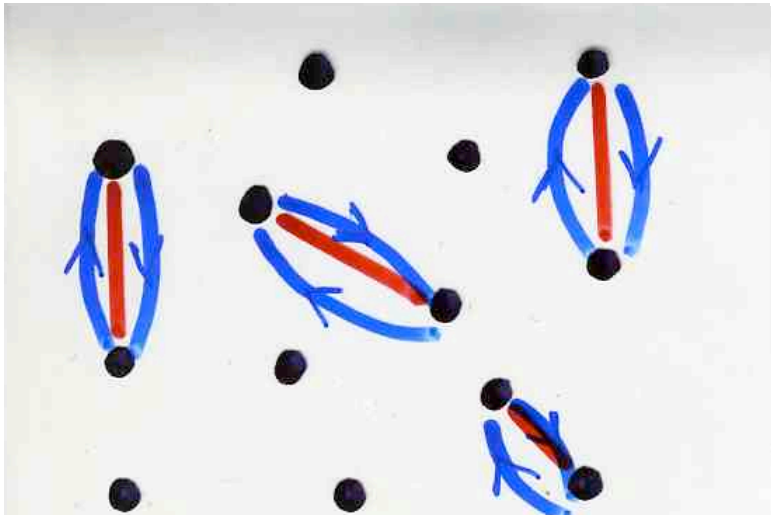


$$\frac{N}{D}$$

$$\frac{C^*(G\setminus v\,;x)}{C^*(G\,;x)} = \text{g.f. } \frac{\text{pyramids}}{\text{dimers}},$$

$v \in$ maximal dimer

$$= \text{g.f tree-like paths on } G \quad v \rightsquigarrow v$$



Couplage

2) (Godsil)

tree-like paths in G $\xrightarrow{\text{bijection}}$ paths in tree $T_u(G)$

u

Lemma. G, u

There exist a tree T, r root

ω $\Longleftrightarrow$ η

tree-like on G — path on T

$u \bullet \rightsquigarrow$ $\qquad$ $r \bullet \rightsquigarrow$

$|\omega| = |\eta|$

$T_u(G)$

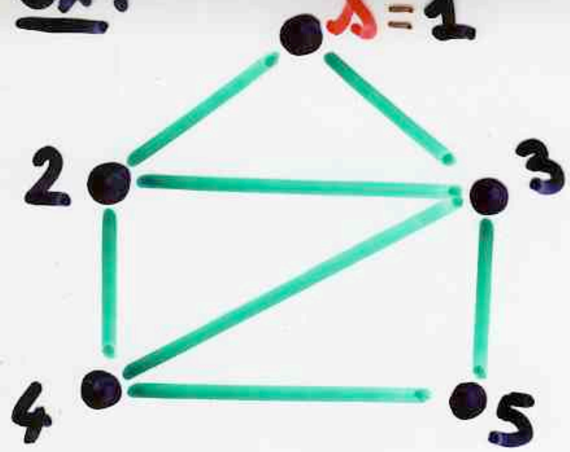
<u>vertices</u> $= \{$ self-avoiding paths η starting from u $\}$

