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HEAPS OF PIECES, I : BASIC DEFINITIONS AND COMBINATORIAL LEMMAS

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Abstract. We introduce the combinatorial notion of heaps of pieces, which gives a geometric interpretation of the Cartier-Foata's commutation monoid. This theory unifies and simplifies many other works in Combinatorics : bijective proofs in matrix algebra (MacMahon Master theorem, inversion matrix formula, Jacobi identity, Cayley-Hamilton theorem), combinatorial theory for general (formal) orthogonal polynomials, reciprocal of Rogers-Ramanujan identities, graph theory (matching and chromatic polynomials). Heaps may bring new light on classical subjects as poset theory. They are related to other fields as Theoretical Computer Science (parallelism) and Statistical Physics (directed animals problem, lattice gas model with hard-core interactions). Complete proofs and definitions are given in sections 2, 3,4,5. Other sections give a summary of possible applications of heaps.

1. Introduction

Following some work of Foata [24] on combinatorial properties of rearrangements of sequences, Cartier and Foata [9] introduced in 1969 the monoids generated by an alphabet A with relations ab = ba, for all pairs of letters a,b of A such that (a,b) ∈ C, where C is a fixed subset of A x A. The basic properties of these monoids, especially the so-called *flow monoid* and *rearrangement monoid*, appear nowadays to be a classical model in combinatorics (see for example the corresponding chapters of the books of Lallement [39] or Lothaire [40]). These monoids are sometimes called *free partially abelian monoids*. For short, we propose to call them *commutation monoids*.

This model has been used to prove combinatorially (i.e. with bijections) some classical formulae of matrix algebra : the celebrated MacMahon Master theorem in Cartier-Foata [9], the inversion matrix formula in Foata [26] and the Jacobi identity in Foata [27]. More recently, Gessel [30] has shown how to deduce, from the commutation monoid model, Stanley's relation between *chromatic polynomials* and *acyclic orientations* of graphs. Very recently, a new active area of research has grown up in Theoretical Computer Science, using commutation monoids as an algebraic and combinatorial model for *parallelism* problems and concurrency access to data bases, see §10 below.

In this paper, we introduce another model : the notion of *heaps of pieces*. This model will appear to be equivalent to the commutation monoid model. At the beginning, the reader may have certain doubts about the interest of presenting this new version of the commutation monoid with heaps of pieces. These doubts will probably be reinforced after reading the abstract definitions 2.1,2.4, 2.5 and 2.7 below where the heaps model seems more complicated than the commutation monoid.

Once the reader has overpassed these abstract preliminaries, heaps give a powerful "geometric" visualization of the commutation monoids. Many basic lemmas and bijections become really simple. The heaps model appears to be related to other domains, as for example Statistical Physics, although the relationship with commutation monoids was not obvious. Using the heaps model, we have solved combinatorially some open questions about the *directed animals* model introduced by physicists in 1982 (see a survey in Viennot [47]).

Now we give with an example an intuitive introduction to the notion of heaps. Suppose we have an 8 × 8 chessboard and some dimers. Each dimer is a piece of wood which can cover two consecutive cells of the chessboard. Suppose we put the dimers, one by one, on the chessboard. Each time, one choose a "geographical position" for the next dimer (i.e. two consecutive cells of the chessboard). Then the dimer is put vertically above this position and lowered until it touches the chessboard, covering the two cells of the chosen geographical position, or until it touches one (or two) other dimers previously placed. Placing the new dimer under other dimers is not allowed. In other words, it must be possible to remove the dimers one by one, whithout moving the other dimers, as in the game called "Mikado". What is seen on the chessboard is the visualization of the mathematical notion of *heaps of dimers*, (see Fig. 1).

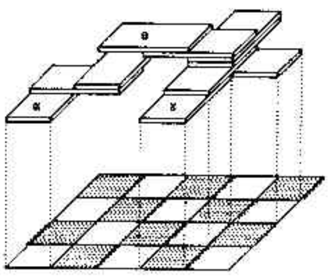


Fig.1. Heap of dimers (or equivalence class of a commutation monoid)

When we consider such heaps of dimers, we thus forget some informations about the exact order of placements of the dimers. For example, between the two dimers α and γ of figure 1, one cannot tell which one was placed first. Nevertheless, looking this same figure, one can say that the dimer α was put before the dimer β. In other words, we define the relation $\alpha \leqslant \beta$ iff it is impossible to remove the dimer α from the heap whithout removing the dimer β. The relation $\leqslant$ is a partial order relation. A heap will be a *poset* (partially order set) satisfying certain axioms relating the order relation $\leqslant$, called "*to be above*", and another relation called *concurrency relation*. Here this relation is defined on the set of "geographical positions" for dimers (there are 2×8×7=112 such positions).Two positions are concurrent iff they have one (or two) cells in common.

If we take as alphabet the set A of the 112 possible "geographical positions" for dimers on the chessboard, a word w of letters in A is an encoding of the placements of the dimers of the heap (remembering the order of placements). Forgetting this exact order corresponds to consider the word w up to the commutations ab = ba , where the geographical positions a and b are disjoint (i.e. not in concurrence). The heap of dimers is exactly the geometric visualization of the equivalence class of the word w in the corresponding commutation monoid.

This paper is the first of a series devoted to the theory of heaps and its various applications. It contains two parts. In sections 2,3,4,5 we give the basic definitions and lemmas of the theory, with complete proofs. Sections 6,7,8,9,10 present a summary of the other papers [15],[17],[50],[51] and related works.

Heaps are defined in §2, together with the *heap monoid* H(P,𝒞) related to a set of basic pieces P equipped with a concurrency relation.

In §3, we show the equivalence between the heap monoids and the *commutation monoids*.

In §4, we show that every heap monoid can be realized with a concurrency relation analogous to the one described above with dimers. Basic pieces are subsets of a set, each of these subsets being equipped with a certain combinatorial structure. We also show that every poset can be "realized" as a heap of pieces. This section is just a preliminary step of a promising area of research : studying *posets theory* with the heaps point of view, in particular *realizations* of family of posets as a family of heaps H(P,𝒞).

Basic lemmas about *heaps generating functions* are given in §5: inversion lemma (in fact the equivalent of the Möbius function of the commutation monoid, defined by Cartier, Foata [9]), heaps with given maximal pieces and the logarithmic property "log(heap)=pyramid". A *pyramid* is a heap having only one maximal piece (as in figure 1).

After the work of Cartier, Foata [9] and Foata [26],[27] giving combinatorial proof of classical *matrix algebra* theorems, and also works of Jackson [36], Straubing [46] and simplifications of Zeilberger [51], we present in Dulucq, Viennot [18] an ultimate step, unifying all these bijections as simple consequences of a few basic properties of heaps. A summary is given in §6.

After Flajolet [22], the author has proposed in [47] (survey in [48]) a combinatorial theory of formal *orthogonal polynomials* with weighted paths. Some part of this theory can be simplified and reinterpreted with heaps of pieces. This is summarized in §7. Closely related is a property of Andrews [2] about the "reciprocal" of the famous Rogers-Ramanujan identities. Andrew's interpretation can also be deduced from heaps basic properties.

In §8 we give some relations between heaps and *graph theory* : *chromatic polynomials* and acyclic orientations of graphs from Gessel [30] and a summary of Desainte-Catherine, Viennot [15] relating heaps and *matching polynomials* of graphs. Godsil's tree-like paths [31] fit very well with the heaps model.

In §9 we present a brief summary of Viennot [50],[51] giving two applications of heaps theory in *statistical physics* : the combinatorial solution of the *directed animal* problem and a combinatorial interpretation of the *density of a gas* with hard-core interactions.

In §10, we give a flavor of the connections with *parallelism* problems in *Theoretical Computer Science*.

2. Basic terminology for heaps

Let P be a set equipped with a symmetric and a reflexive binary relation $\mathscr{C}$ (i.e. $a\mathscr{C}b \Leftrightarrow b\mathscr{C}a$ and $a\mathscr{C}a$ for every $a,b \in P$). The elements of P are called *basic pieces*. The relation $\mathscr{C}$ is called the *concurrency relation*.

Definition 2.1. A *labeled heap* with pieces in P is a triple $(E,\leq,\varepsilon)$ where $(E,\leq)$ is a finite *poset* (i.e., partially ordered set) with order relation denoted by $\leq$ and ε is a map $\varepsilon : E \to P$ satisfying the two following conditions

(i) for every $\alpha,\beta\in E$ such that $\varepsilon(\alpha)$ $\mathscr{C}$ $\varepsilon(\beta)$, then α and β are comparable (i.e $\alpha\leq\beta$ or $\beta\leq\alpha$).

(ii) for every $\alpha,\beta\in E$ such that $\alpha<\beta$ and β covers α (i.e. $\alpha\leq\gamma\leq\beta \Rightarrow \gamma=\alpha$ or $\gamma=\beta$) then $\varepsilon(\alpha)$ $\mathscr{C}$ $\varepsilon(\beta)$.

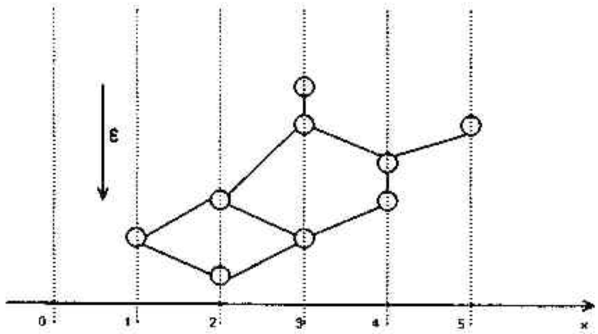
The elements of E will be called *pieces*. When $\alpha\leq\beta$, we will say that the piece β is *above* the piece α.

Remark that P is not necessarily finite but it is important to set down that all the heaps we consider in this theory are *finite*.

