Escuela de Investigación CIMPA
"Álgebra, Combinatoria y Física"

Facultad de Ingeniería, Universidad de Valparaíso, Chile

Valparaíso, 20-31 Enero 2014 -

30 Enero 2014

curso II

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Ch 2

generating functions for heaps, heaps and paths

3 basic lemma

- $(\text{heaps}) = \dfrac{1}{(\text{trivial heaps})}$
- $\log(\text{heaps}) = \text{pyramids}$
- path = heap

The inversion lemma

1/D

weight valuation $v(E)$

- $v : P \longrightarrow \mathbb{K}[x, y, \ldots]$

  basic piece

- $v(\alpha, i) = v(\alpha)$

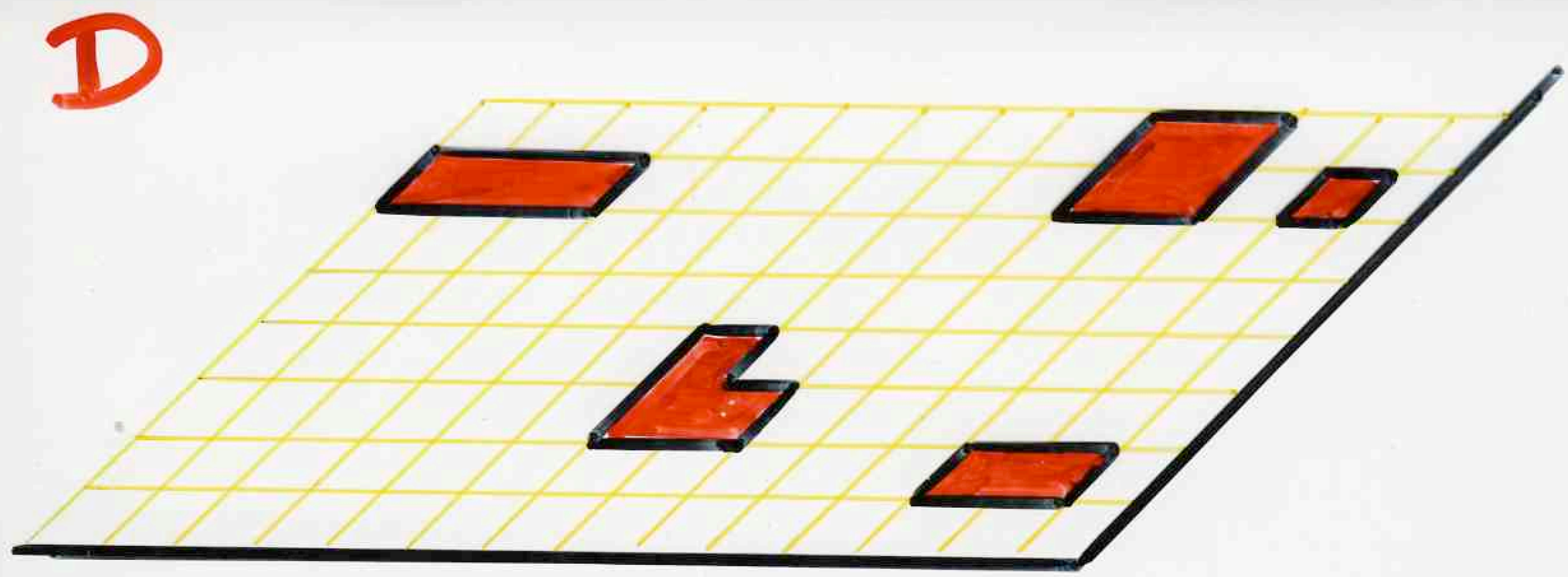
  piece

- $v(E) = \prod_{(\alpha, i) \in E} v(\alpha\ i)$

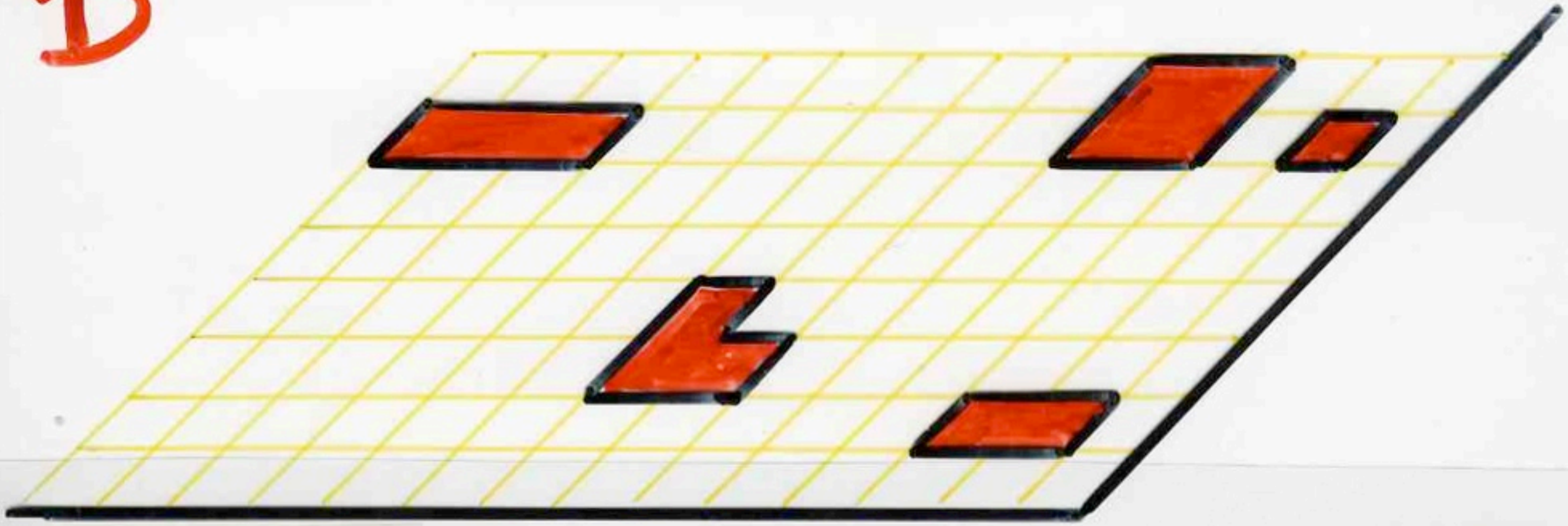
  heap

Inversion lemma

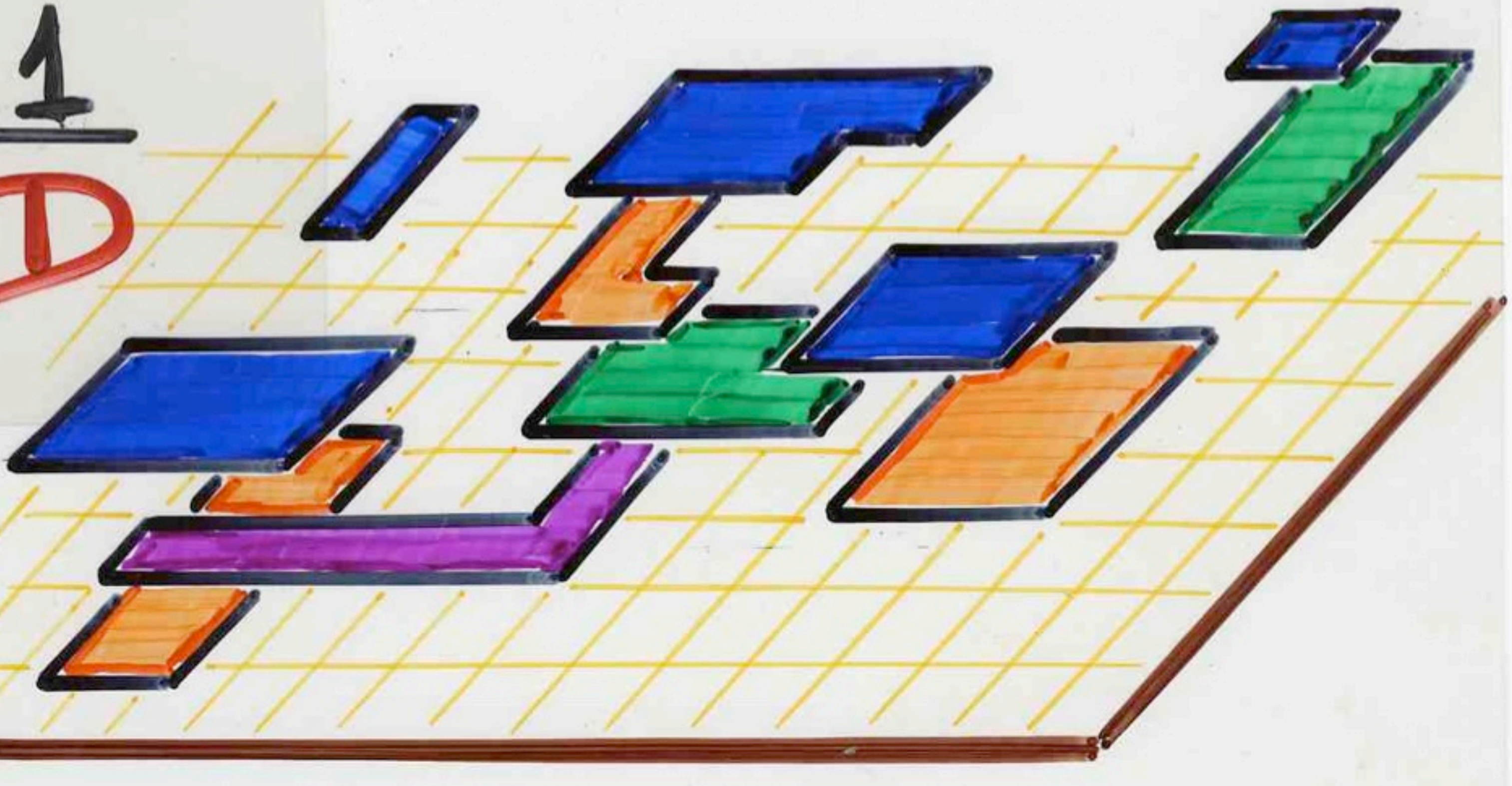
$$\sum_{\substack{E \\ \text{heaps}}} v(E) = \frac{1}{\displaystyle\sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)}$$