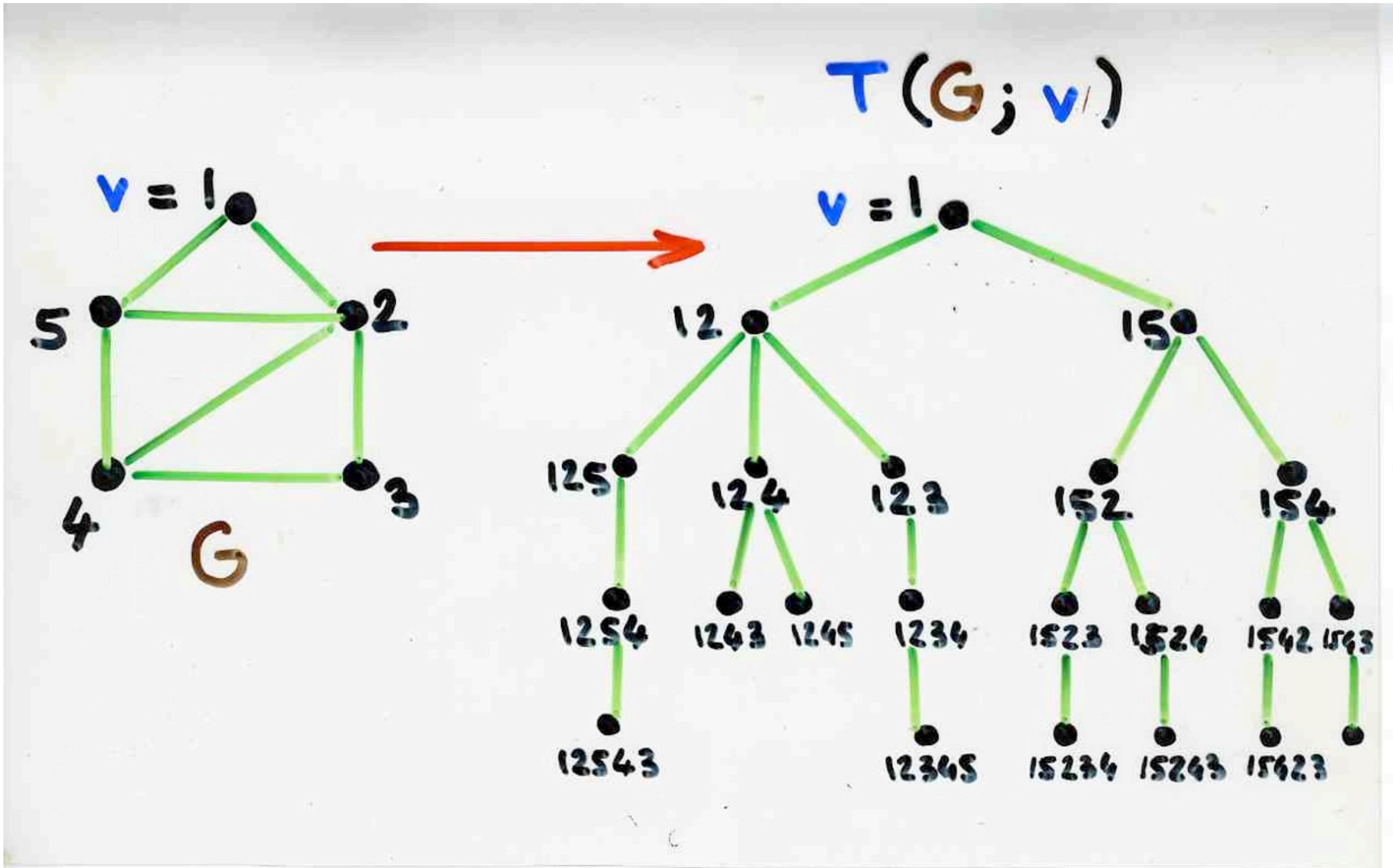
<u>edges</u> $=$ η — η'

iff $\quad \eta = (s_0, \dots, s_k)$

$\eta' = (s_0, \dots, s_k, s_{k+1})$

<u>ex</u>:





T(G; v)
v = 1
5
2
4
3
G
v = 1
12
15
125
124
123
152
154
1254
1243
1245
1234
1523
1524
1542
1543
12543
12345
15234
15243
15423

There exist tree T, root v

$$\frac{C^*(T \setminus v\, ; x)}{C^*(T\, ; x)} = \frac{C^*(G \setminus v\, ; x)}{C^*(G\, ; x)}$$