Example 2.2. Let $B = [0,8] \times [0,8]$. A *cell* (or *elementary square*) is the set of points (x,y) of B such that $i<x<i+1$, $j<y<j+1$ for certain i, j of $[0,7]$. The set P of basic pieces is the set of subsets of B formed by the union of two cells joined by an edge. The concurrency relation $\mathscr{C}$ is defined by $a\mathscr{C}b$ iff $a\cap b \neq \emptyset$. A heap E with pieces in P was visualized on Fig.1.

Here the map ε is the projection associating to each dimer of the heap its "geographical position", i.e. an element of P (see the heuristic introduction in §1).

Example 2.3. Let $P = \mathbb{Z}$ be the set of integers . The concurrency relation $\mathcal{C}$ is defined by : $i\mathcal{C}j$ iff $|j-i|\leqslant 1$ for $i,j\in P$. The poset E is defined on Fig.2 by its *Hasse diagram*, (i.e. an edge goes upward from α to β iff β covers α). The map ε is defined on Fig.2 by: for $\alpha\in E$ lying on the vertical line $x=i$, then $\varepsilon(\alpha)=i$.



The reader will verify that the axioms (i) and (ii) are satisfied.

Fig.2. A heap of pieces.

Equivalent definitions for heaps

a) Conditions (i) and (ii) can be replaced by (i) and (ii') where (ii') is the following condition

(ii') for every $\alpha,\beta\in E$ with $\alpha\leqslant\beta$, there exists a sequence $\alpha=\alpha_1\leqslant\ldots\leqslant\alpha_k=\beta$ of pieces of E such that $\varepsilon(\alpha_i)\ \mathcal{C}\ \varepsilon(\alpha_{i+1})$ for every i, $1\leqslant i<k$.

b) A second formulation is (i) and (ii") where (ii") is the following condition

(ii") the order relation $\leqslant$ is the transitive closure of the relation $\mathcal{C}'$ defined by : for $\alpha,\beta\in E$, $\alpha\ \mathcal{C}'\ \beta$ iff $\alpha\leqslant\beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$.

c) A third formulation is the following. Let ε: $E\rightarrow P$ be a map of the set E in P. Let $G(E,\varepsilon,\mathcal{C})$ be the graph which vertices are the elements of E and with an edge between α and β iff $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$. Then defining an order relation $\leqslant$ on E such that $(E,\leqslant,\varepsilon)$ satisfies conditions (i) and (ii), is nothing but defining an *acyclic orientation* of the graph $G(E,\varepsilon,\mathcal{C})$ (i.e. an orientation of each edge such that the graph does not contain cycles).

Subheap. Let $(E,\leqslant,\varepsilon)$ be a heap and F a subset of E. Let ε' be the restriction of ε to F. Let $\mathcal{R}$ be the relation defined on F by $\alpha\ \mathcal{R}\ \beta$ iff $\alpha\leqslant\beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$. Let $\leqslant'$ be the transitive closure of $\mathcal{R}$. Then $(F,\leqslant',\varepsilon')$ is called a *subheap*.

Definition 2.4. Let $(E,\leqslant,\varepsilon)$ and $(E',\leqslant',\varepsilon')$ be two heaps of pieces in P with the same concurrency relation $\mathcal{C}$. We say that they are *isomorphic* iff there exists a bijection $\varphi:E\rightarrow E'$ which is a poset isomorphism (i.e. $\alpha\leqslant\beta$ in E iff $\varphi(\alpha)\leqslant'\varphi(\beta)$ in E'), and such that $\varepsilon=\varepsilon'\circ\varphi$.

Definition 2.5. A *heap* of pieces (in French : *empilement de pièces*) in P with concurrency relation $\mathscr{C}$ is a labeled heap (definition 2.1) defined up to a heap isomorphism (or equivalence class, for isomorphism, of labeled heaps).

In the following, a heap will be denoted by one of its representative $(E,\leq,\varepsilon)$ or E for short. We will sometimes use the notation $E = (E,\leq,\varepsilon)$. We will again say that the elements of E are *pieces*, and call the order relation $\leq$ as "to be above". The set of all finite heaps with pieces in P and concurrency relation $\mathscr{C}$ is denoted by $H(P,\mathscr{C})$.

Lemma 2.6. Any automorphism φ of a labeled heap $(E,\leq,\varepsilon)$ is trivial (i.e. is the identity map of E).

Proof. For any basic piece $a\in P$, $\varepsilon^{-1}(a)$ is a finite chain of E (from (i) and the reflexivity of $\mathscr{C}$). The relation $\varepsilon=\varepsilon\circ\varphi$ implies that φ preserves this chain. As it is a poset automorphism, we deduce that φ is the identity map.

Let a_n be the number of heaps of $H(P,\mathscr{C})$ having n pieces. From lemma 2.6, we deduce that the number b_n of labeled heaps, with set of labels any set of n elements as for example $E = \{1,2,\ldots,n\}$, is $b_n=n!a_n$. When enumerating heaps (resp. labeled heaps) we will use the ordinary (resp. exponential) generating function $\sum_{n\geq 0} a_n t^n$ (resp. $\sum_{n\geq 0} b_n t^n/n!$).

In fact, labeled heaps are an example of Joyal's *species* [38]. Heaps are the corresponding *type of species*.

Definition 2.7. Let E and F be two heaps of $H(P,\mathscr{C})$. the *product* $H=E\circ F$ (or *superposition* of F over E) is the heap H defined by the following : if $E = (E,\leq_E,\varepsilon)$, $F = (F,\leq_F,\varepsilon')$, $H = (H,\leq_H,\varepsilon'')$, then

(i) $H=E+F$ (disjoint union of E and F)

(ii) ε'' is the unique map $\varepsilon'':H\to P$ which restriction to E (resp. F) is ε (resp. ε').

(iii) the order relation $\leq_H$ is the transitive closure of the following relation $\mathscr{R}$ for $\alpha,\beta\in H$, $\alpha\,\mathscr{R}\,\beta$ iff

- $\alpha,\beta\in E$ and $\alpha\leq_E\beta$

or - $\alpha,\beta\in F$ and $\alpha\leq_F\beta$

or - $\alpha\in E,\beta\in F$ and $\varepsilon(\alpha)\,\mathscr{C}\,\varepsilon'(\beta)$.

Remark that E and F are subheaps of $E\circ F$.

Such a definition is compatible with isomorphisms and thus is well defined on the set $H(P,\mathscr{C})$ of heaps.

This product of heaps is associative and $H(P,\mathscr{C})$ is a *monoid*, called the *heap monoid*, which neutral element is the *empty heap* denoted by $\emptyset$.

An element α of P will be identified with a heap reduced to a single piece. The heap $E\odot\alpha$ is said to be obtained by *adding* (or *putting*) the (basic) piece α *above* the heap E. Any heap E is a product (in general in several different ways) of its (basic) pieces. Remark that for any two basic pieces.

(1) $\alpha,\beta\in P$, $\alpha\odot\beta=\beta\odot\alpha$ iff $\alpha \not\mathscr{C} \beta$ (i.e. α and β are not in concurrency).

The product of heaps is a left and right *simplifiable* product, that is : $E\odot F=E\odot F' \Rightarrow F=F'$ and $E\odot F=E'\odot F \Rightarrow E=E'$. If $E\odot\alpha=F$, for $\alpha\in P$ and $E,F\in H(P,\mathscr{C})$, we will say that E is obtained by *deleting* the piece α from the top of the heap F.

Definition 2.8. A *trivial* heap is a heap such that the order relation $\leqslant$ is trivial, that is no pieces are above another.

We will denote by $T(P,\mathscr{C})$ the set of trivial heaps of $H(P,\mathscr{C})$. If the concurrency relation $\mathscr{C}$ is "empty", every heap is trivial and the heap monoid $H(P,\mathscr{C})$ is isomorphic to the free commutative monoid generated by P.

Lemma 2.9. <u>Any heap $E \in H(P,\mathscr{C})$ can be written in a unique way as a product of trivial heaps $E = T_1\odot\ldots\odot T_p$ satisfying the condition</u> :

(2) <u>for any $1\leqslant j<p$, any pieces of T_{j+1} is above a piece of T_j</u> .

It suffices to take T_1 as the subheap formed by the minimum elements of E. Then one can write $E = T_1\odot E_1$. Repeating recursively this factorization, we get the unique factorization satisfying (2).

This factorization can be characterized in another way. It is the unique factorization into a product of trivial heaps, each factor having maximum cardinality.

3. The Cartier-Foata commutation monoid

Let A be a set and A^* be the *free monoid* generated by A, that is the set of words $u=a_1a_2...a_p$ with *letters* a_i in the set A (called *alphabet*), together with the multiplicative law *concatenation* of two words : $u=a_1..a_p$ and $v=b_1..b_q$, $uv=a_1..a_pb_1..b_q$. The *empty word* is denoted by e.

Let C be a symmetric and antireflexive relation on A (i.e. $a\not{C}a$ for every $a\in A$).

Definition 3.1. The *commutation monoid* $Co(A,C)$ is the quotient of the free monoid by the congruence $\equiv_C$ generated by the (commutation) relations :

(3) for every $a,b \in A$ with $a\ C\ b$, then $ab \equiv_C ba$.

The words u and v are equivalent iff one can transform u into v by a sequence of transpositions of two consecutive letters a and b such that a C b. The monoids $Co(A,C)$, introduced by Cartier and Foata [9], are also called *free partially abelian monoids*.

We suppose that the alphabet A is the set P of basic pieces equipped with the concurrency relation $\mathscr{C}$. Let $C = \bar{\mathscr{C}}$ be the complementary relation (i.e. a C b iff a $\not\mathscr{C}$ b). We are going to show that the heap monoid $H(P,\mathscr{C})$ is a commutation monoid isomorphic to $Co(P,C)$.

We define the map $\psi : P^* \to H(P,\mathscr{C})$ by the relation

(4) for $w = \alpha_1\alpha_2\ldots\alpha_n \in P^*$, $\psi(w) = \alpha_1\circledast\alpha_2\circledast\ldots\circledast\alpha_n \in H(P,\mathscr{C})$.