D



$\frac{1}{D}$



Extension of the inversion lemma

N/D

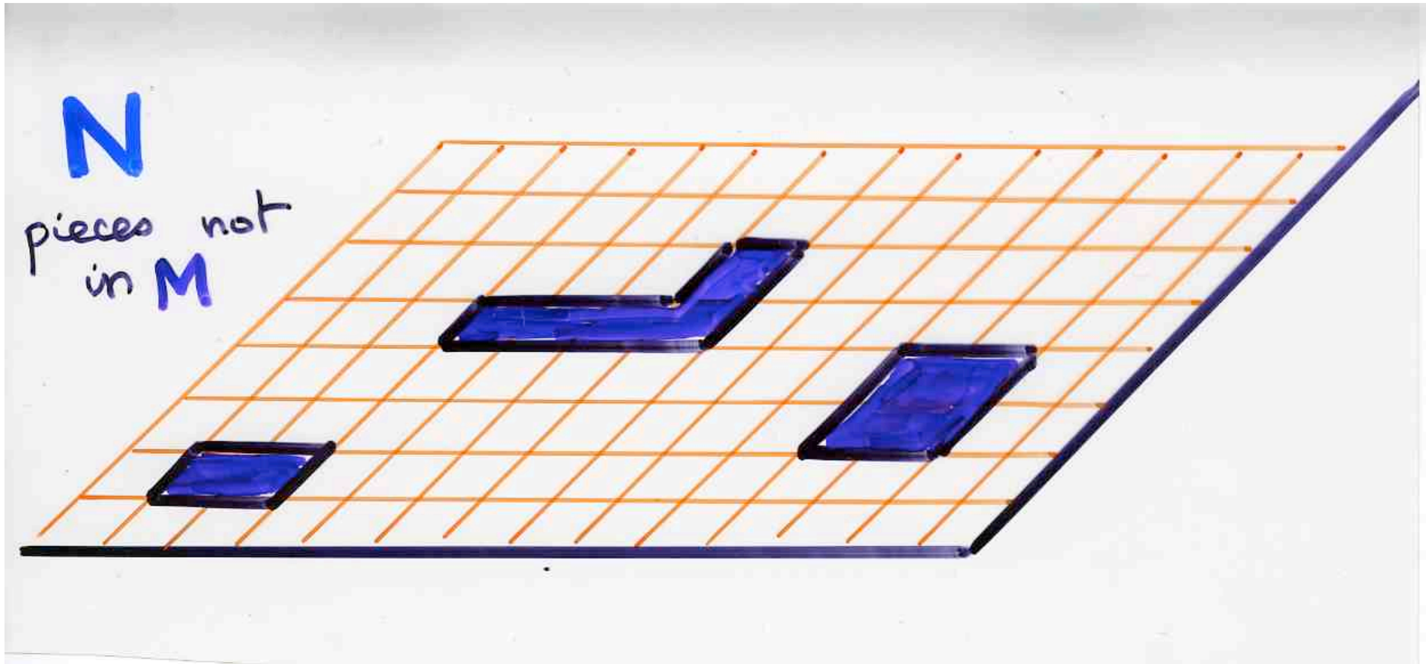
extension of the inversion lemma

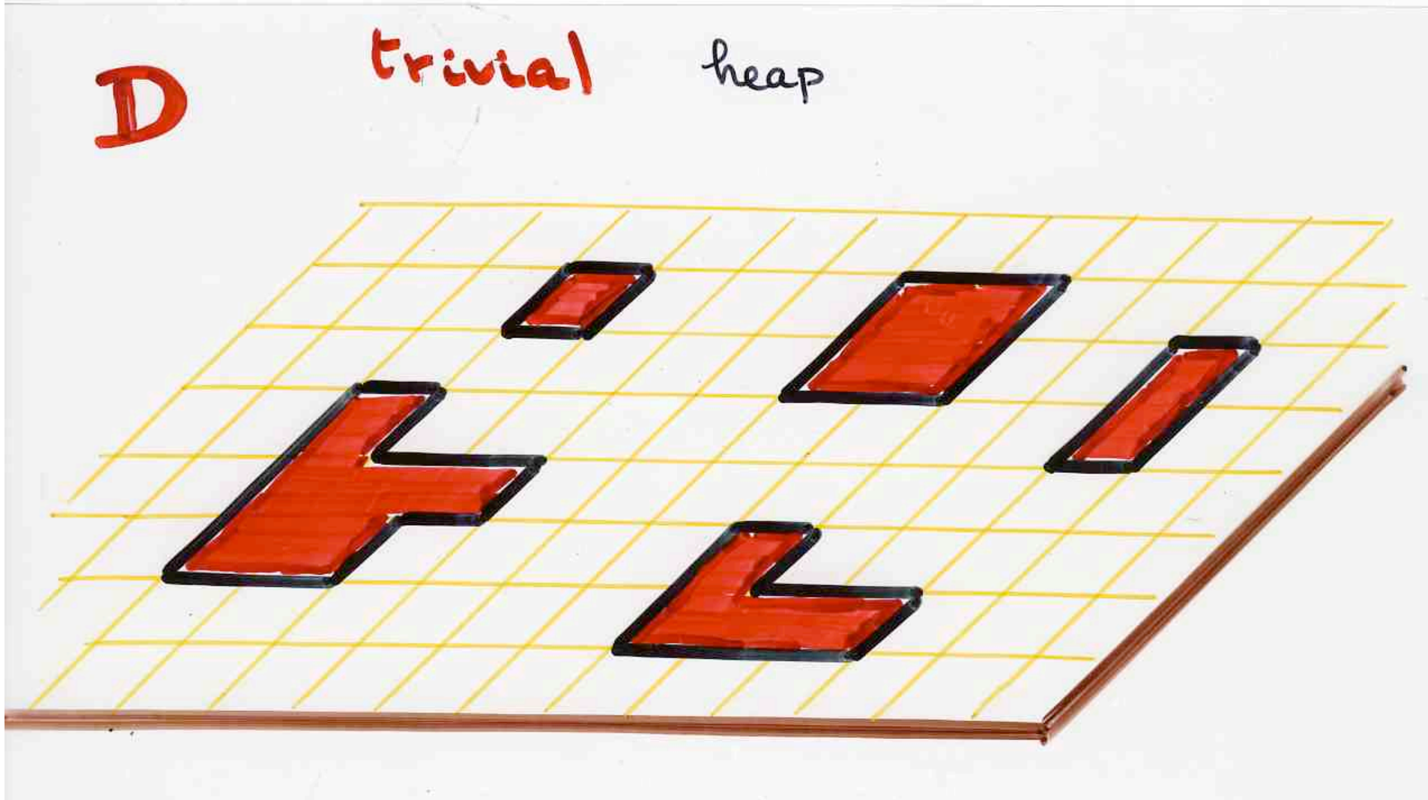
$M \subseteq P$

$$\sum_{\substack{E \\ \Pi(\text{maximal pieces}) \in M}} v(E) = \frac{N}{D}$$

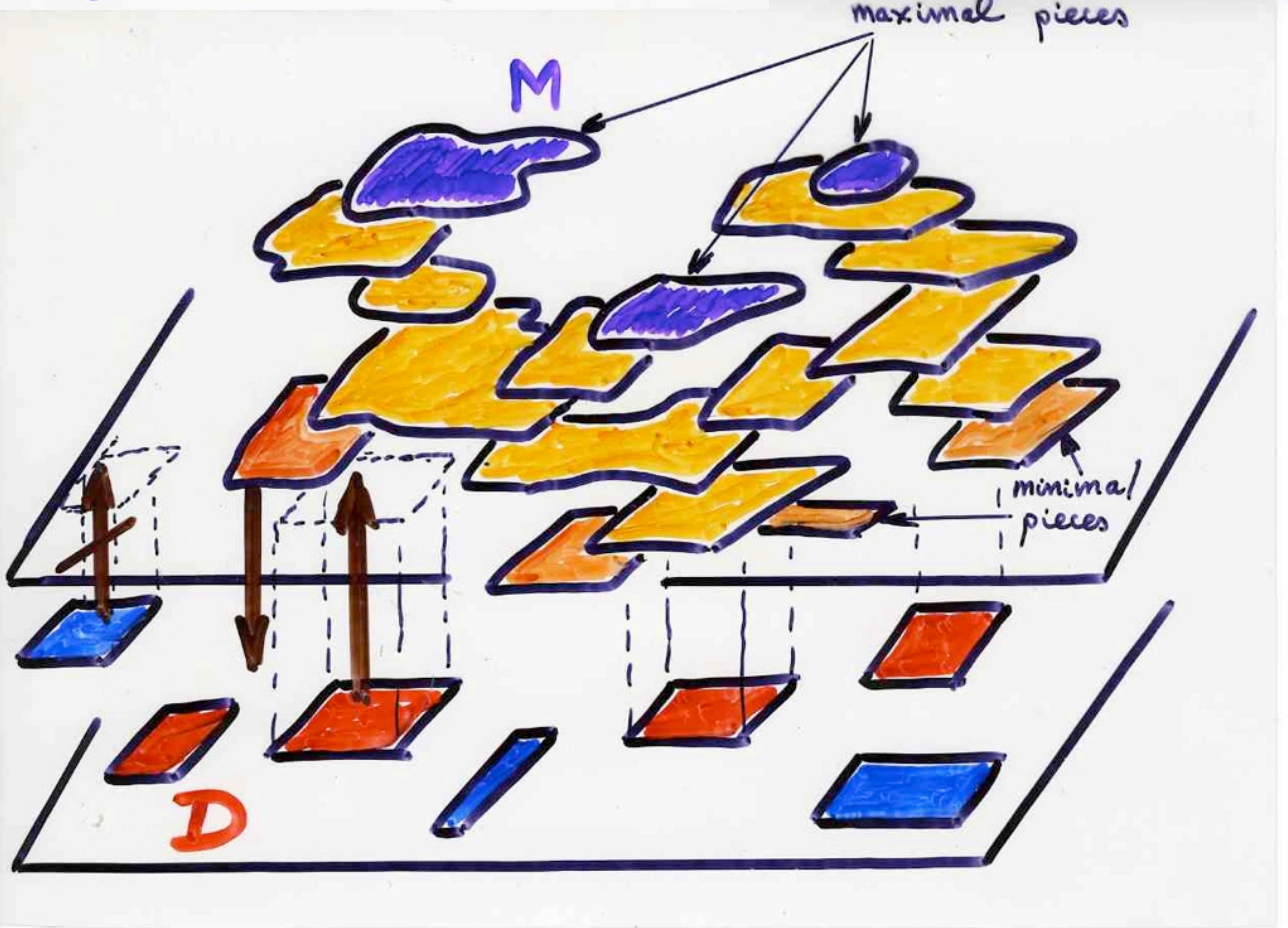
$$D = \sum_{\substack{F \\ \text{trivial heaps}}} (-1)^{|F|} v(F)$$

$$N = \sum_{\substack{F \\ \text{trivial heaps} \\ \text{pieces} \notin M}} (-1)^{|F|} v(F)$$


N
pieces not
in M


D
trivial heap

Proof by involution



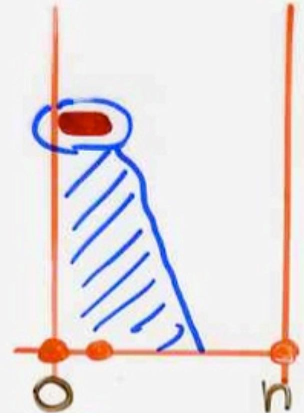
exercise: (example)