3) $C^*(T ; t) = \chi^*(T ; t)$ polynôme caractéristique de l'arbre T

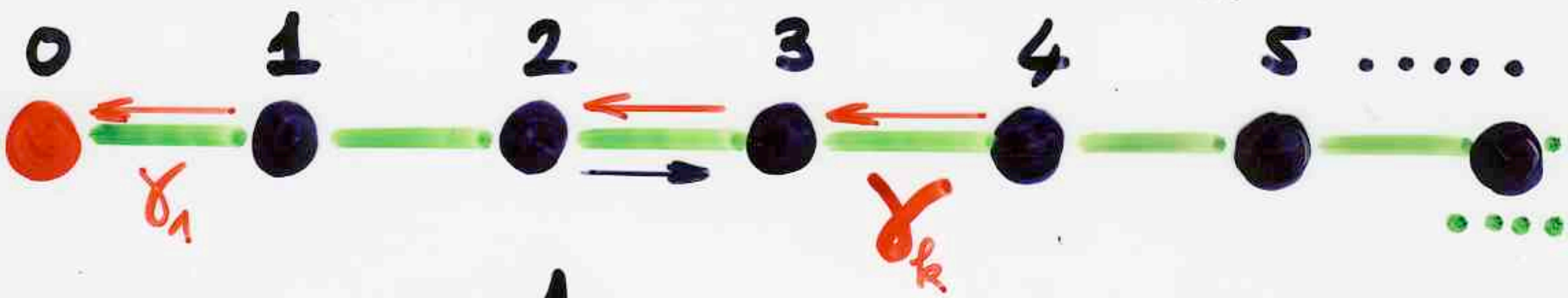
- valeurs propres d'une matrice symétrique
- zéros réels, par récurrence sur $|S|$

Prop. For every graph G the zeros of the matching polynomial $C(G; x)$ are real numbers

Heilman, Lieb (1972) Gruber, Kunz (1971)

Godsil, Gutman (1981)