In other words ψ is the unique morphism (of monoids) such that for $\alpha\in P$, $\psi(\alpha)$ is the heap identified with the basic piece α.

Let $(E,\leq)$ be a poset having n elements. A *natural labeling* of the poset $(E,\leq)$ is a bijection $f : E \to [n]=\{1,2,\ldots,n\}$ such that

(5) for every $\alpha,\beta \in E$, $\alpha\leq\beta \Rightarrow f(\alpha) \leq f(\beta)$.

Another equivalent definition is the so-called *linear extension* of a poset.

Lemma 3.2. Let $(E,\leq,\varepsilon)$ be a heap of $H(P,\mathscr{C})$. For $u = \alpha_1\ldots\alpha_n \in \psi^{-1}(E)$, let $\lambda(u) = f$ be the labeling $f:E\to[n]$ defined by $\varepsilon(f^{-1}(i)) = \alpha_i$. The map λ is a bijection between the set of words $\psi^{-1}(E)$ and the set $\mathscr{L}(E)$ of natural labelings of E.

Proof. a) From the definition 2.7 of the product of heaps, the heap $E = \alpha_1\circledast\alpha_2\circledast\ldots\circledast\alpha_n$ is obtained by adding vertices $s_1,s_2,\ldots s_n$ to the empty heap, with the map ε defined by $\varepsilon(s_i) = \alpha_i$, and the order relation $\leq$ defined by

(6) $\leq$ is the transitive closure of the relation $\mathscr{R}$ defined by $s_i \mathscr{R} s_j$ iff $i\leq j$ and $\alpha_i \mathscr{C} \alpha_j$.

Thus the map $f=\lambda(\alpha_1..\alpha_n)$ defined by $f(s_i) = i$ is a natural labeling of E.

b) Conversely, let $f: E \to [n]$ be a natural labeling of E. Let $t_i=f^{-1}(i) \in E$ $(1\leq i\leq n)$ and $\beta_i=\varepsilon(t_i)$. Let F be the heap $F=\psi(\beta_1\ldots\beta_n)$. We can identify the vertices of the two heaps E and F. We show that these heaps are isomorphic. If $s \leq_E t$, then from heap axiom (ii'), there exists a sequence $t_{i_1} = s \leq_E \ldots \leq_E t_{i_k} = t$ of vertices of E such that for j, $1\leq j<k$, $t_{i_j} \mathscr{C} t_{i_{j+1}}$. As the map $t_i\to i$ is a natural labeling of E, then $i_1\leq\ldots\leq i_k$. From the definition of the product $\beta_1\circledast\beta_2\circledast \ldots \circledast\beta_n$, we deduce $t_{i_j} \leq_F t_{i_{j+1}}$ and thus $s \leq_F t$. The heaps E and F are isomorphic and $\beta_1\circledast\ldots\circledast\beta_n \in \psi^{-1}(E)$.

Combining a) and b) the map λ is a surjection from $\psi^{-1}(E)$ onto $\mathscr{L}(E)$. As it is obviously an injection, the lemma is proved.

□

Lemma 3.3. For every heap $E \in H(P,\mathscr{C})$, the set of words $\varphi^{-1}(E)$ is an equivalence class for the commutation relation $\equiv_C$.

Proof. a) If α and β are two basic pieces not in concurrency, the two heaps $\alpha\circ\beta$ and $\beta\circ\alpha$ are trivial (definition 2.8.) . Thus $\alpha\circ\beta=\beta\circ\alpha$. For $u,v \in P^*$, we deduce that $u \equiv_C v$ implies $\varphi(u) = \varphi(v)$.

b) Conversely let $u=\alpha_1...\alpha_n$ and $v=\beta_1...\beta_n$ be two words such that $\varphi(\alpha_1...\alpha_n)=\varphi(\beta_1...\beta_n)$ is the heap E. As in the proof a) of lemma 3.2, let $s_1,...,s_n$ be the vertices of $E = (E,\leq,\varepsilon)$ with $\varepsilon(s_i)=\alpha_i$. From (6), the vertex s_i of E is minimal (for $\leq$) iff no pieces α_j, $1\leq j<i$ are in concurrency with α_i, that is α_i commutes with all the letters located at its left in the word $u = \alpha_1..\alpha_n$. We can write $u \equiv_C u_1u_1'$, where u_1 is the word containing all the letters (commuting two by two) of u corresponding to minimal elements of E.

Similarly, we can write $v \equiv_C v_1v_1'$, where v_1 is the word containing all the letters (commuting two by two) of v corresponding to minimal elements of E.

Thus $u_1 \equiv_C v_1$ and $\varphi(u_1')=\varphi(v_1')$ is the subheap E_1 obtained from E by deleting all its minimal elements (see lemma 2.9).

By a recurrence on the common length of the words u and v, we deduce that u and v are equivalent modulo $\equiv_C$.

□

Combining lemmas 3.2 and 3.3, we deduce

Proposition 3.4. Let $H(P,\mathscr{C})$ be a heap monoid with pieces in P and concurrency relation $\mathscr{C}$. Let C be the complementary relation of $\mathscr{C}$. The morphism of monoid φ: $P^* \rightarrow H(P,\mathscr{C})$ defined by (4) induces an isomorphism $\bar{\varphi}$ between the monoid $H(P,\mathscr{C})$ and the commutation monoid $Co(P,C)$.

It may be useful to restate the definition of this isomorphism $\bar{\varphi}:Co(P,C) \rightarrow H(P,\mathscr{C})$, together with its main properties coming from the proof of lemmas 3.2 and 3.3.

a) Let $\bar{u}$ be an element of Co(P,C).Choose any representative $u=\alpha_1\alpha_2...\alpha_n$ of this class of words. Then the heap $\varphi(u)=\alpha_1\circ\alpha_2\circ...\circ\alpha_n$ is independent of the choice of $u\in\bar{u}$ and will be denoted by $\bar{\varphi}(\bar{u})$.

b) Conversely, if $E = (E,\leq,\varepsilon)$ is a heap of $H(P,\mathscr{C})$, taking any natural labeling $f: E \rightarrow [n]$ of the poset $(E,\leq)$, we define a word $u=u_1...u_n$ with $u_i=\varepsilon(f^{-1}(i))$.Let $\bar{u}$ be the equivalence class of u for $\equiv_C$. Then the map $E \rightarrow \bar{u}$ is the reverse bijection of the bijection $\bar{\varphi}:Co(P,C) \rightarrow H(P,\mathscr{C})$.

c) Let $\bar{u}$ be a commutative class of Co(P,C) and $u=\alpha_1...\alpha_n$ be one of its representants. We define a poset $([n],\leq)$ in the following way. The vertices are the integers 1,2,..,n. The order relation $\leq$ is defined by the following relation

(7) $i\leq j$ iff there exists a sequence $1\leq i_1=i<...<i_k=j\leq n$ such that the letters α_{i_j} and $\alpha_{i_{j+1}}$ do not commute (for $1\leq j<k$).

Defining the map $\varepsilon: [n] \to P$ by $\varepsilon(i)=\alpha_i$, we have now a labeled heap $E(u) = ([n],\leqslant,\varepsilon)$ which is a representant of the heap $\bar{\varphi}(\bar{u})$. This labeling f of the vertices of $\bar{\varphi}(\bar{u})$ by the integers $1,2,\ldots,n$ is a natural labeling. The map $u \to f$ is a bijection between the words of the equivalence class $\bar{u}$ and the natural labelings (or linear extensions) of the poset underlying the heap $\bar{\varphi}(\bar{u})$.

Remark that the order relation defined by (7) is the transitive closure of the relation defined by Cori and Métivier from the directed graph denoted by $\Gamma(u)$ in [12]. Also, to give the labeled heap $E(u) = ([n],\leqslant,\varepsilon)$ is equivalent to give the so-called *"dependency graph"* of the word u introduced by Perrin in [43].

If we restate lemma 2.9 in terms of commutation monoids, we get the classical property (see Cartier, Foata [9]) :

Corollary 3.5. Let u be a word of P^* and $Co(P,C)$ be a commutation monoid. Then $\bar{u}$ can be written in a unique way $\bar{u}=\bar{u}_1\ldots\bar{u}_p$ where each u_j is a block of letters commuting two by two, and for each pair of consecutive blocks $u_j u_{j+1}$, any letter of u_{j+1} does not commute with at least a letter of u_j.

This unique factorization is called the *normal form* of u in [43] and V-factorization in Cartier-Foata [9]. In fact this corollary also comes from part b) of the proof of lemma 3.3.

4. Graphs, Heaps and Posets

Let P and B be two sets and let $\pi: P \to \mathcal{P}(B)$ be a map from P into the set of non empty subsets of B. We define the concurrency relation $\mathscr{C}$ by the relation.

(8) for $a,b \in P$, $a \,\mathscr{C}\, b$ iff $\pi(a) \cap \pi(b) \neq \emptyset$.

In this fundamental example of heaps, the heap monoid $H(P,\mathscr{C})$ will also be denoted by $H(P,\pi,B)$. The set B is called the *basis*. The subset $\pi(a)$ is called the *support* of the basic piece $a \in P$.

Let $E = (E,\leqslant,\varepsilon)$ be a heap of $H(P,\pi,B)$ and $\alpha \in E$ be a piece of E. The subset $\pi\circ\varepsilon(\alpha)$ will also be called the *support* of the piece α. We say that two pieces $\alpha,\beta \in E$ (resp. basic pieces $a,b \in P$) are disjoint if their support are *disjoint*. In the contrary, that is $\varepsilon(\alpha) \,\mathscr{C}\, \varepsilon(\beta)$ (resp. $a \,\mathscr{C}\, b$) they are said to be *intersecting*. Two heaps E and F are said to be *intersecting* iff one piece of E intersects one piece of F.