heaps of dimers on a strip

semi-pyramids of dimers on a strip

g.f. half-pyramids on $[0, n]$

$$= \frac{F_n(t)}{F_{n+1}(t)}$$



g.f. bounded Dyck paths

$a_{n,k}$ = nombre de couplages de $\{1, 2, \ldots, n\}$ ayant k dominos

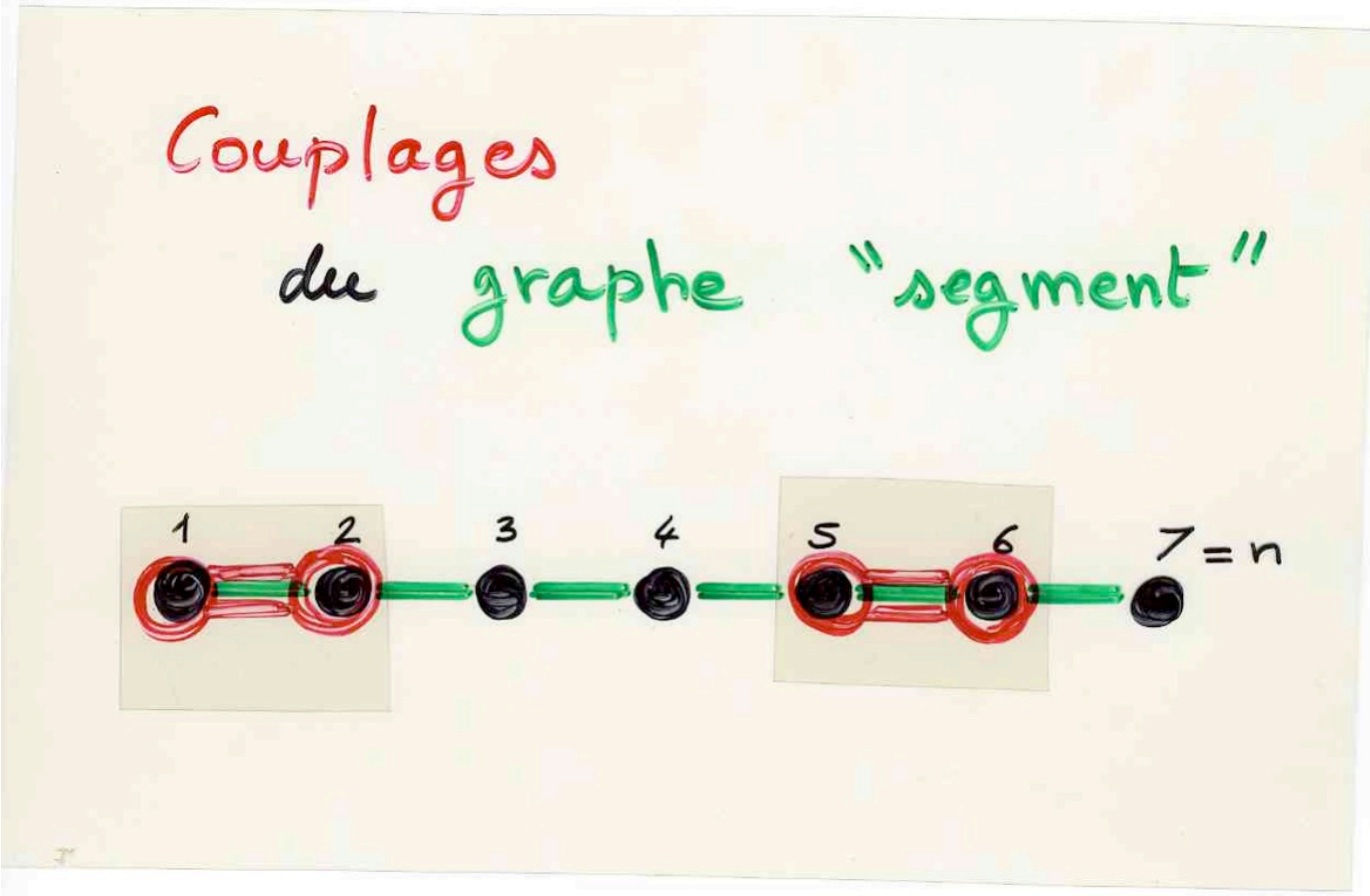
matchings

k dimers

$$P_n(x) = \sum_{k=1}^{n} a_{n,k}\, x^k$$

$$F_n(x) = P_n(-x)$$

Fibonacci polynomial



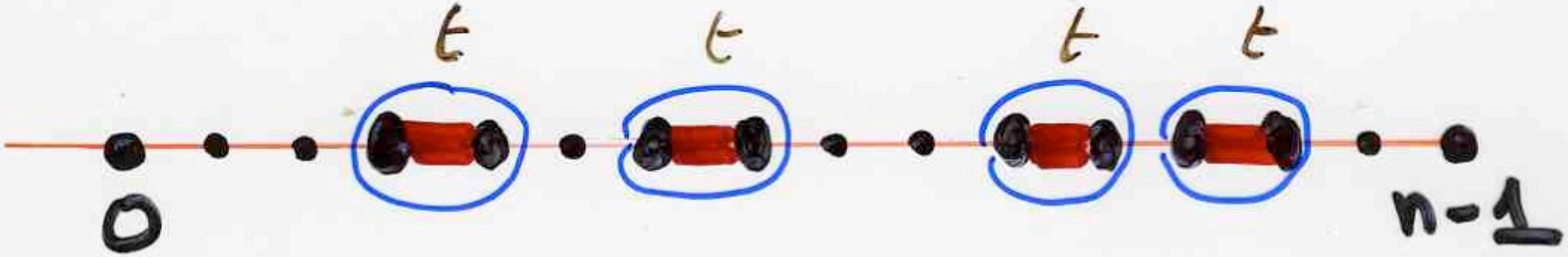
matching of the “segment” graph

Fibonacci polynomials

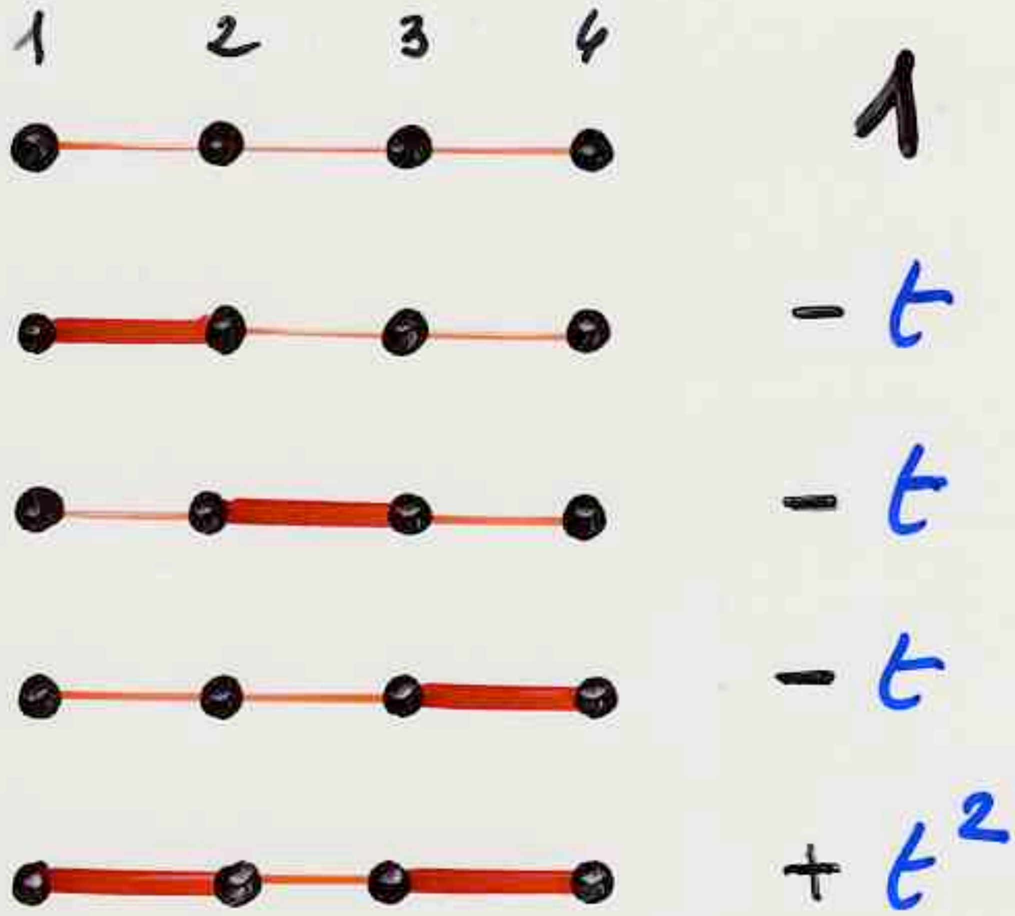
$F_0 = F_1 = 1$

$F_n = F_{n-1} - t F_{n-2}$

1
$1 - t$
$1 - 2t$
$1 - 3t + t^2$



trivial heap of dimers



$$F_4(t) = 1 - 3t + t^2$$

exercise

$a_{n,k}$ = nombre de couplages de $\{1, 2, \dots, n\}$ ayant k dominos

matchings

k dimers

$$a_{n,k} = \binom{n-k}{k}$$

addition +

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$$\sin((n+1)\theta) = \sin\theta \; U_n(\cos\theta)$$

$$U_n(x) = F_n^*(2x)$$

* reciprocal polynomial

Tchebychef polynomials 2nd kind

example

x^3

$-x$

$-x$

$$F_3(x) = x^3 - 2x$$

$$\sin(4\theta) = \sin\theta\,[8\cos^3\theta - 4\cos\theta]$$

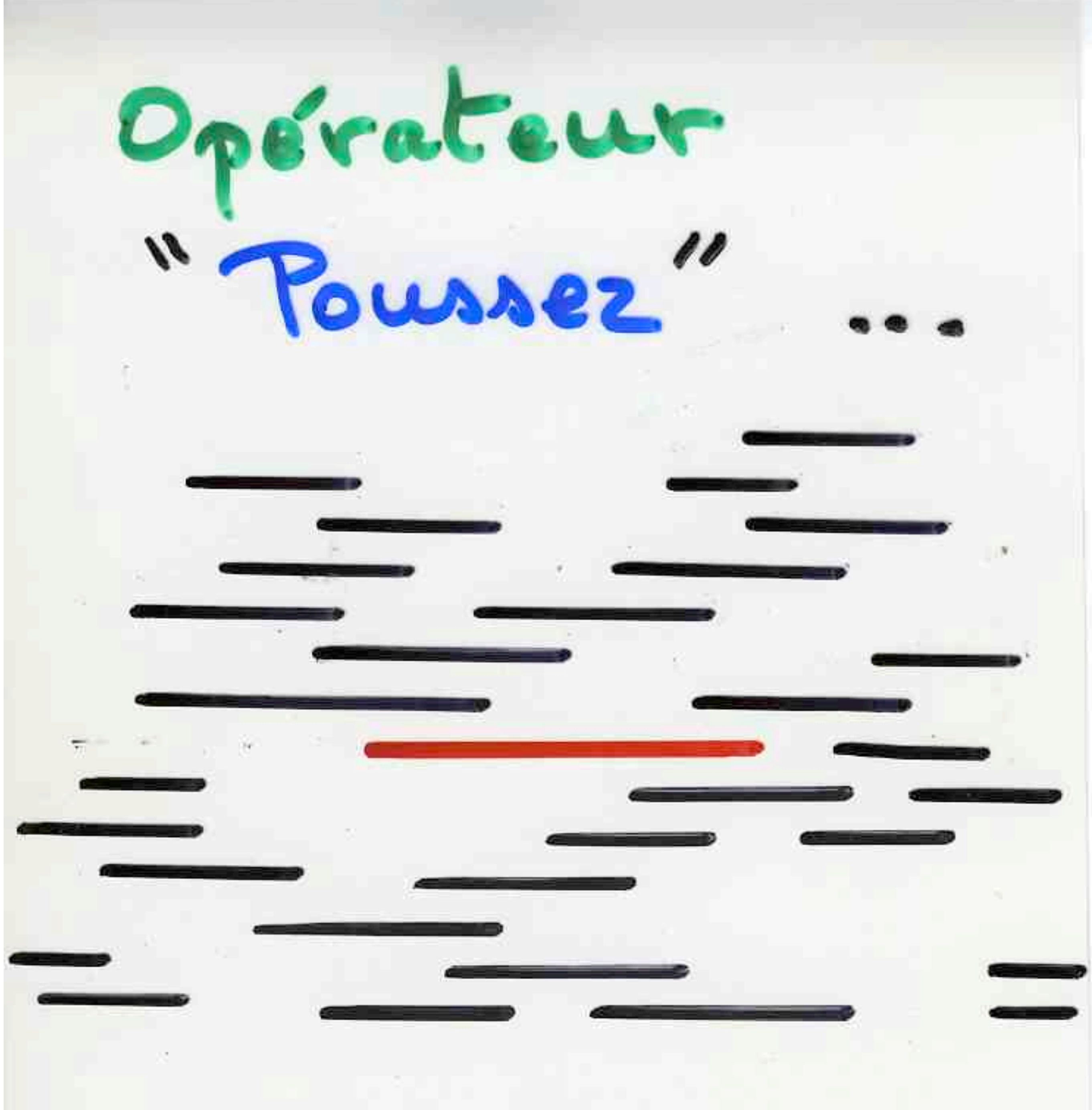
The logarithmic lemma

- **logarithmic lemma**

$$v(\text{piece}) = t\, \underbrace{w(\text{piece})}_{\text{polynomial not containing } t}$$

$$t \frac{d}{dt} \log\left(\sum_{\text{heap } E} v(E)\right) = \sum_{\substack{P \\ \text{pyramid}}} v(P)$$