Tree-like continued fraction

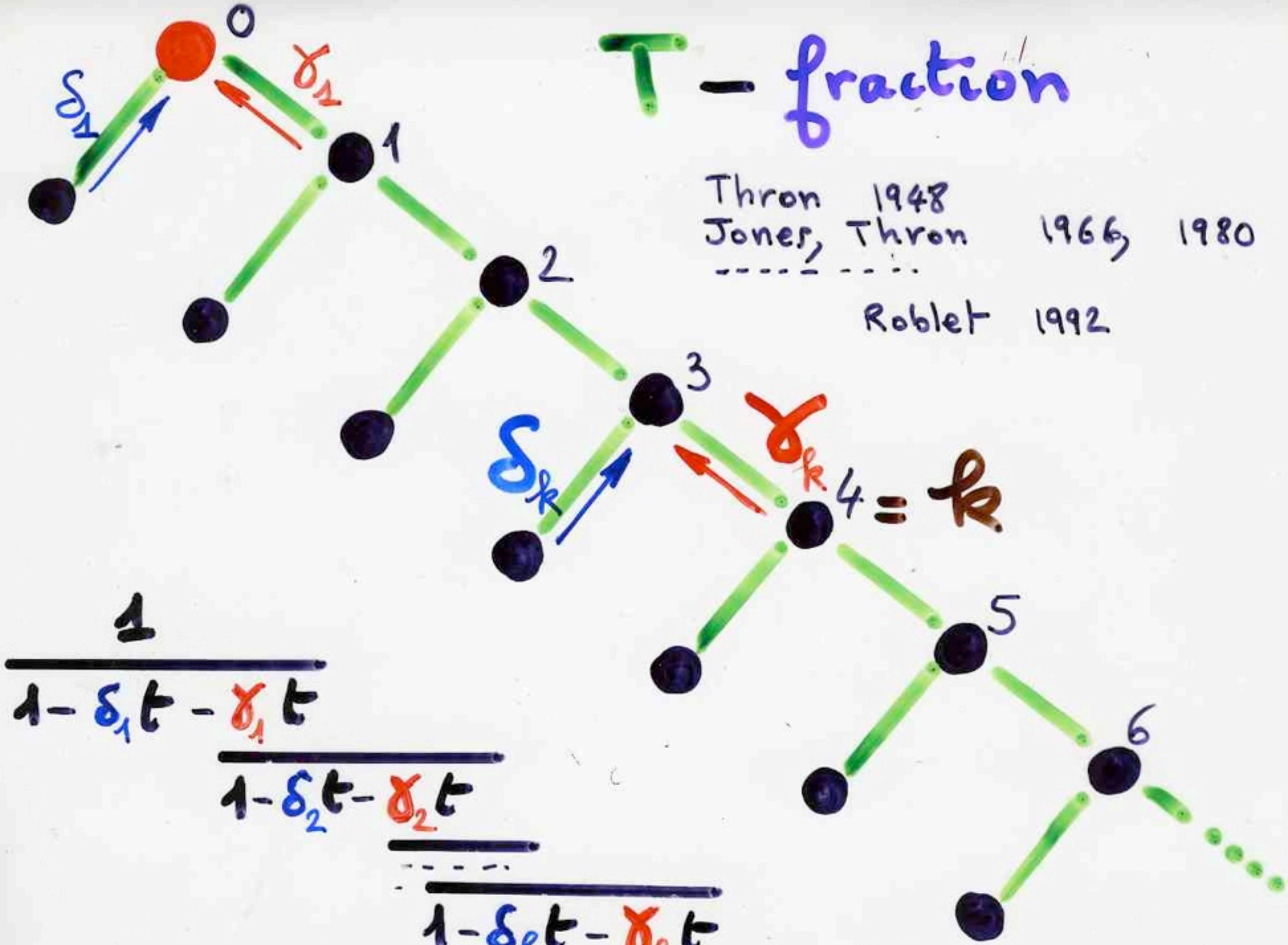


$$\cfrac{1}{1-\cfrac{\gamma_1 t}{1-\cfrac{\gamma_2 t}{\cfrac{\cdots\cdots\cdots}{1-\cfrac{\gamma_k t}{\cdots\cdots}}}}}$$