Example 4.1. Let $B = \mathbb{Z}$ and P be the set of *dimers*, that is the set of subsets of the form $\{i,i+1\}$, $i \in \mathbb{Z}$. We define π as the restriction to P of the identity map of $\mathcal{P}(B)$. The heap displayed on Fig.3 is" isomorphic" to the heap of Fig.2. (here the term "isomorphic" would be an extension of definition 2.4 to the case of two heaps with different set of basic pieces, see below just before remark 4.4).

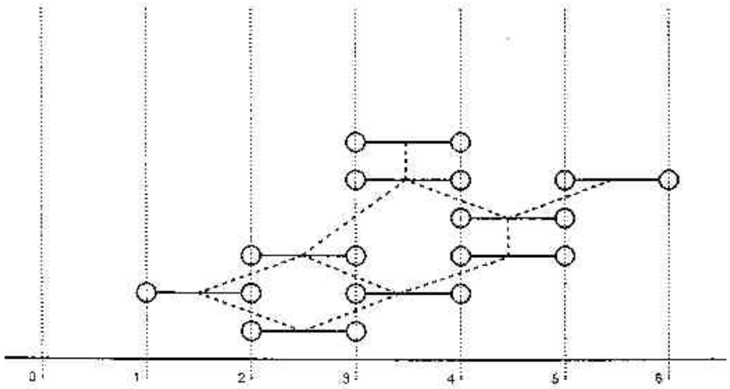


Fig.3. A heap of dimers on $\mathbb{Z}$.

Example 4.2. Let $B = [0,8] \times [0,8]$ and P as in example 2.2. Let π be the restriction to P of the identity map of $\mathcal{P}(B)$. Heaps of $H(P,\pi,B)$ were considered in example 2.2 and are visualized on Fig.1 of the introduction.

In fact, any heap monoid $H(P,\mathcal{C})$ can be identified with a heap monoid $H(Q,\pi,B)$. For that, we need the following definition.

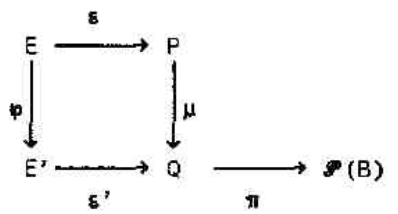
Definition 4.3. Let $\mathcal{C}$ be a concurrency relation (i.e. symmetric and reflexive) on the set P. The *concurrency graph* is the graph $G(\mathcal{C})$ with vertices in P and with edges $\{a,b\} \in A$ iff $a \,\mathcal{C}\, b$ and $a \neq b$.

Let $B = P \cup A$. For each $a \in P$, we define the subset $\mu(a)$ of B by

$$\mu(a) = \{a\} \cup \{\{a,b\} \in A\}. \tag{9}$$

The map μ is a bijection between P and $\mu(P)=Q$. Now $a \,\mathcal{C}\, b$ iff $\mu(a) \cap \mu(b) \neq \emptyset$. Let π be the restriction to Q of the identity map of $\mathcal{P}(B)$.

Any heap $(E,\leq,\varepsilon) \in H(P,\mathcal{C})$ is "*isomorphic*" to a heap $(E',\leq',\varepsilon')$ of $H(Q,\pi,B)$, i.e. there exists a poset isomorphism $\varphi : E \to E'$ and a bijection $\mu : P \to Q$ "*preserving*" the concurrency relations of P and Q, such that the following diagram is commutative



Remark 4.4. The construction of the map μ defined by (9) is related to the so-called *line graph* (or *median graph*) of the concurrency graph $G(\mathcal{C})$.

If the graph $G(\mathcal{C})$ is represented by points of $\mathbb{R}^d$ joined by segments, then one can represent $\mu(a)$ as the set of points formed by the vertex a and the middle of the edges containing this vertex a. The pieces look like *starfishes* (see Fig.4).

The monoid $H(Q,\pi,B)$ constructed above from P and $\mathcal{C}$ will be called a *starfish monoid*.

Proposition 4.5. Every heap monoid is isomorphic to a starfish monoid.

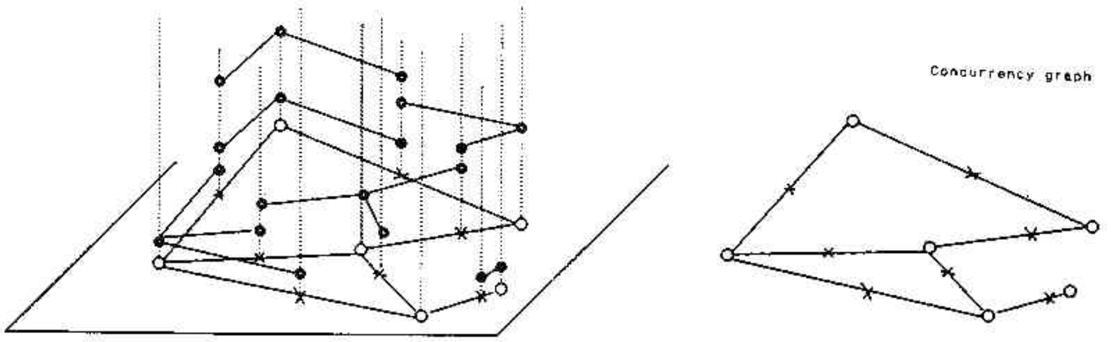


Fig.4. A heap of starfishes.

Proposition 4.6. Every poset $(E,\leqslant)$ can be represented as the underlying poset of a heap $(E,\leqslant,\varepsilon)$.

For the proof of this proposition, we need the following definitions.

Let $(E,\leqslant)$ be a poset and $\mathcal{F} = (C_i)_{i\in I}$ be a family of chains. We say that $\mathcal{F}$ *strongly covers* E iff, for every pair α,β of elements of E such that β covers α (i.e. α and β are connected by an edge in the Hasse diagram of E), then there exists a chain C_i of $\mathcal{F}$ containing both α and β.

Let $\mathcal{F} = (C_i)_{i\in I}$ be such a family (always exists). We take as basis the set $B=I$. The basic pieces are the subsets of I and π is the identity of $P = \mathcal{P}(I)$. We define the map $\varepsilon : E \longrightarrow P$ by $\varepsilon(\alpha)=\{i\in I, \alpha\in C_i\}$. The concurrency relation $\mathcal{C}$ is defined by (8).

The triple $(E,\leqslant,\varepsilon)$ satisfies condition (i) of definition 2.1 of labeled heap : if $\alpha,\beta\in E$, $\varepsilon(\alpha)\,\mathcal{C}\,\varepsilon(\beta)$ implies that α and β belongs to a same chain C_i of $\mathcal{F}$. Condition (ii) of definition 2.1 follows from the strongly covering property. □

It would be interesting to represent some known families of posets as families of heaps $H(P,\pi,B)$. Is it possible to give a poset characterization of the posets underlying heaps of a given heaps monoid Many questions arise about *representations* of posets with heaps. Here we will mainly be interested in heaps as a tool for combinatorial enumeration and combinatorial interpretation of classical results or identities. Nevertheless we will mention the following property.

Let $E = (E,\leqslant,\varepsilon)$ be a heap of $H(P,\pi,B)$. For $x\in B$, the *fiber* of E over x is the set defined by

$$F_x(E) = \{\alpha \in E,\ x \in \pi\circ\varepsilon(\alpha)\} . \tag{10}$$

Such fibers are chains. The family $\{F_x(E)\}_{x\in B}$ strongly covers the poset $(E,\leq)$.

The minimum cardinality of the basis set B such that the poset $(E,\leq)$ is realized as a heap of $H(P,\pi,B)$ is the minimum number of chains strongly covering E. This number is not less that the minimum number of chains covering E. This last number is more classical in poset theory and, from Dilworth's theorem is known to be equal to the maximum cardinality of *antichains* of E (set of elements two by two incomparable).

The reader may ask the interest of introducing the map $P \xrightarrow{\pi} \mathcal{P}(B)$ instead of simply introducing the basic pieces as subsets of B. We will need basic pieces where a combinatorial structure is defined on their support. An important example will be heaps of *cycles*. Here P is the set of all cycles of B (in the sense of cycle of permutation : that is a circular permutation on a subset of B). The map π associates to a cycle its-underlying set of vertices. An example is displayed on Fig.5. The order $\leq$ between the cycles is defined by the fibers (corresponding to the vertical lines).

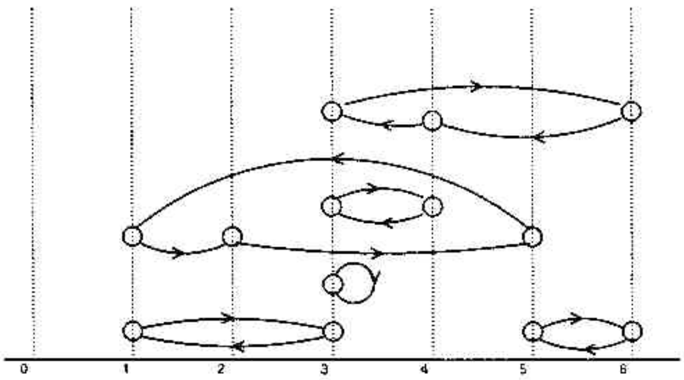


Fig.5. A heap of cycles on $\mathbb{N}$.

5. Heap Generating functions

Let $H(P,\mathscr{C})$ be a heap monoid with basic pieces in P and concurrency relation $\mathscr{C}$. Let $\mathbb{K}[[X]]$ be the algebra of formal power series with variables in a set X (not necessarily finite) and with coefficients in the commutative ring $\mathbb{K}$. We define a *valuation* (or *weight* function) as a map $v : P \longrightarrow \mathbb{K}[[X]]$ which associates to every basic piece $\alpha\in P$ a power series $v(\alpha)$ *having no constant term*. This last condition is necessary for the summability of heaps generating functions. In general, most examples of the theory will be such that $v(\alpha)$ is a monomial in variables from X.