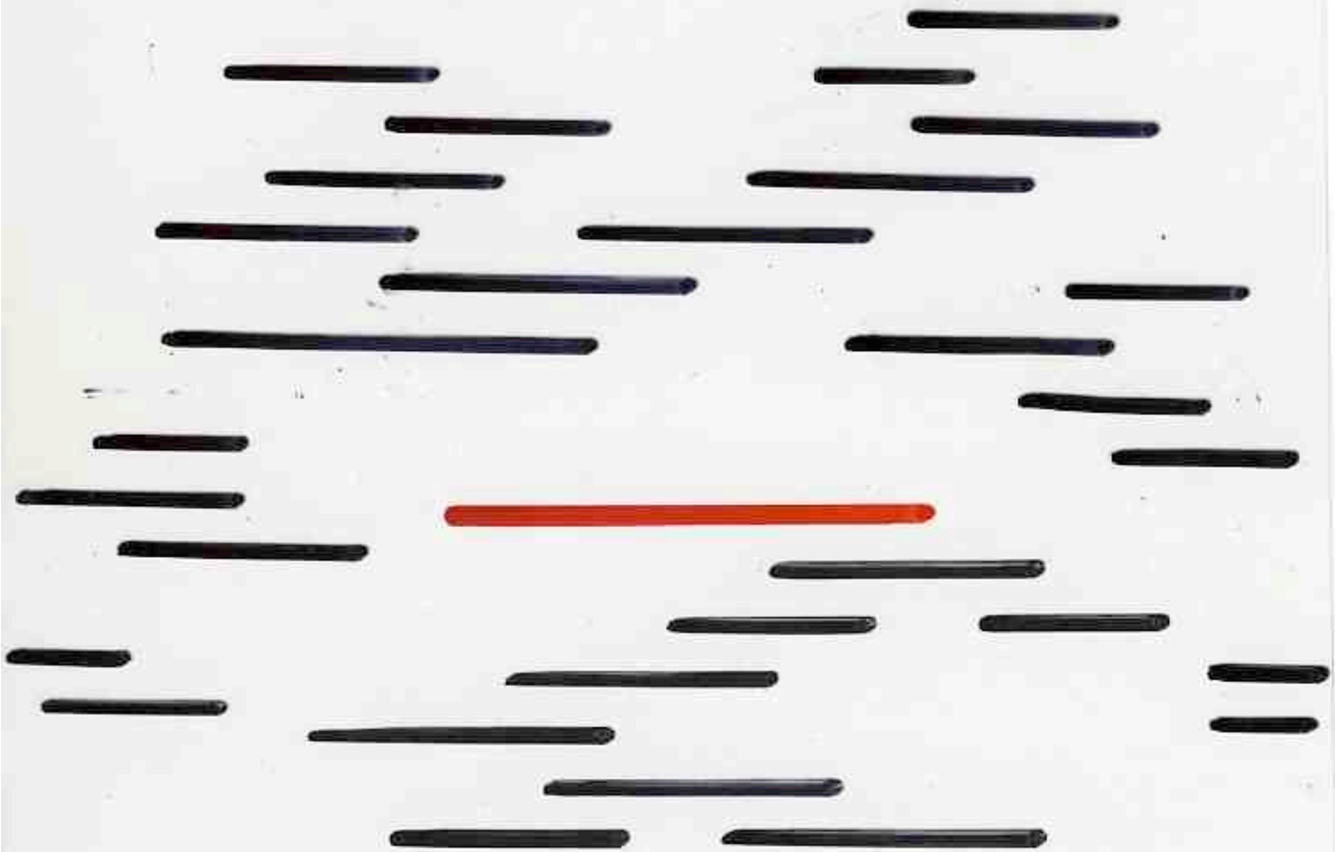
Pointed Heap = Pyramid x Heap



Push operator

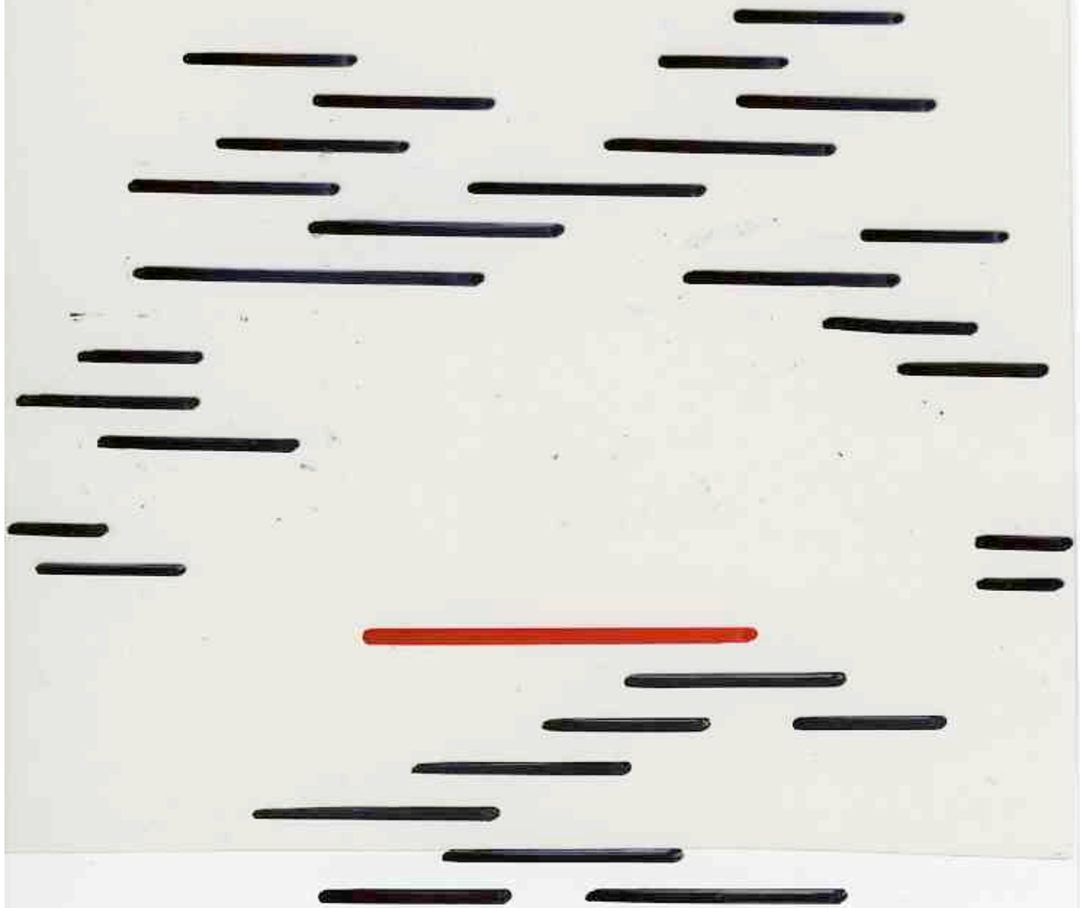
Opérateur

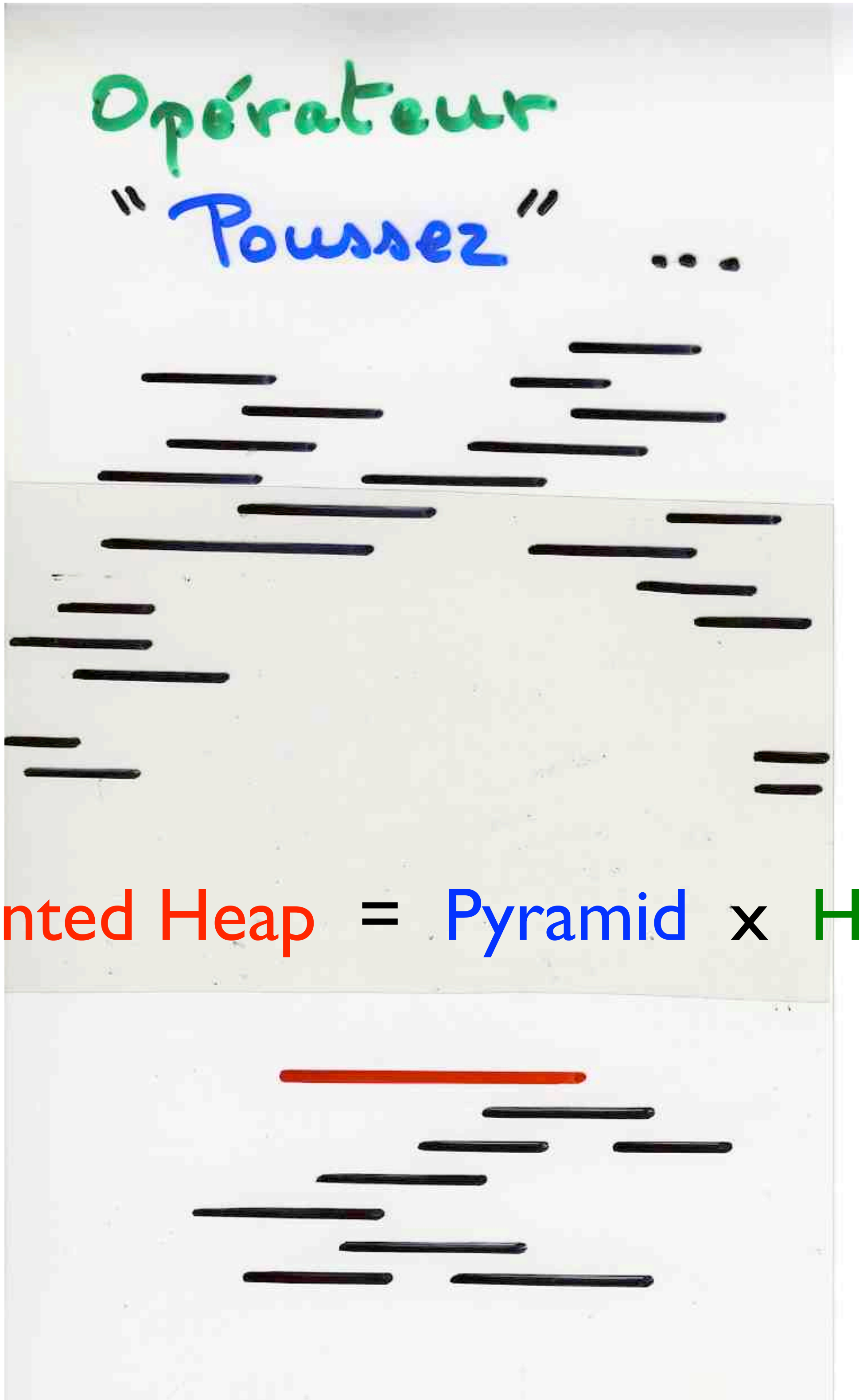
"Poussez" ...



Opérateur

"Poussez"...





Pointed Heap = Pyramid x Heap

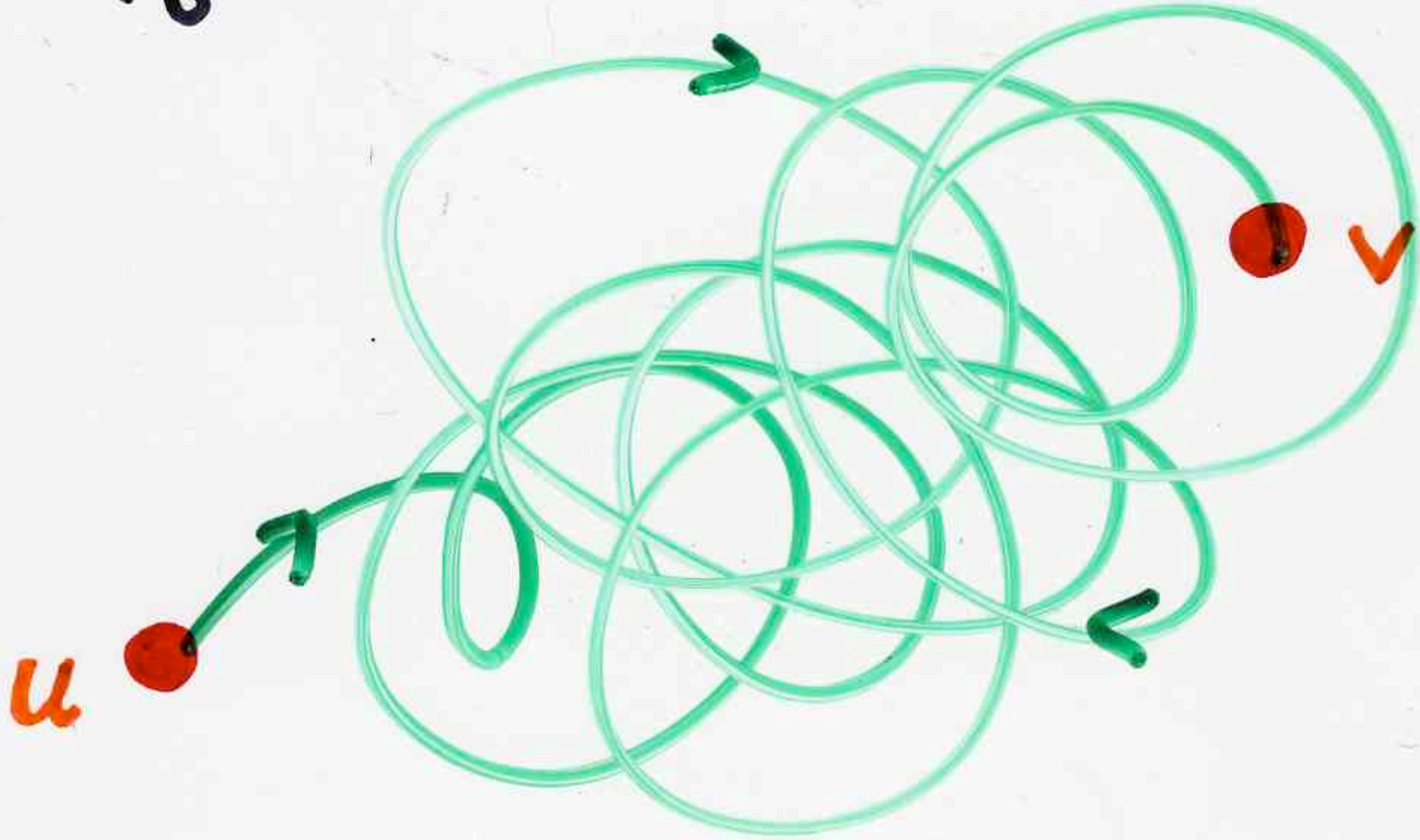
The third lemma:

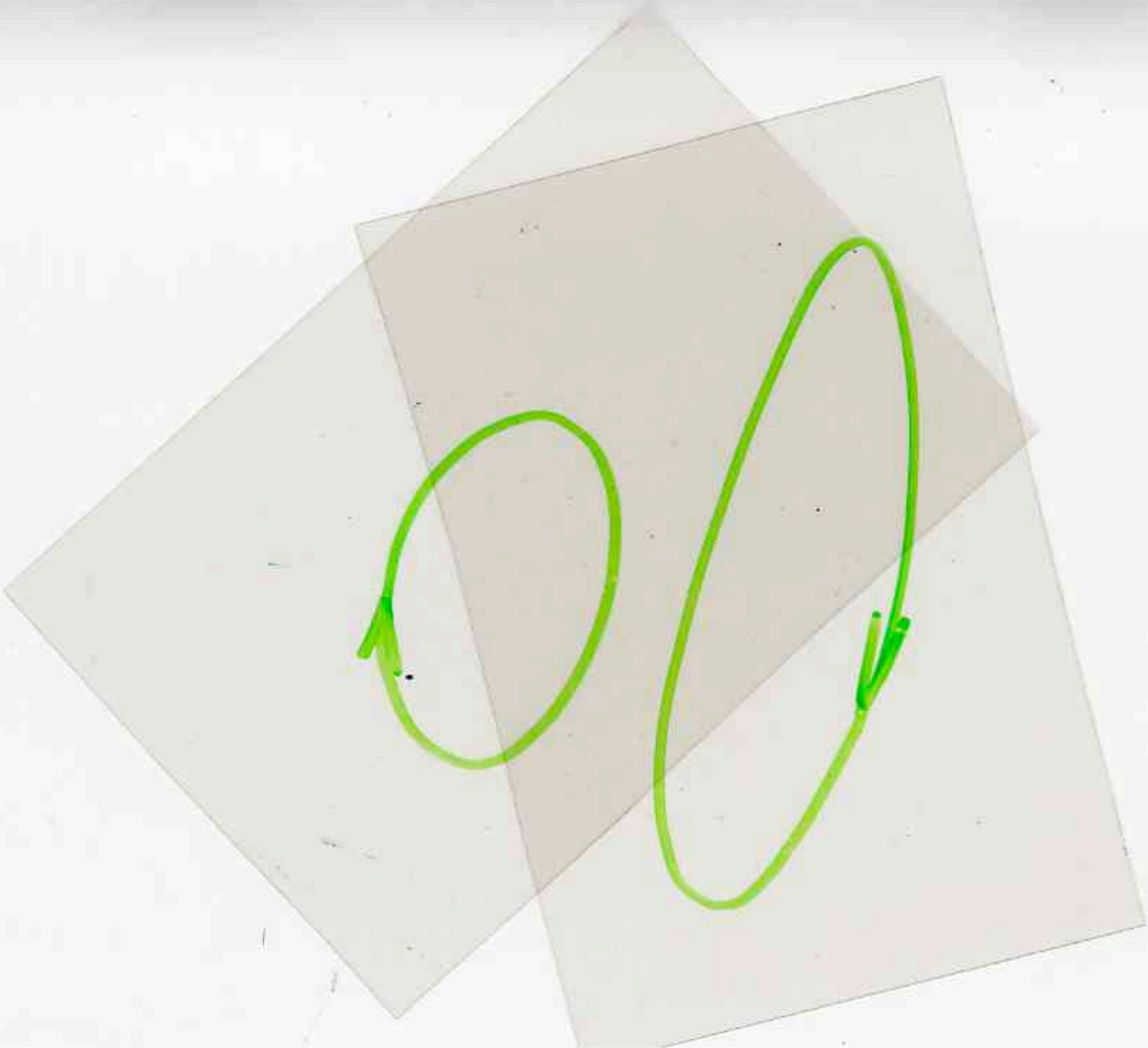
Paths and heaps of cycles

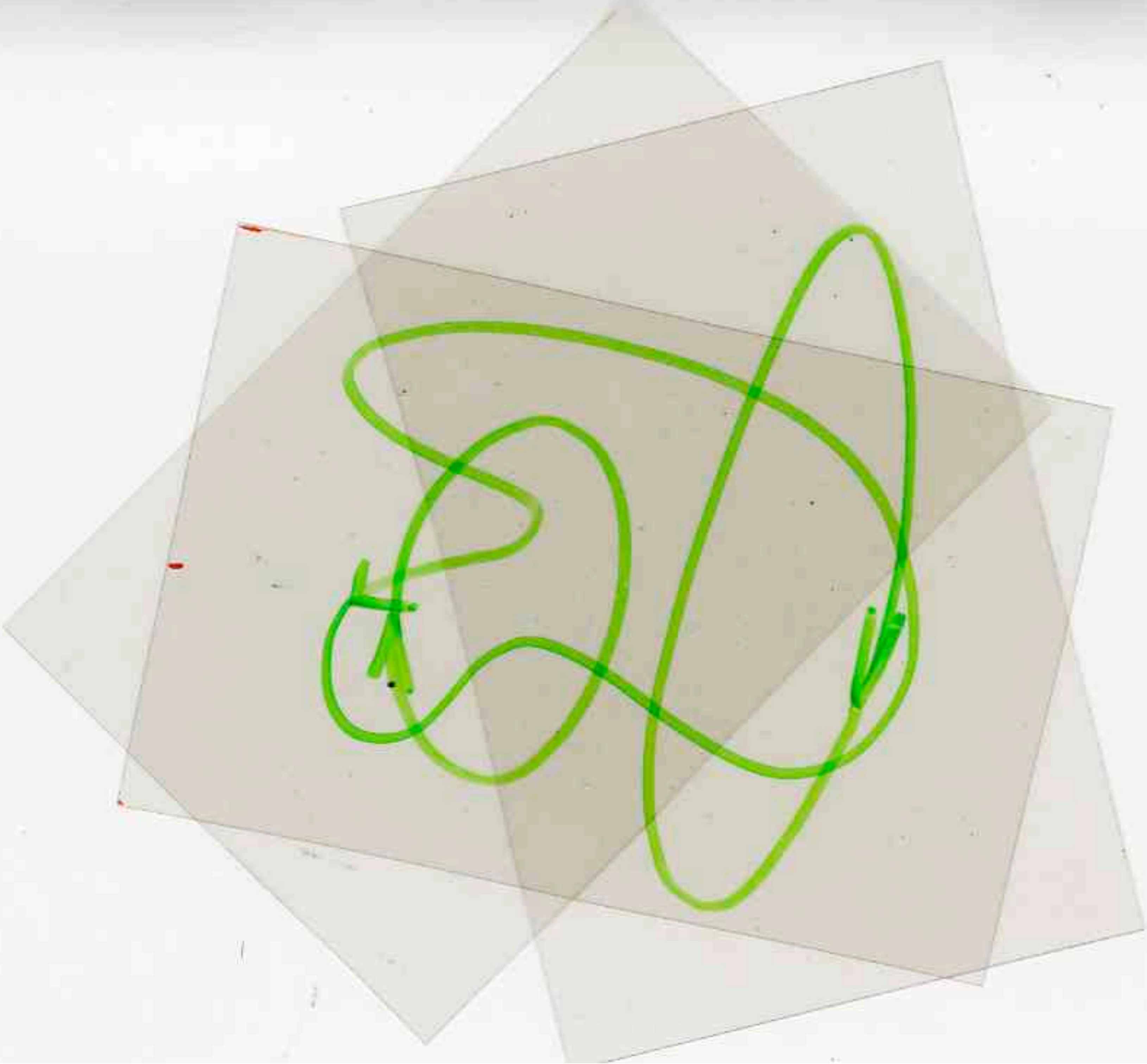
Path = Heap

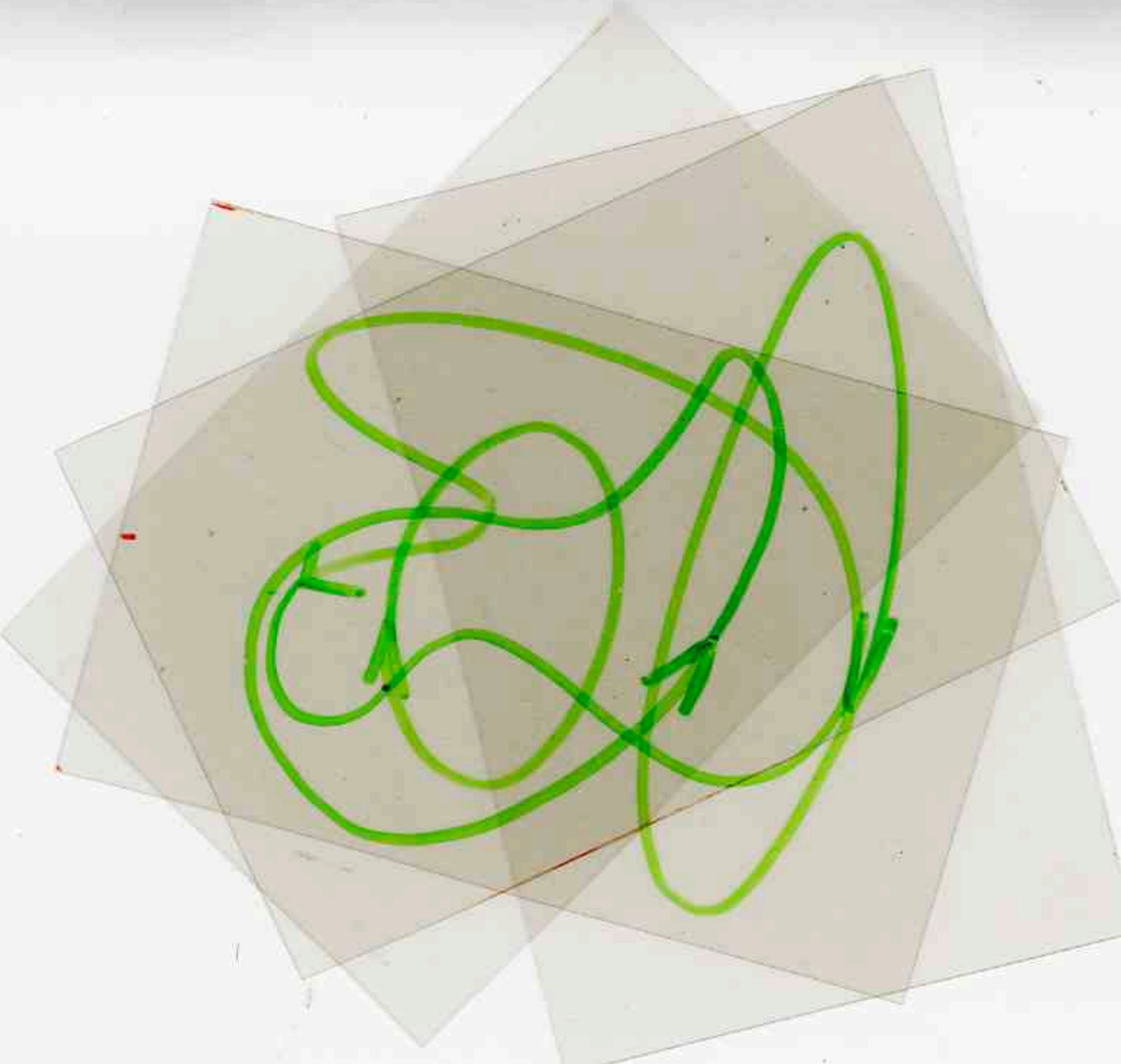
(of cycles)

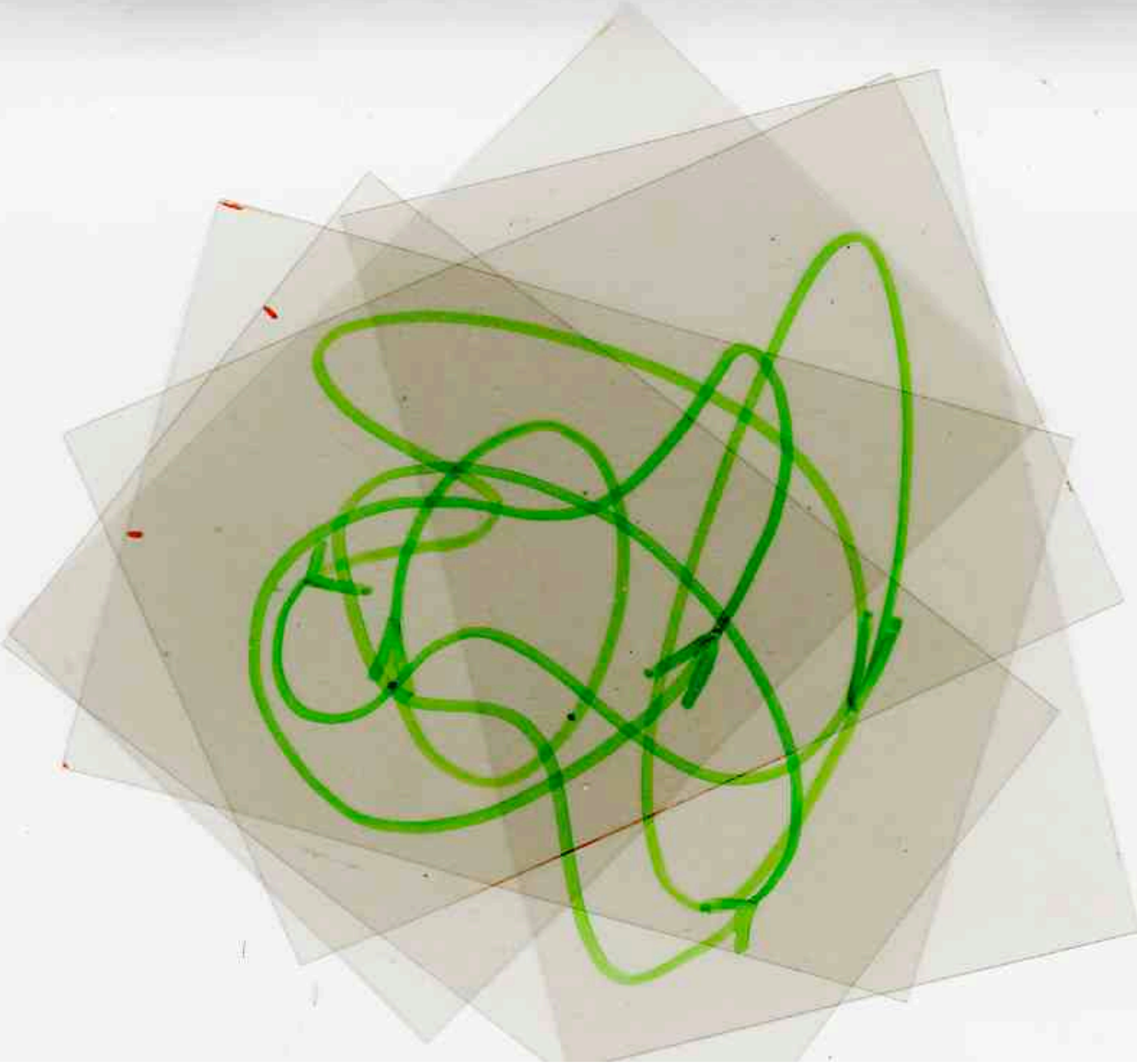
Proof:

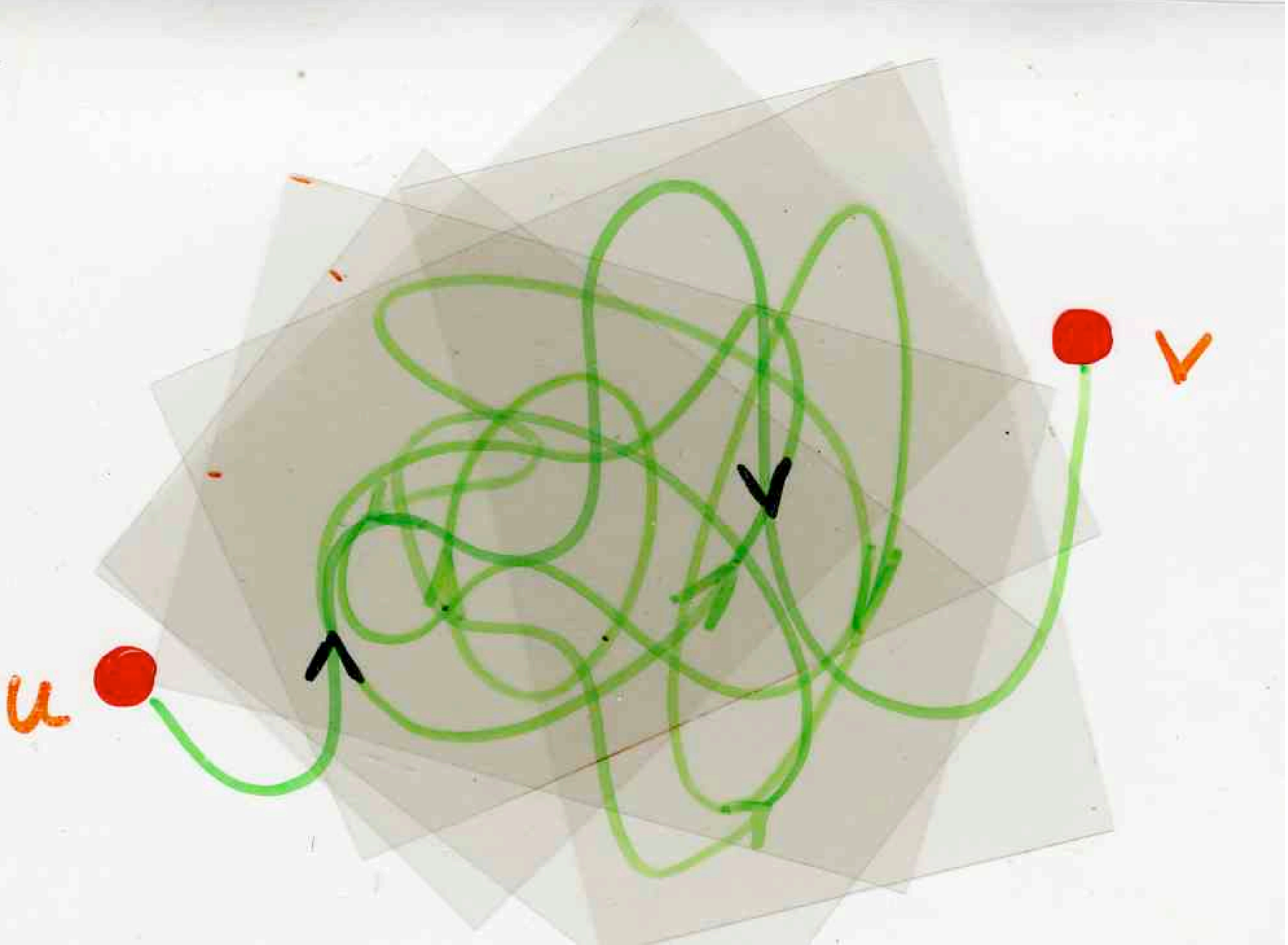


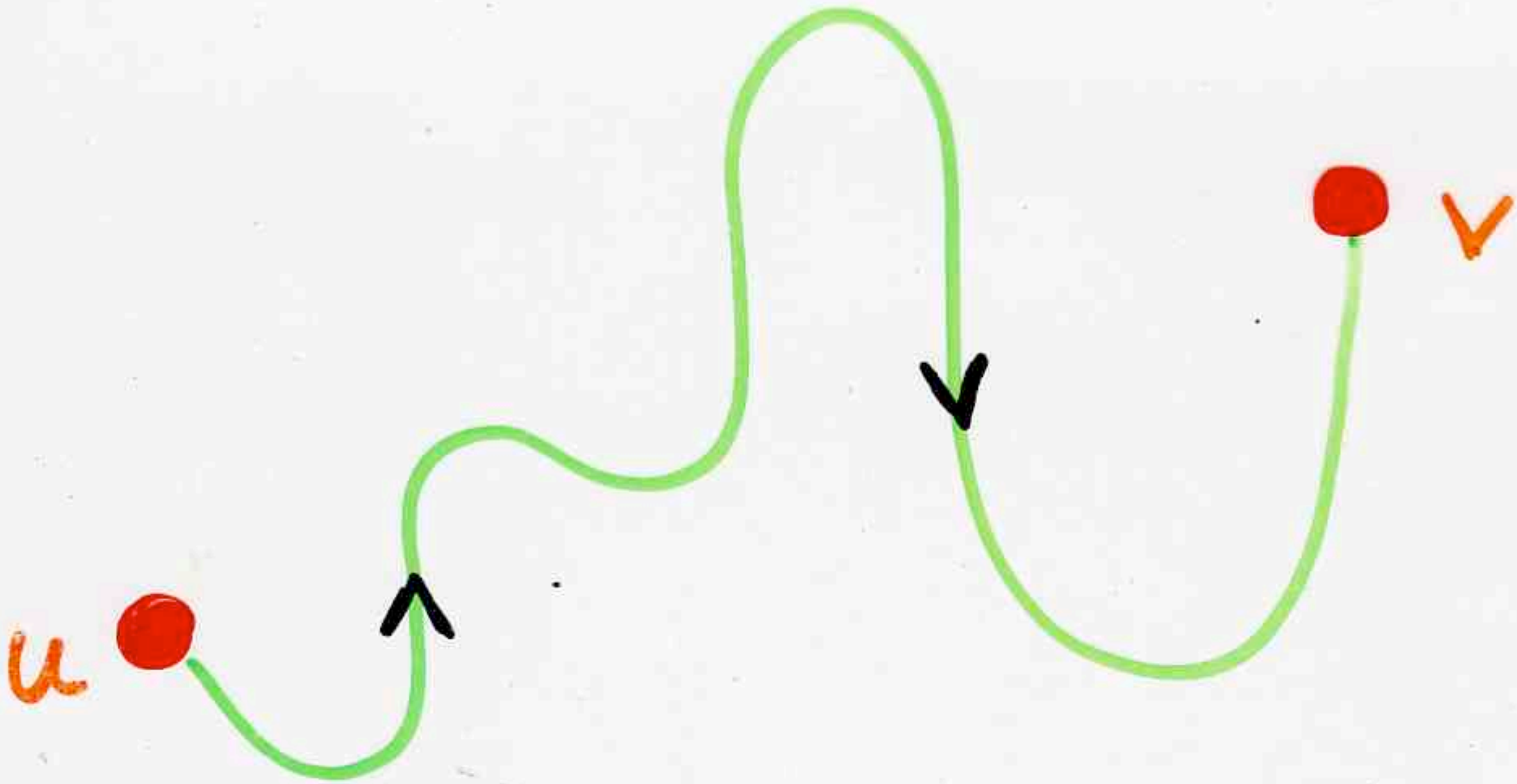









u
v


u
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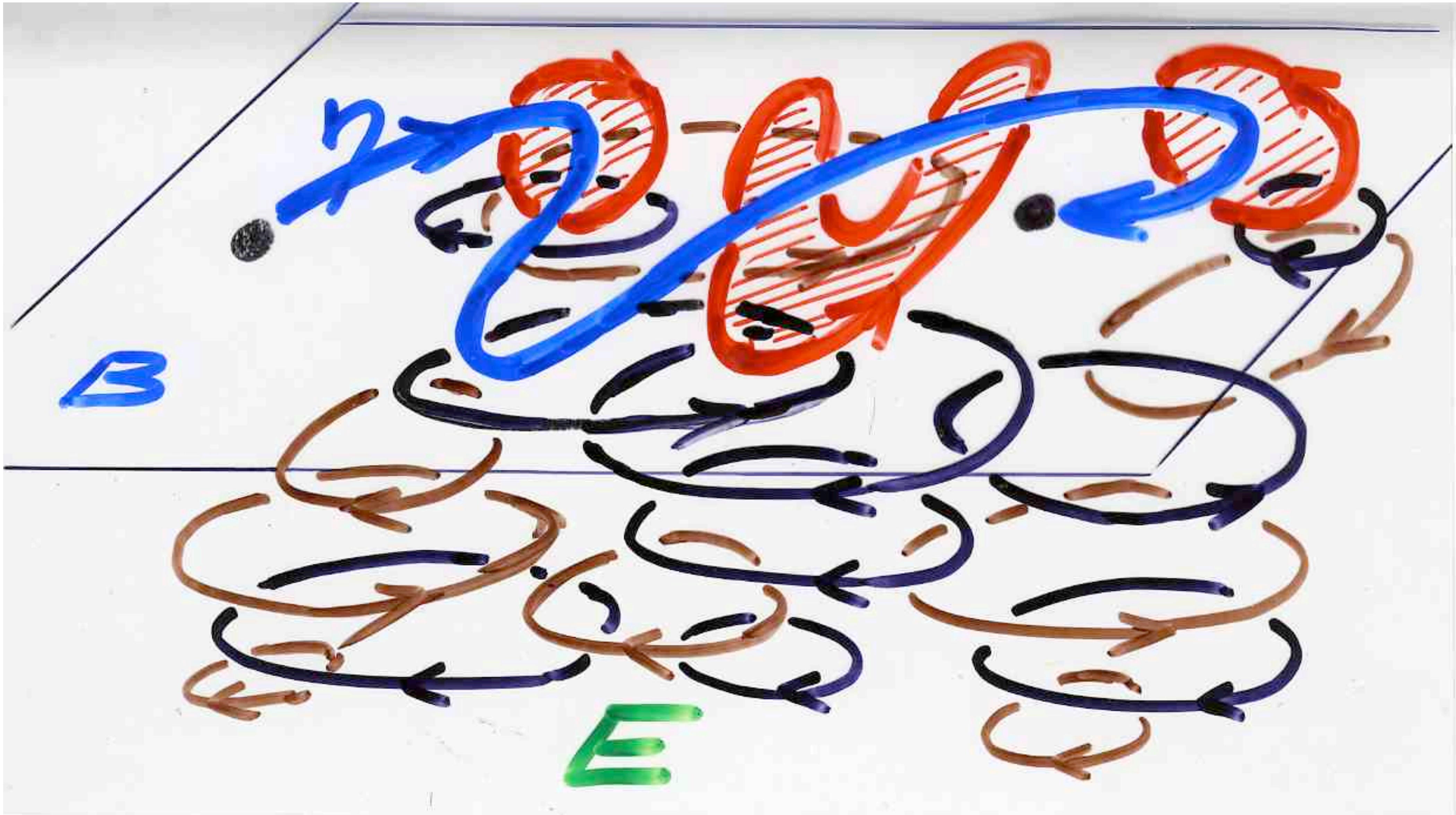

B
E

Bijection

Paths ω $\rightarrow$ (η, E)
$u \rightsquigarrow v$

- η self-avoiding path going from u to v
- E heap of cycles, $\pi(\alpha)$, $\alpha \in \max(E)$ intersects η

$\omega = (s_0 = u, \ldots, s_n = v)$ path on B
$u \rightsquigarrow v$

$\omega \rightarrow (\eta ; \{\gamma_1, \ldots, \gamma_{r_n}\})$

η: self-avoiding path $u \rightsquigarrow v$ ("coupe")

$\{\gamma_1, \ldots, \gamma_{r_n}\}$: sequence of cycles

$$\omega \longrightarrow (\eta \,;\, \{\dot{\gamma}_1, \dots, \dot{\gamma}_{r_n}\})$$

coupe(ω) suite(ω)

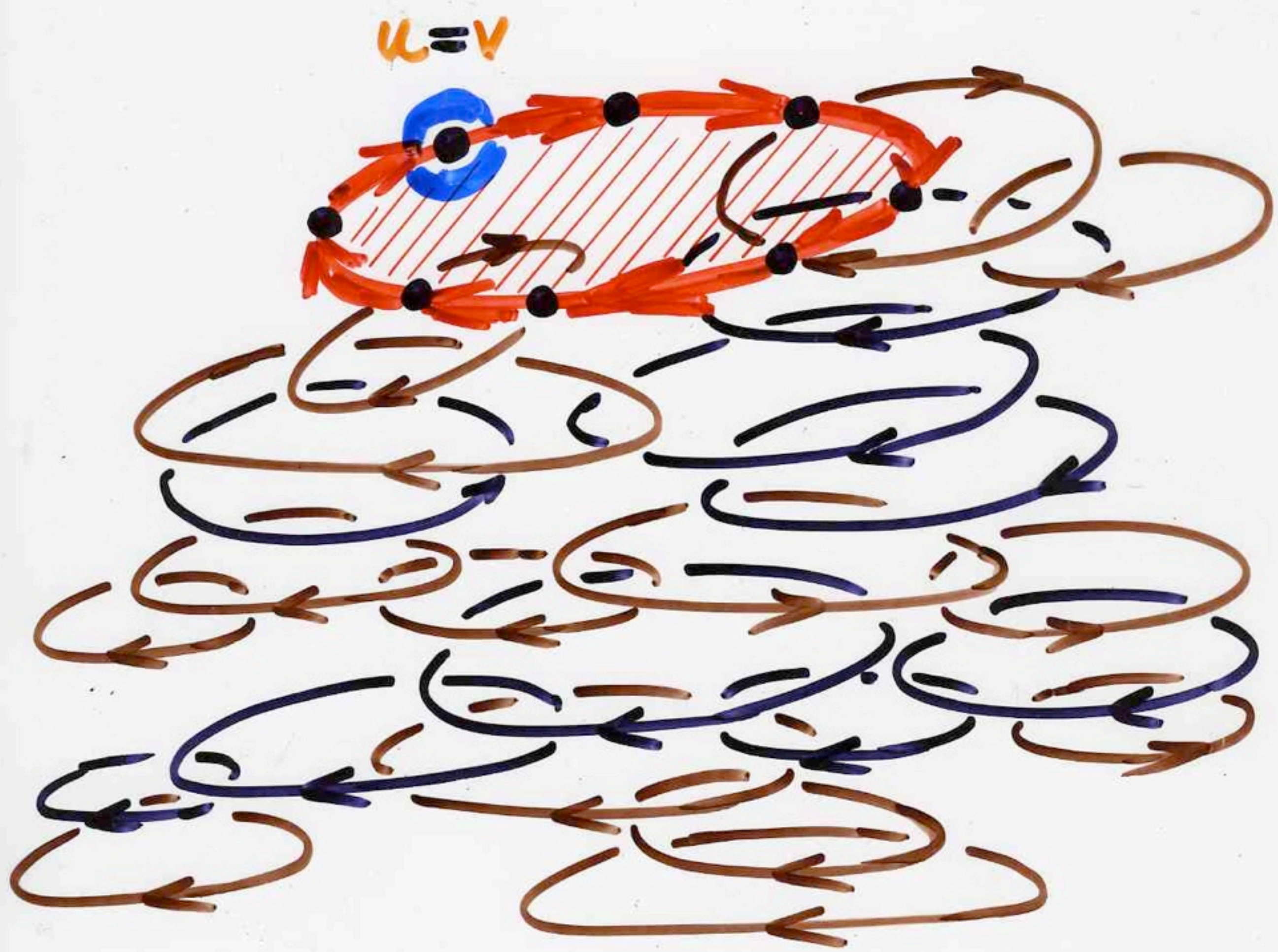
$$\downarrow$$

cycles heap

$$(\eta \,;\, (\gamma_1 \bullet \cdots \bullet \gamma_{r_n}))$$

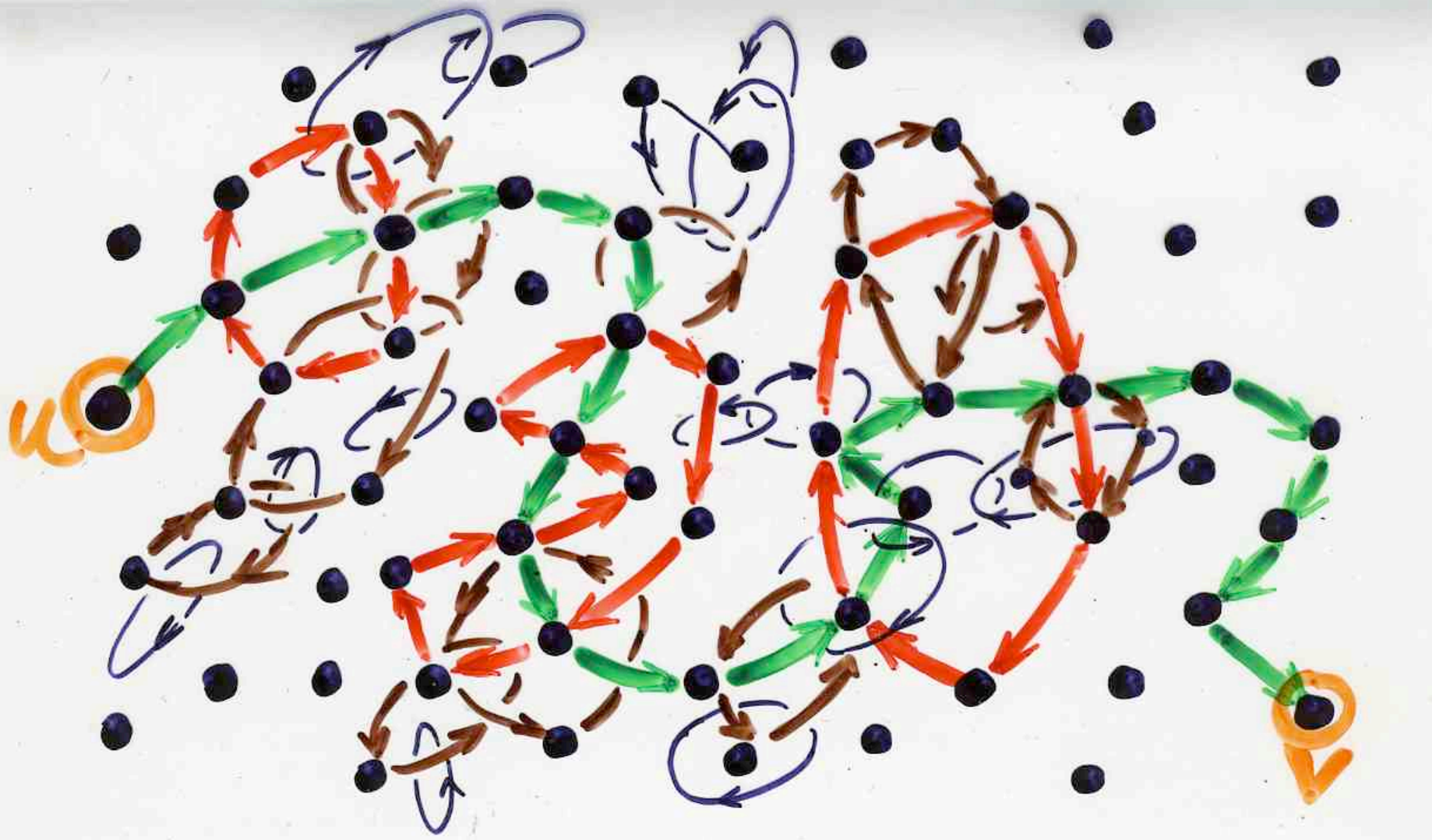
or pyramid $(\gamma_1 \bullet \cdots \bullet \gamma_{r_n} \bullet \eta) = \mathrm{Pyr}(\omega)$