S - fraction

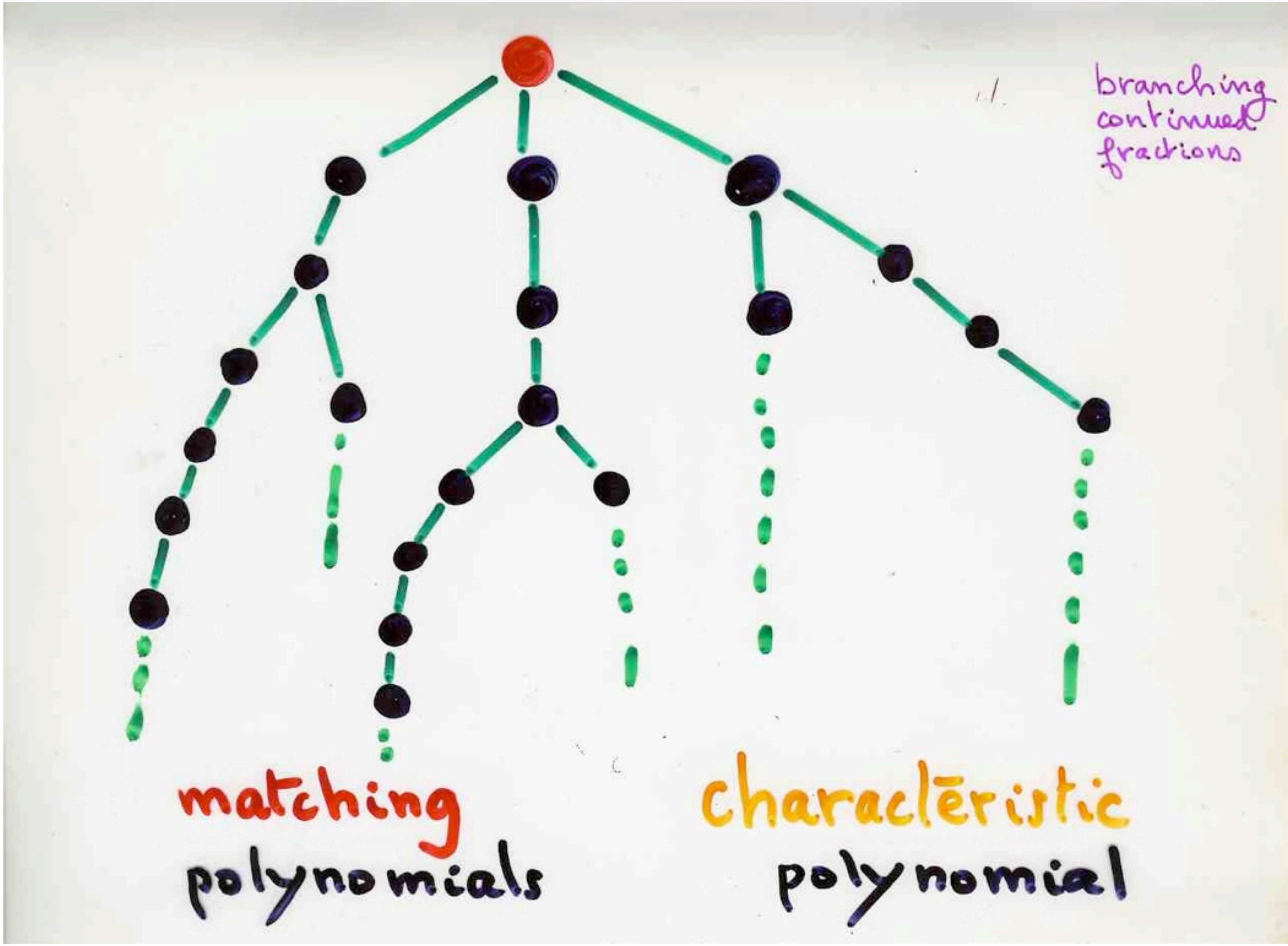
T - fraction

Thron 1948
Jones, Thron 1966, 1980
- - - - - - - -
Roblet 1992



$$\cfrac{1}{1-\delta_1 t-\cfrac{\gamma_1 t}{1-\delta_2 t-\cfrac{\gamma_2 t}{\cdots\cfrac{}{1-\delta_k t-\cfrac{\gamma_k t}{\cdots\cdots}}}}}$$

two-point Padé approximant
at 0 and ∞ .