The *valuation* (or *weight*) $v(E)$ or a heap $E=(E,\leq,\varepsilon)$ is the product of the valuations of its pieces $v(E) = \prod_{\alpha\in E} v(\varepsilon(\alpha))$.

In all this work, we suppose that the valuation v satisfies the following condition

(11) for every monomial μ in the variables X, there exists a finite number of heaps E of $H(P,\mathscr{C})$ such that the coefficient of μ in the series $v(E)$ is $\neq 0$.

This condition, which is satisfied when the set P of basic pieces is finite, implies the summability of the heaps generating function $\sum_{E\in H(P,\mathscr{C})} v(E)$.

Proposition 5.1. (Inversion lemma) Let $H(P,\mathscr{C})$ be a heap monoid with a valuation v satisfying (11). The generating function of the weighted heaps of $H(P,\mathscr{C})$ is given by

$$\sum_{E\in H(P,\mathscr{C})} v(E) = \frac{1}{\sum_{F\in T(P,\mathscr{C})} (-1)^{|F|}\, v(F)}, \tag{12}$$

where $T(P,\mathscr{C})$ denotes the set of trivial heaps (definition 2.8).

The identity (12) is equivalent to the identity

$$\sum_{(E,F)} (-1)^{|F|}\, v(E)\, v(F) = 1, \tag{13}$$

where the summation is over all pairs (E,F) of $HT=H(P,\mathscr{C}) \times T(P,\mathscr{C})$.

Let $M(E,F)$ be the set of pieces formed by the pieces of F and the maximal pieces of E which are not in concurrence with pieces of F. Let L be a non-empty trivial heap. In the summation (13), we select only pairs (E,F) such that $M(E,F) = L$. We can write $L=L_1\circ L_2$ with $E=E_1\circ L_1$ and $F=L_2$ (see Fig.6). Thus, we have the identity

$$\sum_{\substack{(E,F)\in HT \\ M(E,F)=L}} (-1)^{|F|}\, v(E)\,v(F) = v(L) \sum_{E_1} v(E_1) \left[\sum_{\substack{L_1,L_2\in T(P,\mathscr{C}) \\ L=L_1\circ L_2}} (-1)^{|L_2|} \right],$$

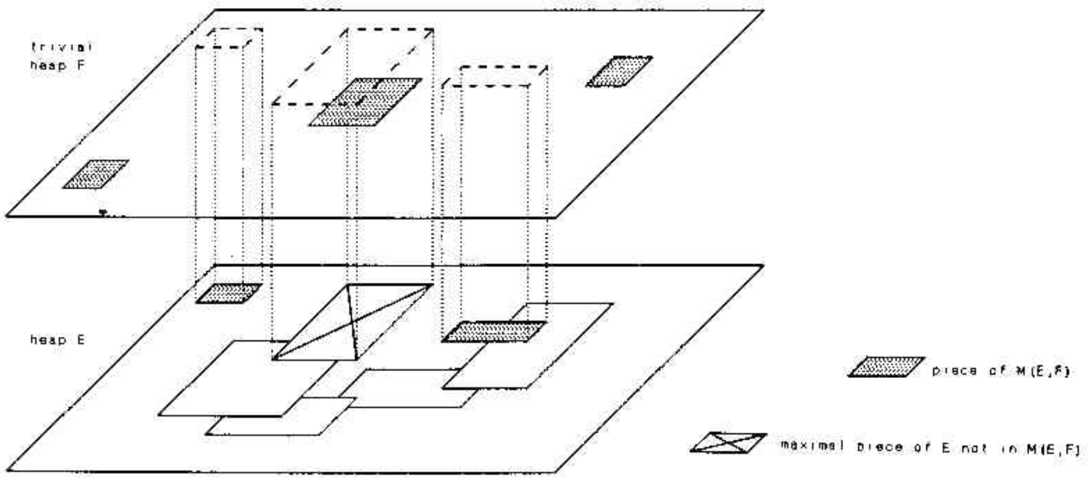
where the first summation of the right hand-side is over heaps $E_1\in H(P,\mathscr{C})$ such that all their maximal pieces are in concurrence with at least a piece of L. The second summation of the right-hand side is 0.

Thus, the only non-vanishing term in (13) is the pair corresponding to $M(E,F) = \emptyset$, that is $E=\emptyset$, $F=\emptyset$. Its weight is **1**.

□

Remark 5.2. If the reader prefers a proof with bijections, it would be possible to define a sign-reversing involution on the set HT. The idea is simply to "transfer" a piece α of F on the top of E, or vice-versa (see Fig.6). We totally order the set P of basic pieces. For a pair $(E,F) \in HT$, $(E,F) \neq (\emptyset,\emptyset)$, we take the smallest piece α of $M(E,F)$. If α is a piece of E, then $E = E_1 \circ \alpha$ and we define $\Psi(E,F) = (E_1, \alpha \circ F)$. If α is a piece of F, then $F = F_1 \circ \alpha$ $(= \alpha \circ F_1)$ and we define $\Psi(E,F) = (E \circ \alpha, F_1)$. The map $\Psi(E,F) \rightarrow (E',F')$ is an involution such that

$$(-1)^{F} v(E)v(F) = -(-1)^{F'} v(E')v(F').$$



The concurrency relation $\mathscr{C}$ is the intersection relation

Fig.6. Proof of proposition 5.1.

In term of commutation monoid, relation (12) is nothing but expressing the *Möbius function* of that monoid (see theorem 2.4 of Cartier, Foata [9]). Möbius inversion of posets is a classical chapter of Combinatorics which has been popularized by Rota [42]. Content, Lemay, Leroux [11] present a synthesis of Rota and Cartier-Foata's Möbius inversion. We give the following extension, which generalizes a proposition of Desainte-Catherine [14], [15].

Proposition 5.3. Let $H(P,\mathscr{C})$ be a heap monoid with a valuation v satisfying (11). Let M be a set of basic pieces of P. The generating function of weighted heaps of $H(P,\mathscr{C})$ such that the maximal pieces are in M is given by.

$$(14) \qquad \sum_{\substack{E \in H(P,\mathscr{C}) \\ \text{Maximal pieces } M}} v(E) = \frac{N}{D}, \qquad \text{with}$$

$$D = \sum_{F \in T(P,\mathscr{C})} (-1)^{|F|} v(F) \qquad \text{and} \qquad N = \sum_{F \in T(P \setminus M,\mathscr{C})} (-1)^{|D|} v(F).$$

where $T(P,\mathscr{C})$ denotes the set of trivial heaps (definition 2.8).

It would be possible to define a sign-reversing involution as in remark 5.2 transfering a piece α from E to F, by taking M(E,F) as the set of minimal pieces of E, not in concurrence with any pieces of F, together with the set of pieces of F which are in M or in concurrence with at least one piece of E. The pairs corresponding to $M(E,F) = \emptyset$ are the pairs involved in the summation for N.

A simpler proof, as suggested by A.Joyal, is to apply proposition 5.1 and the following lemma.

Lemma 5.4. Let $M \subset P$ and $E \in H(P,\mathscr{C})$. Then the heap E has a unique factorization $E = E_1 \circ E_2$ where E_1 is a heap with maximal pieces in M and E_2 is a heap with pieces not in M.

The power serie D appearing as the denominator of generating function (12) and (14) plays an important role in heaps theory. We will call this power serie the *exclusion power serie* for the pieces P and concurrency relation $\mathscr{C}$ (and the valuation v), or for short the *exclusion serie* of the heap monoid. We will denote it by D or $D(P,\mathscr{C})$ or $D(P,\mathscr{C},v)$. If P is finite, D is a polynomial, the *exclusion polynomial*.

Example 5.5. Let $A = (a_{ij})$ be an n×n matrix. Let B = [n], P be the set of cycles on B and $\pi : P \longrightarrow \mathscr{P}(B)$ the map associating to a cycle its underlying set. The concurrency relation is defined by (8). The valuation of the cycle $\gamma = (x_1 \ldots x_m)$ is the product $\lambda^m a_{x_1 x_2} \ldots a_{x_{m-1} x_m} a_{x_m x_1}$

The letters λ and a_{ij} can be considered as formal variables in X. Then it is almost the definition of the determinant to say that $D(P,\mathscr{C}) = \det(I - \lambda A)$.

Thus the *characteristic polynomial* of the matrix A can be considered as the reciprocal of an exclusion polynomial.

Example 5.6. Let $B = \mathbb{N}$ be the basis and P be the set of monomers $\{i\}$, $i \geq 0$ and dimers $\{i,i+1\}$, $i \geq 0$, with concurrency relation the intersection relation as in §4.

Let $\{b_k\}_{k \geq 0}$ and $\{\lambda_k\}_{k \geq 1}$ be two sequences of the ring $\mathbb{K}$. The monomer $\{i\}$, $i \geq 0$ is weighted $b_i x$. The dimer $\{i-1,i\}$, $i \geq 1$, is weighted $\lambda_i x^2$.

If we restrict the basis to be $B_n = [0,n-1]$, with pieces in B_n, let $P_n(x)$ be the corresponding exclusion polynomial. These polynomials satisfy the three-terms linear recurrence relation

(15) $P_{n+1}(x) = (x-b_n)\, P_n(x) - \lambda_n\, P_{n-1}(x)$, with $P_0(x)=1$, $P_1(x)=x-b_0$.

From Favard's theorem the sequence $\{P_n(x)\}_{n \geq 0}$ is a sequence of formal *orthogonal polynomials* and conversely,any orthogonal polynomials are obtained this way. (see for example Chihara [10], Viennot [47]).

Example 5.7. Let G = (V,A) be a graph with vertices in V and edges in A. Let P = A with concurrency relation $\mathscr{C}$ be the intersection relation. The weight of an edge is x^2. Then the *matching polynomial* of the graph G (see for example [20],[32],[35]) is the reciprocal of the corresponding exclusion polynomial.