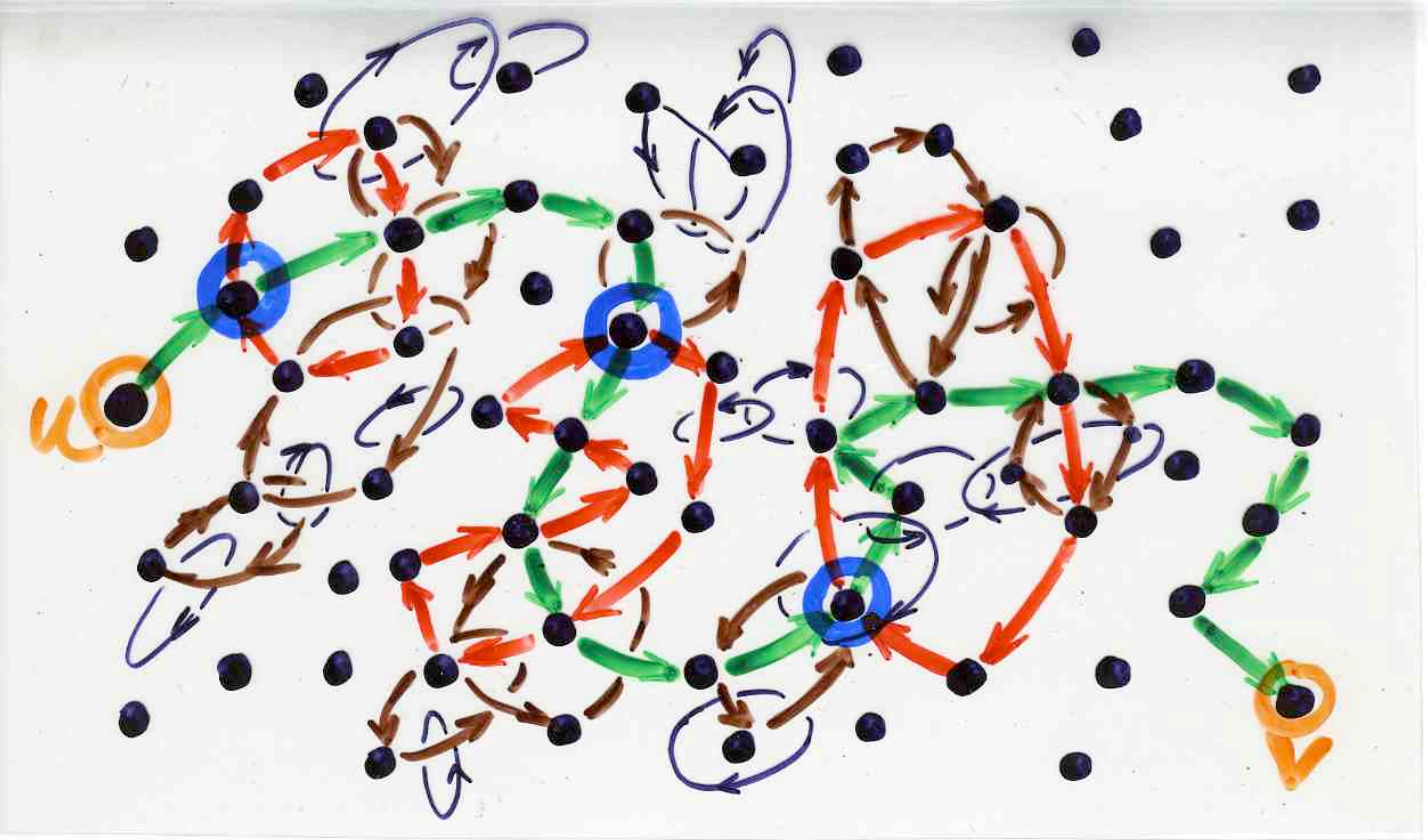
$\omega \longrightarrow \mathrm{Pyr}(\omega)$ bijection

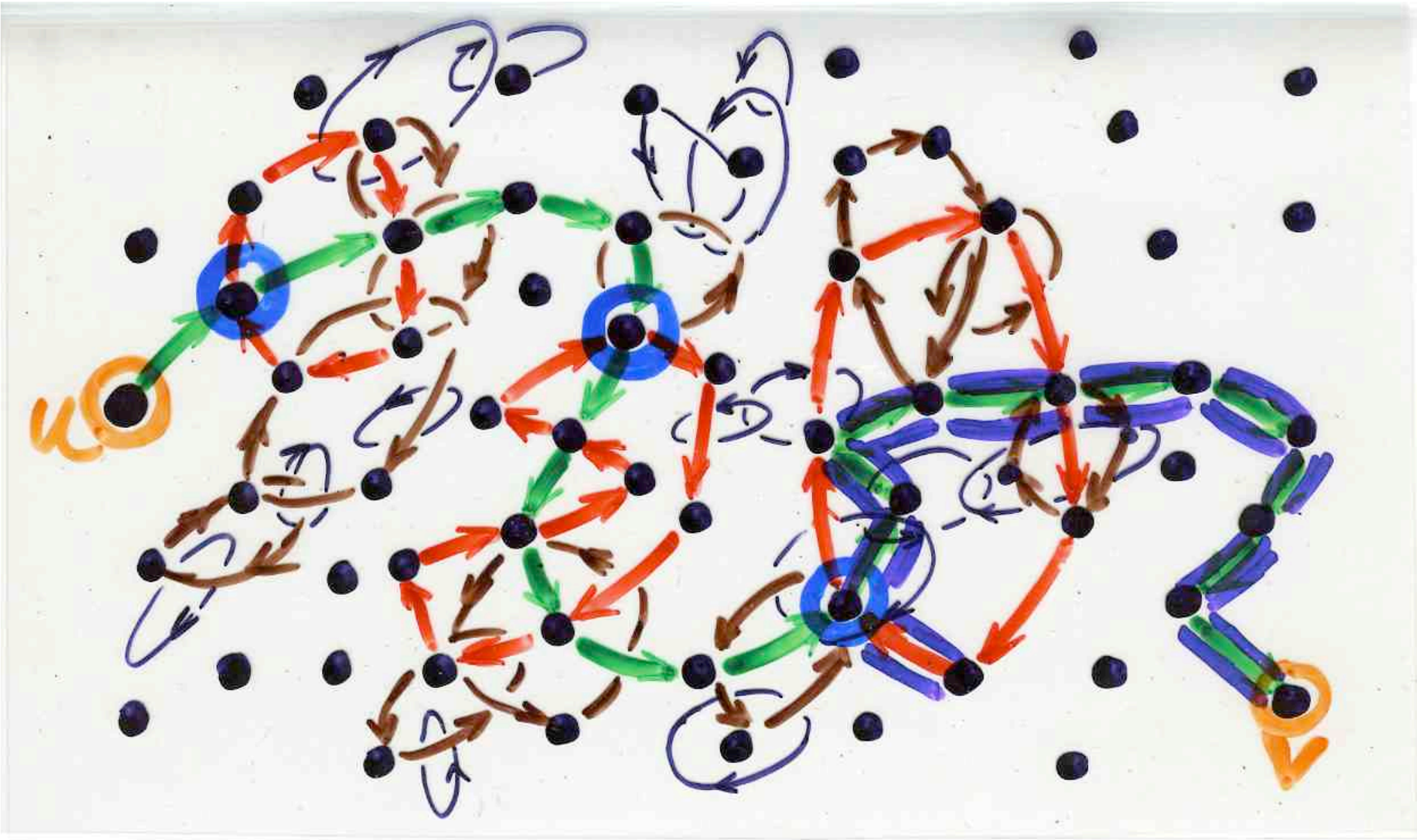


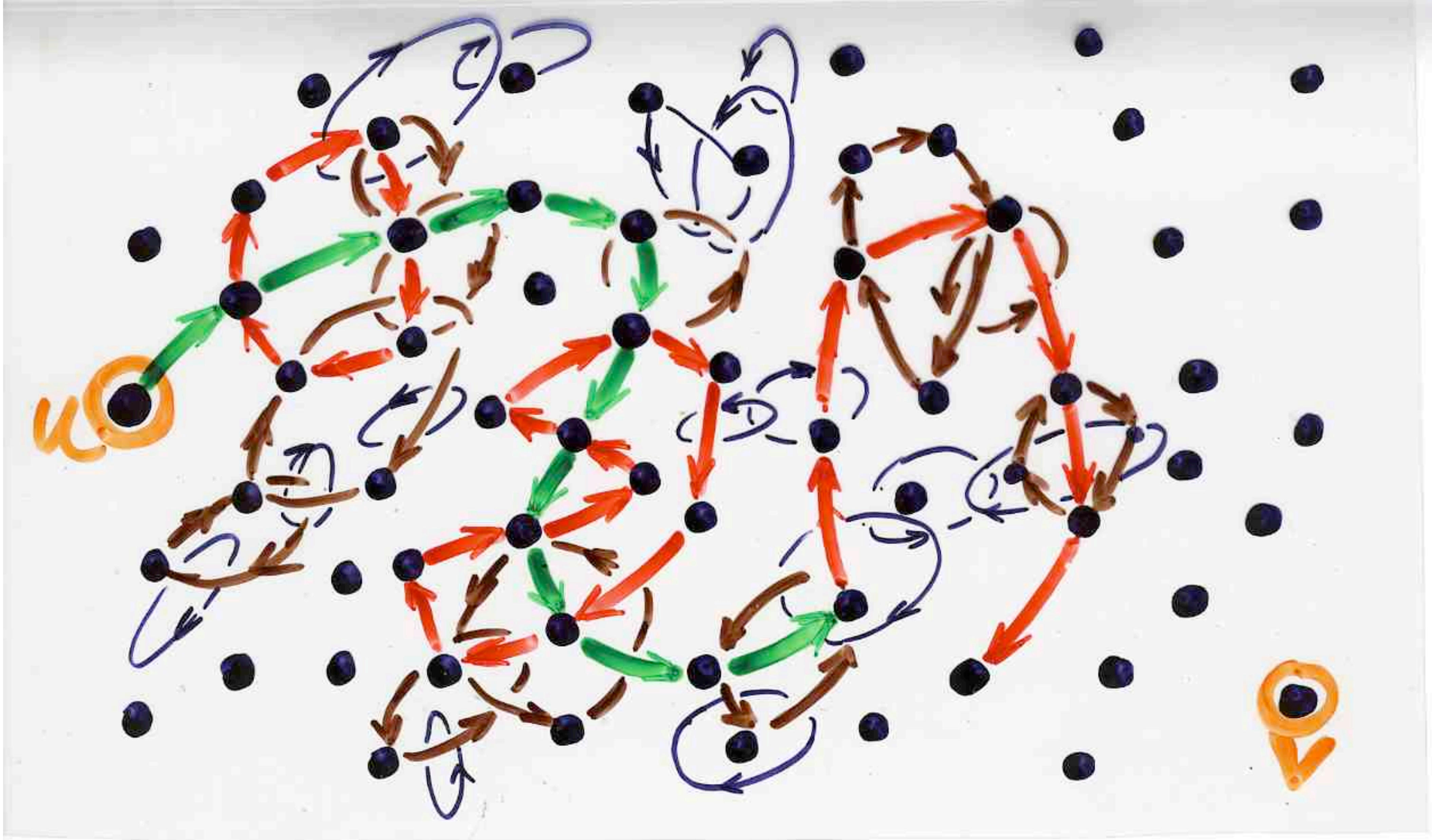
Rooted cycles pyramid

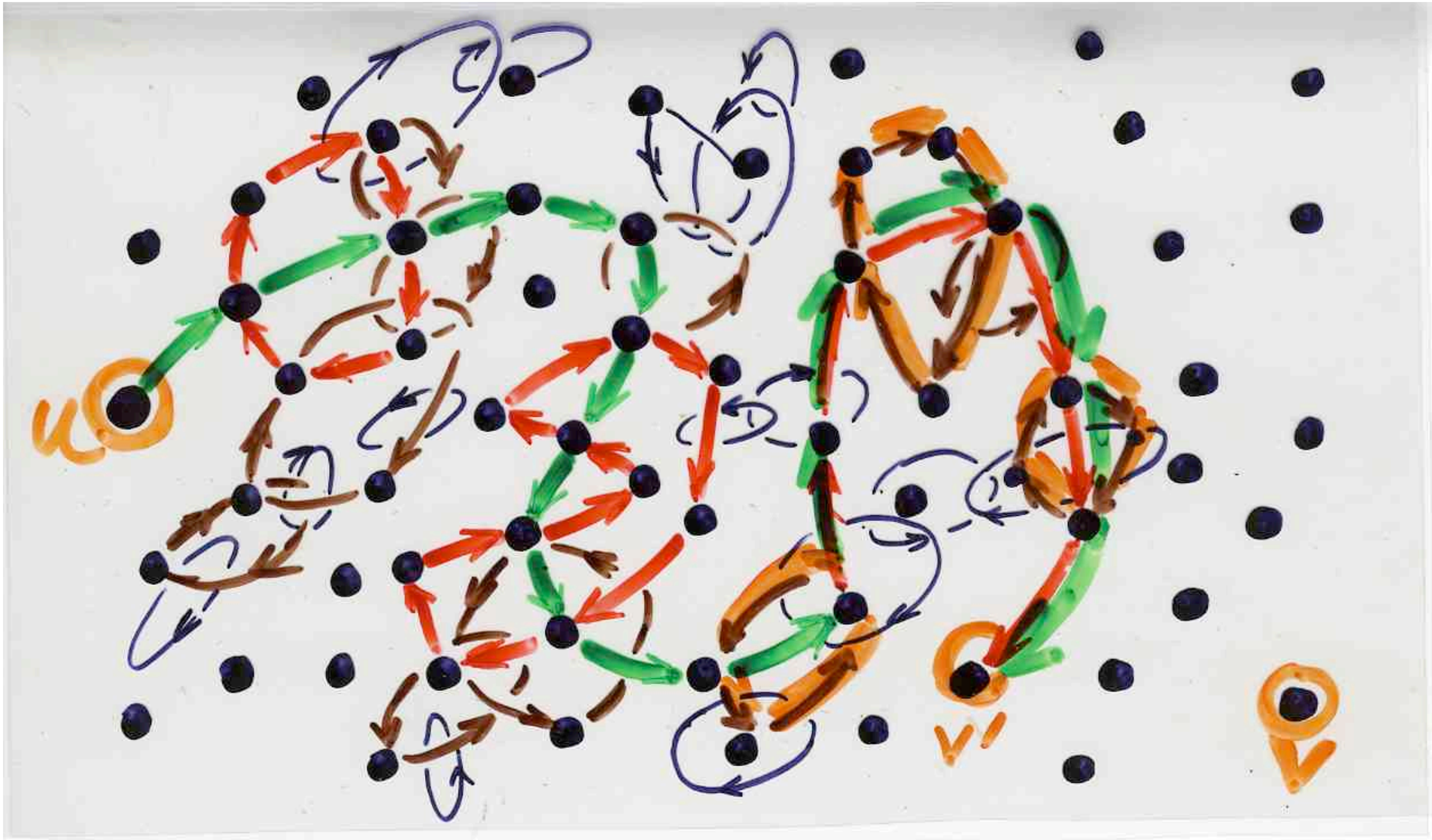
path -- heap of cycles:
inverse bijection