branching continued fractions
matching polynomials
characteristic polynomial

Skorobogat'ko, Dronjuk,
Bobik, Ptašnik 1967
oscillating mechanical systems

Pustomel'nikov 1969
differential equation on a tree

Cori, Vauquelin
Françon, Arquès
planar maps

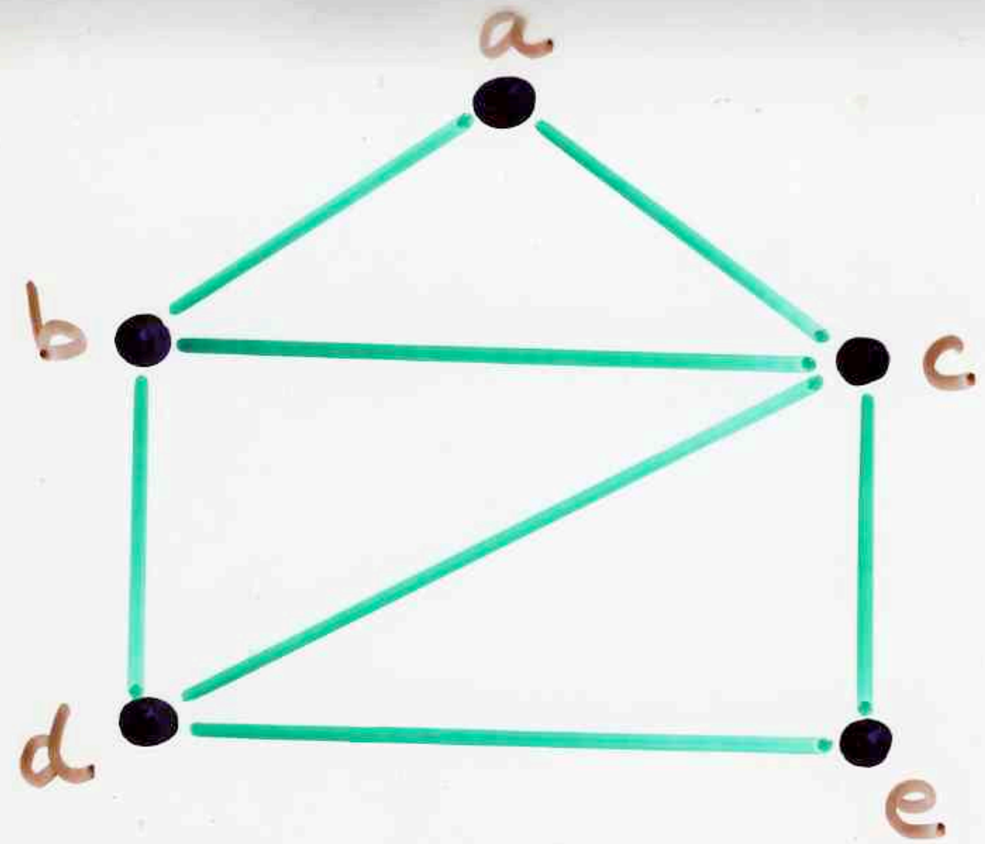
chromatic polynomial
and
acyclic orientations of a graph