Example 5.8. Let $G = (V,A)$ be a graph with vertices in V and edges in A. The set of basic pieces is $P=V$ with concurrency relation $\mathscr{C}$ defined by : $a \mathscr{C} a$ and $a \mathscr{C} b$ iff $\{a,b\} \in A$ (i.e. G is the concurrency graph of $\mathscr{C}$). Each vertex is weighted x. We propose to call the corresponding exclusion polynomial the *independency polynomial* of the graph G. It is less classical than the matching polynomial, but it appears in some statistical mechanics models (see below §9,b). In fact, up to a change of variable, the matching polynomial of the graph G is the independency polynomial of its so-called *line graph* .

We interpret below the logarithm of the generating function of weighted heaps. We suppose that the ring $\mathbb{K}$ is the field $\mathbb{Q}$ of rational numbers. We need the following definition.

Definition 5.9. A *pyramid* is a heap having a unique maximal piece.

Proposition 5.10. Let $H(P,\mathscr{C})$ be a heap monoid with a valuation v satisfying (11). Then

$$\log \left[\sum_{E \in H(P,\mathscr{C})} v(E) \right] = \sum_{F} \frac{v(F)}{|F|}, \tag{16}$$

where the second summation is over all pyramids of $H(P,\mathscr{C})$.

Condition (11) implies the summability of both sides of identity (16).

Here we work with exponential generating function, that is labeled heaps :

$$\sum_{E \in H(P,\mathscr{C})} v(E) = \sum_{n \geq 0} \left(\sum_{|E|=n} n! \frac{v(E)}{n!} \right).$$

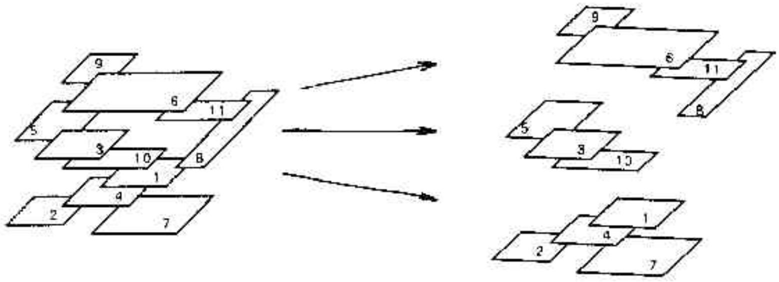
This generating function is the exponential generating function for labeled (by $1,2,\ldots,n$) weighted heaps. We decompose such heaps E into pyramids in the following way .

We select the piece α_1 of E with minimal label (i.e. 1). Let E_1 be the subheap formed by all the pieces below α_1 (order ideal). E_1 is a pyramid and in fact E can be factorized $E=E_1 \circ E_1'$. Another way to define E_1 is to say that there exists a unique factorization $E=E_1 \circ E_1'$ such that E_1 is a pyramid with maximal piece α_1. We select the piece α_2 of E_1' with minimal label and get a factorization $E=E_1 \circ E_2 \circ E_2'$. Recursively we have a factorization of the heap E into a product of labeled pyramids with the property

(17) the piece with minimal label is the maximal piece of the pyramid.

Conversely, from the set $\{E_1,\ldots,E_k\}$ of such pyramids, one can reconstruct E by taking their product in the increasing order of the label of their maximal element.

In the context of species of Joyal [38], or of Foata's "*composé partitionnel*" [25], a heap is an "*assembly*" of labeled pyramids satisfying (17). Their exponential generating function is the right-hand side of (16). The proposition comes from standard result on assembly of species or on "composé partitionnel". □



heap assembly of pyramids

Fig.7. log(heap) = pyramid.

In the next sections, we give a summary of the possible applications of heaps theory to enumerative and interpretative combinatorics. This will be done in details in the papers [15],[18],[50],[51].

6. Flow monoid and combinatorial proofs in linear algebra

(summary of [18])

We use the notations of §4. The basis is B. The set of basic pieces is $P = B \times B$. The projection $\pi : P \longrightarrow \mathcal{P}(B)$ is defined by

(18) for any $(s,t) \in P = B \times B$, $\pi(s,t) = s$.

Definition 6.1. The *flow monoid* is the heap monoid $F(B)=H(P,\pi,B)$ defined by (18).

This corresponds to the flow monoid introduced by Cartier, Foata [9] : the edges (s,t) and (s',t') commute iff $s \neq s'$.

The heaps of $H(P,\pi,B)$ are called *flows*. Such a flow E is defined by its fibers. The fiber $F_s(E)$ over $s \in B$ is isomorphic to a word of B_s^* with $B_s = \{s\} \times B$. In fact there is no order relation between two elements of distinct fibers and the flow monoid is isomorphic to a direct product of free monoids $H(P,\pi,B) \approx \prod_{s \in B} B_s^*$ (see Fig.8).

Definition 6.2. A *rearrangement* is a flow $E=(E,\leq,\varepsilon)$ of $H(P,\pi,B)$ such that for every $s \in B$, the fiber $F_s(E)$ defined by (10) satisfies

(19) $|F_s(E)| = |\{\alpha \in E,\ \varepsilon(\alpha) = (t,s)\}|$.

In other words the number of edges (t,s) of E coming in s is the same as the number of edges (s,t) starting from s.

The rearrangements form a submonoid R(B) of the flow monoid F(B).

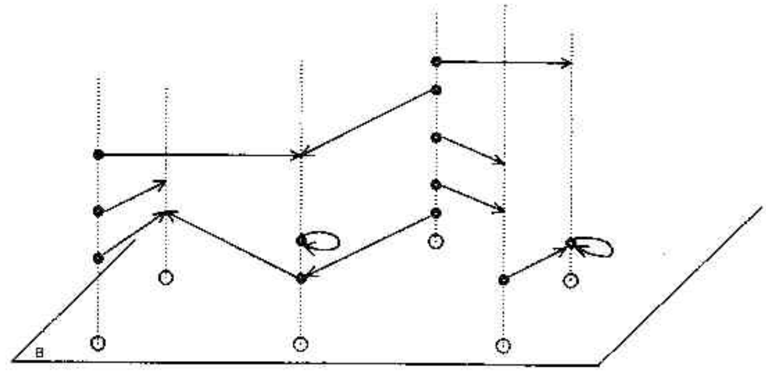


Fig.8. A flow.

The two following propositions are typical examples of bijections transforming a heap into another heap of bigger pieces obtained by "gluing" the small pieces (or conversely "breaking" pieces into smaller pieces).

A *path* ω of B is any sequence $\omega=(s_0,s_1,\ldots,s_n)$ of points of B. We consider the heap monoid SPCy(B) = $H(Q,\pi,B)$ which pieces Q are *cycles* (Cy) on B or *self-avoiding* paths (SP) on B (i.e. no two vertices appear twice in ω) and the projection π is the map associating to a piece its underlying set of vertices of B. A path ω can be identified with the flow $(s_0,s_1)\circledast(s_1,s_2)\circledast\ \ldots\ \circledast(s_{n-1},s_n)$, (product of heaps).

A cycle $\gamma = (s_1,\ldots,s_n)$ (see the definition at the end of §4 and also see example 5.5) can be identified with the rearrangement $(s_1,s_2)\circledast\ldots\circledast(s_{n-1},s_n)\circledast(s_n,s_1)$. The submonoid Cy(B) of SPCy(B) is formed by heaps of cycles.

Proposition 6.3. Let u,v $\in$ B. There exists a bijection between paths ω of B going from u to v and pyramids E of SPCy(B) such that all pieces are cycles of B, except the maximal piece, which is a self-avoiding path η going from u to v. This bijection is such that the number of edges (s,t) in ω (or elementary steps) in the same as the number of edges (s,t) contained in the cycles and the paths η of the pyramid E.

This bijection is particularly useful for the enumeration of certain families of heaps (see below the directed animal problem).

Proposition 6.4. There exists an isomorphism of monoids Ψ:Cy(B)$\rightarrow$R(B) between the heap monoid of cycles and the heap monoid of rearrangements. Moreover, for any s,t$\in$B, Ψ preserves the number of edges (s,t) in each heap.

Each bijection of propositions 6.3 and 6.4 is obtained by "breaking" the heap of cycles (and self-avoiding path) into its elementary components : the edges (s,t) considered as elements of the flow monoid.

Combining the above propositions with the propositions of §5 gives combinatorial proofs of classical identities in linear algebra (see [9],[26],[27],[36],[46],[52]).

Let $A = (a_{ij})$ be an $n \times n$ matrix and $X = \begin{pmatrix} x_1 \\ . \\ x_n \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ . \\ y_n \end{pmatrix} = AX$.

Paths and cycles are weighted as in example 5.5 by

$v(\omega) = v(s_0,s_1) \ldots v(s_{n-1},s_n)$ and $v(i,j) = a_{ij}$.

Corollary 6.5. (MacMahon Master theorem). <u>The coefficient of</u> $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ <u>in the formal serie</u> $1/\det(I-AX)$ <u>is the same as the coefficient of</u> $x_1^{\alpha_1} \ldots x^{\alpha_n}$ <u>in the polynomial</u> $y_1^{\alpha_1} \ldots y_n^{\alpha_n}$.

This is a combination of proposition 5.1, example 5.5 and proposition 6.4.

Corollary 6.6. (Inversion matrix formula) <u>The term</u> (i,j) <u>of the inverse matrix</u> $(I-A)^{-1}$ <u>is</u> $N_{ij}/\det(I-A)$ <u>where</u> N_{ij} <u>is the term</u> (j,i) <u>of the adjoint matrix (cofactor)</u>.

This is a combination of proposition 5.3, example 5.5 (together with a companion formula for the cofactor) and proposition 6.3.

Corollary 6.7. (Jacobi identity)

$$\frac{1}{\det(I-A)} = \exp(\operatorname{Tr}(\log(I-A)^{-1})). \tag{20}$$

This identity comes from a combination of proposition 5.1, example 5.5 and proposition 5.3 (in a slightly more general version).