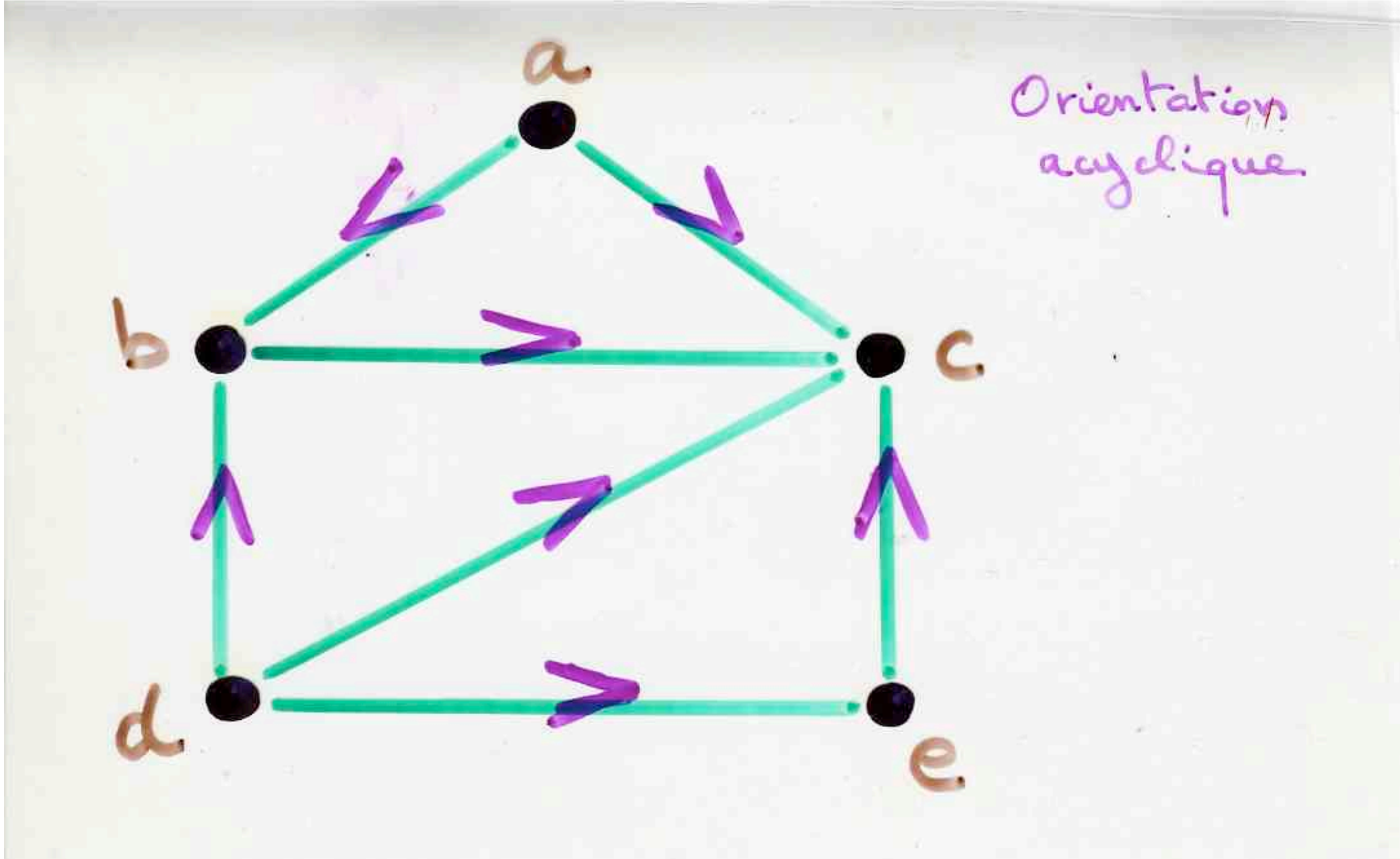
- **Nombre d'orientations acycliques**

(Stanley) $= (-1)^n \underbrace{\Gamma(-1)}_{\text{chromatique}} \quad \left(= T(2,0) \right)$

Stanley, 197.

Gessel, 1985


a
b
c
d
e


a
b
c
d
e
Orientation acyclique

Preuve (Gessel)

- $\Gamma(\lambda) = \sum_{1 \le k \le n} a_k(G)\ \lambda(\lambda-1)\cdots(\lambda-k+1)$

  ↓ nb de "partitions colorées" de G en k blocs

$$\Gamma(\lambda) = \sum_{k} b_k(G)\ \frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}$$

  ↓ nb de "partitions colorées ~~ordonnées~~"

- E multilinear heap: every tube has cardinality one
- $b_k(E)$ = number of "layers decompositions" of E ~~to~~ into k layers, i.e.:

$$E = T_1 \bullet \cdots \bullet T_k$$

T_i trivial heap (i^{th} layer)

- <u>Monoïde de commutations</u>

alphabet S

G graphe des non-commutations

– classe multilinéaire :

contient une et une seule fois chaque lettre de S

– V-factorisation d'une classe α

$\alpha = \beta_1 \cdots \beta_k$, β_i classe formée de lettres distinctes commutant 2 à 2

($\rightarrow$ stable de G)

$b_k(\alpha)$ = nb de V-factorisations de la classe α en k blocs

- $$\Gamma(\lambda) = \sum_{1 \leq k \leq n} \Bigg(\sum_{\substack{\alpha \\ \text{classe} \\ \text{multilinéaire}}} b_k(\alpha) \Bigg) \frac{1}{k!} \lambda(\lambda-1)\cdots(\lambda-k+1)$$

- "série chromatique"

$$K(t) = \sum_{k \geq 0} \Big(\sum_{\substack{\alpha \\ \text{classe}}} b_k(\alpha)\, v(\alpha) \Big) t^k$$

$v(\alpha)$ valuation

$$= \frac{1}{1 - t \Big(\sum_{\substack{\alpha \\ \text{classe} \\ \text{stable} \\ \neq \text{vide}}} v(\alpha) \Big)}$$

(d'après Cartier-Foata, inversion de Möbius)

$$K(-1) = \sum_{\substack{\alpha \\ \text{classe}}} (-1)^{|\alpha|}\, v(\alpha)$$

$$\Rightarrow \Gamma(-1) = \sum_{\substack{\alpha \\ \text{classe} \\ \text{multilinéaire}}} (-1)^{|\alpha|}$$

- Ainsi, $(-1)^n \Gamma(-1)$ est le nombre de classes multilinéaires.

- **Bijection**

Classes multilinéaires

$\updownarrow$

Orientations acycliques du graphe G

Remarque :

Prop – ("théorème" des 4 couleurs)

Tout graphe planaire peut être recouvert par un empilement de hauteur ≤ 4

Déf • E recouvre le graphe de concurrence $(P, \mathcal{C})$ ssi pour tout $s \in P$, le tube au-dessus de s est non vide

Déf • hauteur de l'empilement E = niveau maximum des pièces de E = nb de blocs de la forme normale Cartier-Foata