Also, Cayley-Hamilton theorem can be obtained by using a slightly more general form of the identity (14) of proposition 5.3.

7. Orthogonal polynomials

Any sequence $\{P_n(x)\}_{n \geqslant 0}$ of (formal) orthogonal polynomials appears as the sequence of reciprocal of the exclusion polynomials of weighted monomers and dimers on the segment [0,n-1] (see example 5.6).

A combinatorial theory of classical properties valid for any sequences of orthogonal polynomials has been made by Viennot [47], following work of Flajolet [22]. This combinatorial theory is written in terms of certain weighted paths (called *Dyck* and *Motzkin* paths). Some of the bijective proofs can be simplified by using heaps terminology. The Dyck (resp. Motzkin) paths are transformed (by proposition 6.3) into pyramids of dimers (resp. monomers and dimers) on $B = \mathbb{N}$.

The main property is the following. Let $P_n(x)$ be the sequence of polynomials defined by the recurrence (15) and $H(P,\pi,B)$ be the heap monoid of monomers and dimers on $B = \mathbb{N}$, weighted by the sequences $\{b_k\}_{k\geq 0}$ and $\{\lambda_k\}_{k\geq 1}$ of elements of the ring $\mathbb{K}$ as in example 5.6.

Let μ_n be the sequence defined by

(21) $$\mu_n = \sum_{E} v(F),$$

where the summation is over all weighted pyramids of m monomers and d dimers such that $n = m+2d$ and such that the maximal piece contains the value 0 (i.e. this maximal piece is either {0} or {0,1}), see Fig.9.)

Let f be the unique linear functional $f:\mathbb{K}[x] \to \mathbb{K}$ such that $f(x^n) = \mu_n$ $(n\geq 0)$. Suppose that $\lambda_k \neq 0$ $(k\geq 1)$ and that $\mathbb{K}$ has no zero divisors.

Proposition 7.1 - <u>The polynomials</u> $P_n(x)$ <u>defined by the three-terms linear recurrence (15) are orthogonal with respect to the sequence of moments</u> μ_n <u>defined by (21), that is</u> :

(22) $f(P_k P_l) = 0$ <u>if</u> $k \neq l$ <u>and</u> $f(P_k^2) \neq 0$, <u>for every</u> $k, l \geq 0$.

The proof follows from the same generalization of identity (14) of proposition 5.3 mentioned at the end of §6 about a bijective proof of Cayley-Hamilton theorem.

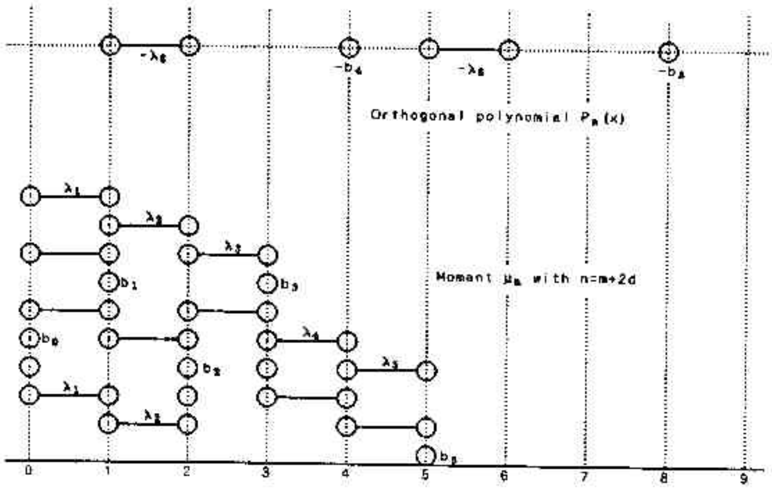


Fig.9. Orthogonal polynomials and moments interpretated as exclusion polynomials and pyramids of monomers-dimers.

Many other properties of general (i.e. formal) orthogonal polynomials can be deduced from heaps basic lemmas. In particular the Jacobi continued fraction expansion (corresponding to Flajolet's theorem about weighted Motzkin paths) here becomes a simple consequence of a decomposition lemma about the pyramids interpretating μ_n into other pyramids. This decomposition is the analog, for ordinary generating functions, of the decomposition given in the ~~proof~~ of proposition 5.10 with exponential generating functions.

Coroisllary 7.2. With the above notation (21) ,

$$(23)\qquad \sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1-b_0 t - \cfrac{\lambda_1 t^2}{1-b_1 t - \cfrac{\lambda_2 t^2}{\cfrac{\cdots\cdots\cdots}{1-b_k t - \cfrac{\lambda_{k+1} t^2}{\cdots\cdots\cdots}}}}}$$

The convergents of the continued fraction (23) are nothing but the generating functions of the pyramids interpretating μ_n and bounded on the segment [0,n]. Thus, applying proposition 5.3, these convergents are

$$(24)\qquad \$P_n^*(t) \;/\; P_{n+1}^*(t),$$

where $P_{n+1}^*(t)$ is the reciprocal $t^{n+1}P_{n+1}(1/t)$ of $P_{n+1}(t)$ and $\$P_n(t)$ is the exclusion polynomial for heaps of monomers and dimers on [0,n] not containing the value 0, that is the n^{th} orthogonal polynomial corresponding to the "shifted" valuations $b'_k = b_{k+1}$, $\lambda'_k = \lambda_{k+1}$.

If we take $b_k=0$ and $\lambda_k = -q^k$, then we get the exclusion power serie D(P,$\mathscr{C}$). We are in the case of an infinite set of pieces and (11) is satisfied. Taking the basis B = $\mathbb{N}$, the exclusion power serie D(q) is the left hand-side of the famous (first) Rogers-Ramanujan identity (see for example Andrews [1]) :

$$(25)\qquad 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} .$$

The left hand side of the second Rogers-Ramanujan identity

$$(26)\qquad 1 + \sum_{n\geq 1} \frac{q^{n^2+n}}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} ,$$

can be interpretated as the exclusion power serie N(q) for weighted trivial heaps of dimers (with valuation $\lambda_k = -q^k$) not containing 0. Proposition 5.1 and 5.3 gives interpretations of the generating functions 1/D(q) and N(q)/D(q) , respectively in terms of heaps and moments pyramids. We can easily deduce Andrews's interpretations [2] with *quasi-partitions*.

8. Heaps and algebraic graph theory

a) Matching polynomials of graphs (summary of [15])

Let G be a graph. The *matching* polynomial of G is the reciprocal of the exclusion polynomial D(G;x) defined in example 5.7 : pieces are dimers on G (i.e. edges) weighted by x^2. Several work has been done on these polynomials, in relation with physics and chemistry, see for example [20],[31],[32],[35].

Proposition 5.1 and 5.3 give combinatorial interpretation of the coefficients of the power series $1/D(G;x)$ and $D(G\setminus M;x)/D(G;x)$ where $G\setminus M$ denotes the graph obtained by deleting from G the set of edges M (resp. set of vertices M).

If M is the set of edges containing a vertex s, then $D(G\setminus M;x)/D(G;x)$ is the generating function for the so-called *tree-like* paths introduced by Godsil [31] in order to give a nice proof of the fact that the roots of matching polynomials are real numbers (Heilmann, Lieb [35]). In Desainte-Catherine, Viennot [15] we deduce bijectively Godsil's result and give some generalizations.

Remark that tree-like paths correspond exactly (via the bijection of proposition 6.3) to pyramids with the restriction that all the cycles have length 2. Such cycles can be identified with dimers of G

b) Chromatic polynomials and acyclic orientations of graphs (from Gessel [30])

Let G be a finite graph with n vertices, $\gamma(G;x)$ be the *chromatic polynomial* of G and $\alpha(G)$ be the number of *acyclic orientations* of G. In [45] Stanley has proved the following identity.

(27) $$\gamma(G;-1) = (-1)^n\,\alpha(G).$$

Gessel [30] has given a nice proof of this identity, using the commutation monoid. Here we just sketch the idea of his proof, translated in terms of heaps.

Let G = (S,A) with set of vertices S (resp. edges A). Let $\mathscr{C}$ be the concurrency relation such that G is its concurrency graph (see §4). Let E be a heap of $H(S,\mathscr{C})$. For $k \geq 1$, we denote by $\beta_k(E)$ the number of factorizations of E in the form $E = T_1 \circ \ldots \circ T_k$ where each T_i is a non-empty trivial heap (remark that condition (2) of lemma 2.9 is not necessarily satisfied). Let v be a valuation on the heap monoid $H(S,\mathscr{C})$ as in §5. A heap E is called *linear* (resp. *covering*) iff each basic piece appears at most (resp. at least) once in E. A linear and covering heap E is a product (in $H(S,\mathscr{C})$) of all the basic pieces S. Let $LC(S,\mathscr{C})$ be the set of such heaps. We have the relation

(28) $$\gamma(G;x) = \sum_{k\geq 0} \frac{1}{k!}\left(\sum_{E\in LC(S,\mathscr{C})} \beta_k(E)\right) x(x-1)\ldots(x-k+1),$$

which leads us to introduce the *complete chromatic power serie* of the graph G.

$$(29) \qquad \Gamma(G;x) = \sum_{k\geq 0} \frac{1}{k!} \left[\sum_{E\in H(S,\mathscr{C})} B_k(E)\, v(E) \right] x(x-1)\ldots(x-k+1).$$

We have

$$(30) \qquad \sum_{E,k\geq 0} B_k(E)\ v(E)\ t^k = \left[1 - t \sum_{F} v(F) \right]^{-1},$$

where the first summation is over all heaps $E\in H(S,\mathscr{C})$ and the second is restricted to non-empty trivial heaps. From relation (12) of proposition 5.1, we deduce (a bijective proof would also be possible)

$$(31) \qquad \Gamma(G;-1) = (-1)^n \sum_{E\in H(S,\mathscr{C})} v(E) .$$

The restriction to linear and covering heaps gives (27).

9. Heaps and Statistical Physics

a) The directed animal problem (summary of Viennot [50])

In 1982, physicists have introduced and studied the following problem. A *directed animal* is a set A of points of $\mathbb{N}\times\mathbb{N}$ such that $(0,0)\in A$ and any point (x,y) of A can be reached by a path going from (0,0) to (x,y), with vertices in A, and using elementary steps North or East. The point (0,0) is called the *source point* and North-East is called *priviligied direction*. The size of the animal A is described by its *width* and *length* (i.e. size of the smallest rectangle containing A with edges parallel or perpendicular to the priviligied direction). Let a_n be the number of directed animals with n points. Considering these animals equidistributed, let ℓ_n (resp. L_n) be the average width (resp. length).

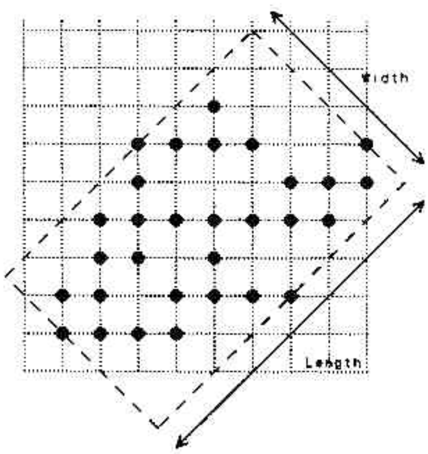


Fig.10. A directed animal, one source point, square lattice.

Physicists expect the following asymptotic behaviour

(32) $$a_n \sim \mu^n n^{-\theta}, \quad \ell_n \sim n^{\nu_\perp}, \quad L_n \sim n^{\nu_{/\!/}}.$$

The constants θ, $\nu_\perp$ and $\nu_{/\!/}$ are called *critical exponents*. Such numbers are of particular importance in the models for *phase transitions* and *critical phenomena*.

A surprising fact is that very simple exact formulae exist for a_n and ℓ_n from which one get immediately μ=3, θ= $\nu_\perp$= 1/2. After many other works (for a survey see [49]) , physics solutions are given by Dhar [16], [17] and Hakim, Nadal [34] following Nadal, Derrida, Vannimenus [41] .

A complete combinatorial solution (for a_n and ℓ_n) can be given by using heaps basic properties and a bijection between directed animals (with one source point) and certain pyramids of dimers on **Z**. This is done in [50], where some conjectures of Dhar [16] are proved. A survey of the directed animal model, with both physics and combinatorial solutions, and relationship with other problems and models, is given in Viennot [49] . The case of directed animals on a *triangular lattice* is easier. A "brute force" bijection between directed animals and certain paths has been given by Gouyou-Beauchamps, Viennot [33] . This bijection is the same as the one obtained using heaps.

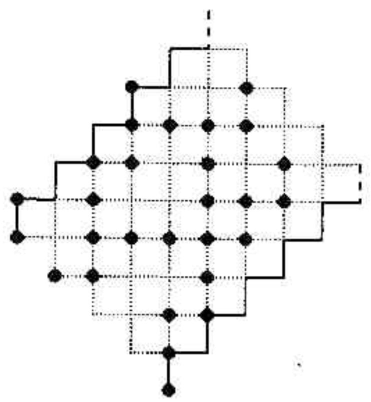


Fig.11. Directed animals on a bounded strip

In the physics solution, Nadal, Derrida, Vannimenus [41], Hakim, Nadal [34] consider directed animals on a *bounded strip* : several source points are now possible (see Fig. 11). The borders may be identified (*circular strip*). Using transition matrices acting on a space of spins, they give a formula for the number of such animals with given source points. This formula is easily obtained from the generating function of such animals, which is a rational serie N(t)/D(t). The polynomials N(t) and D(t) can be deduced from §7 and proposition 5.3, using Tchebycheff polynomials first kind (circular strip) and second kind (bounded strip).

The problem of the existence and determination of the exponent $\nu_{/\!/}$ is still open. It is conjectured [41] to be 9/11.

b) **Combinatorial interpretation of the density of a gas with hard-core interactions** (summary of Viennot [51])

Here we use propositions 5.1 and 5.10 in the case of the independency polynomials of example 5.8. The heaps model put some light on the so-called "*thermodynamic limit*" of the independency polynomials. We obtain a combinatorial interpretation of the *partition function* $Z(t)$ (on an infinite lattice) and of the *density*

(33) $$\rho(t) = t\, d/dt \log Z(t).$$

In fact,

(34) $$-\rho(-t) = \sum_{n\geq 0} a_n t^n,$$

where the a_n are positive integers enumerating certain pyramids.

Using statistical mechanics techniques, Baxter has recently solved [6] the famous *hard hexagon* model. This model has a phase transition for the "*activity*" $t_c=(11+5\sqrt{5})/2$. For $0<t<t_c$, the partition fonction $Z(t)$ is given by the following system of equations

Let $R_I(q)$ (resp. $R_{II}(q)$) be the left hand-side of the first (resp. second) Rogers-Ramanujan identity (25) (resp. (26)). The partition fonction $Z(t)$ is obtained by eliminating q between the two following equations

(35) $$t = -q \left(\frac{R_{II}(q)}{R_I(q)}\right)^5,$$

(36) $$Z = \prod_{n\geq 0} \frac{(1-q^{6n+2})(1-q^{6n+3})^2(1-q^{6n+4})(1-q^{5+1})^2(1-q^{5n+4})^2(1-q^{5n})^2}{(1-q^{6n+1})(1-q^{6n+5})(1-q^{6n})^2(1-q^{5n+2})^3(1-q^{5n+3})^3}.$$

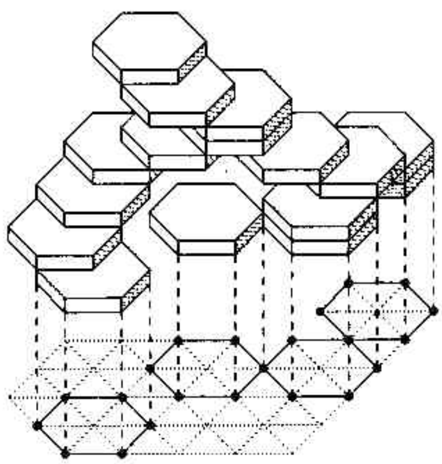


Fig.12. A pyramid of hexagons interpretating the density of the gas in the hard hexagon model.

From heaps basic lemmas, we deduce that the coefficient a_n defined by equations (33), (34), (35) and (36) is the number of pyramids of hexagons on a triangular lattice, formed with n hexagons, as shown on Fig.12. It would be of great interest to prove directly the equations enumerating such pyramids of hexagons, without using Baxter's solution (which has nothing to do with heaps of pieces). Also the combinatorics of heaps proves the equivalence between directed animals problems and hard-core gas model, as shown recently by Dhar [17] using physics arguments.

10. Heaps and parallelism in Computer Science

Finally, commutation monoids have recently appeared in Theoretical Computer Science as a model for *parallelism* and *concurrent access to databases*. This is an active and promising area of research and a meeting on this subject took place in Paris, March 1985, and another is planned in 1986.

A database is a set of objects called *entities*. A *transaction* is any sequence of atomic actions operating on the entities. Several transactions can access concurrently to the same database. An action is identified with a letter and a transaction with a word. Commutations are defined on these letters, describing the possible concurrency access to the database. The model can be developped from an *algebraic* point of view (*rational* and *recognizable* languages in this monoid,...) in analogy with the *free monoid* case (see for example [7],[12],[13],[19],[23],[42]). Another direction introduced by Françon [28],[29] and Arques et al.[4],[5] is *combinatorial*. This direction, following some ideas of Papadimitriou ([42] and related papers) allows the comparison of the performances of concurrency control algorithms with the computation of the cost of *serialization* of an execution or with the determination of Françon's *parallelism ratio*, frequency of *deadlocked executions*, etc... These problems can be reduced to the asymptotic enumeration of certain sets of commutation classes, or enumeration of words in these classes.

The heap monoid model may bring other ideas about these questions. First of all, replacing the alphabet (which letters are a coding of the atomic actions) by a set of basic pieces P, equipped with a basis B and a projection map $\pi: P \to \mathcal{P}(B)$ can be closer to concurrency considerations. For example one can consider the basis B to be the set of entities. An atomic action, symbolized by the basic piece $\alpha \in P$, will operates on the subset $\pi(\alpha) \subset B$ of entities. This atomic action, exactly as the basic piece, is a certain "structure" on the support $\pi(\alpha)$. The concurrency relation of §4 will corresponds to atomic actions having access to common entities. Under this model, two actions will commute iff they operate on disjoint subsets of entities.

Another idea is to use heaps of pieces as a new data structure in Computer Science. This structure appears as a generalization of the *binary tree* structure (which is a poset and thus can be *"realized"* as a heap) and the structure formed by several independant *stacks*. One can implement the heap data structure by its fibers, or by defining *links* between a piece α and the pieces β which are covered by α. This data structure would be of particular advantage in parallel algorithms.

In conclusion, one of the interest of the heaps formulation is to relate some problems coming from completely different fields, as for example the determination of the critical exponents of the directed animals problem in Statistical Physics and the computation of Françon's ratio of parallelism. Both are equivalent to asymptotic enumeration of certains heaps. Some problems are equivalent to enumerate the number of words in a commutation class (as for example *"aggregats"* problems in Statistical Physics). From §3, this is equivalent to enumerate linear extensions of a given poset. This problem is well-known in poset theory and explicit formulae exist only in certain particular cases (standard Young tableaux, standard shifted Young tableaux, trees,...). Of course, the difficulty of the problem remains the same in both points of view, but the spatial intuition, the powerful basic heaps lemmas and the connections made between different domains may be useful

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