u
v

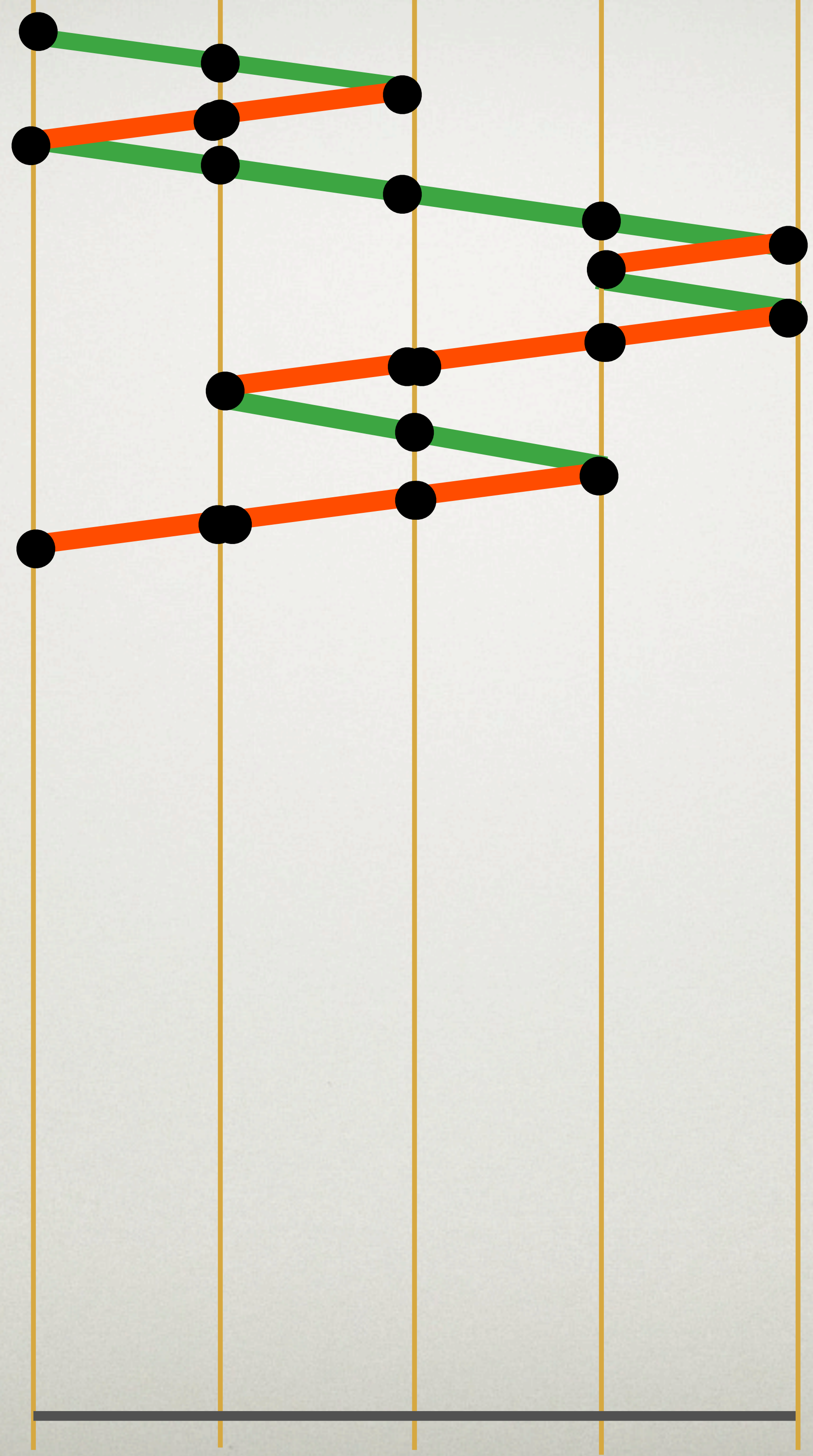

u
v

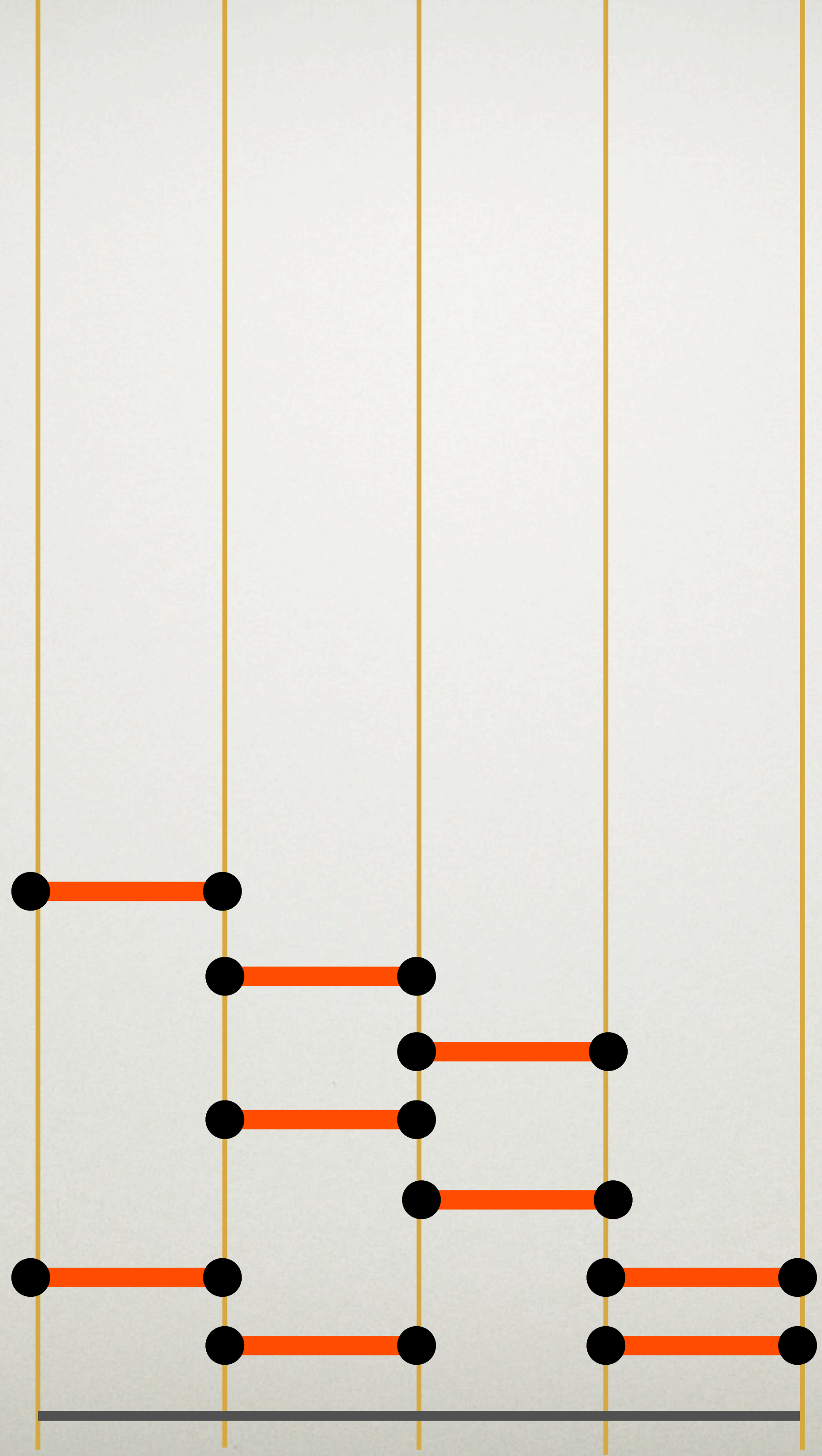

u
v

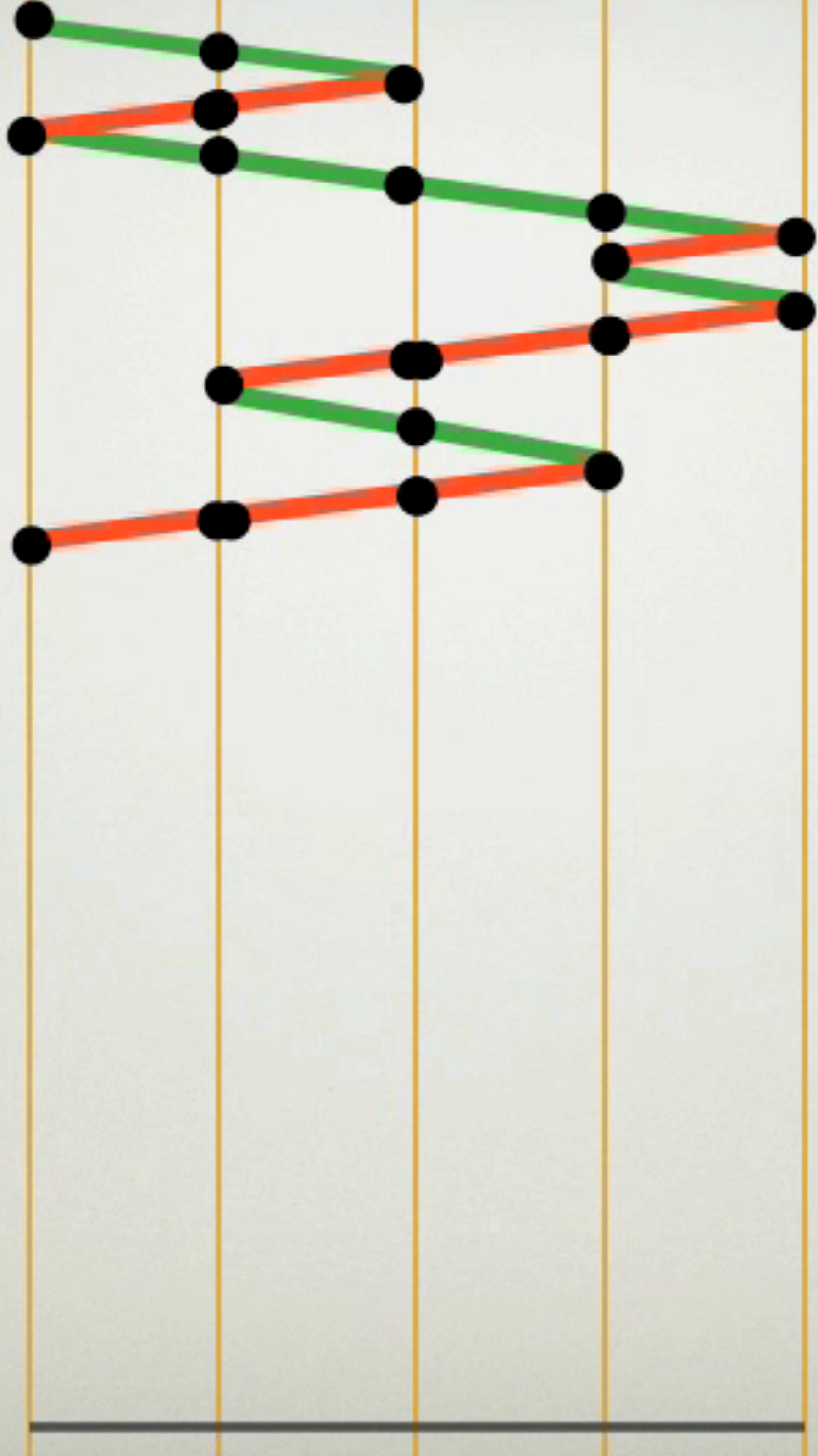

u
v


u
v'
v

example: bijection
Dyck paths
semi-pyramid of dimers







violin:
Gérard
Duchamp

classical linear algebra:
inversion of a matrix

or Cramer's rule
with a transition matrix in physics

Path (or walk)

$$\omega = (s_0, s_1, \dots, s_n) \qquad s_i \in S$$

s_0 starting, s_n ending point

length n

(s_i, s_{i+1}) elementary step

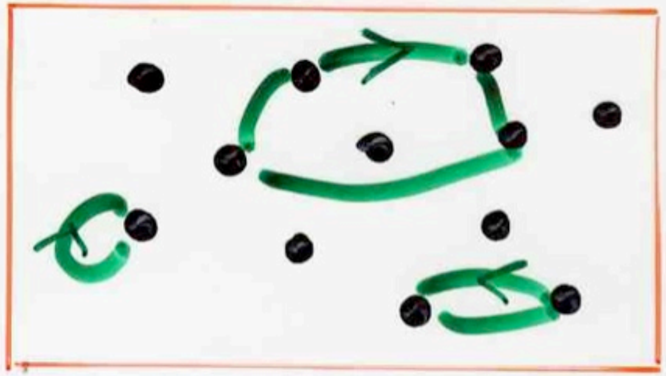
valuation (weight)

$$v(\omega) = \prod_{i=1}^{n} v(s_{i-1}, s_i)$$

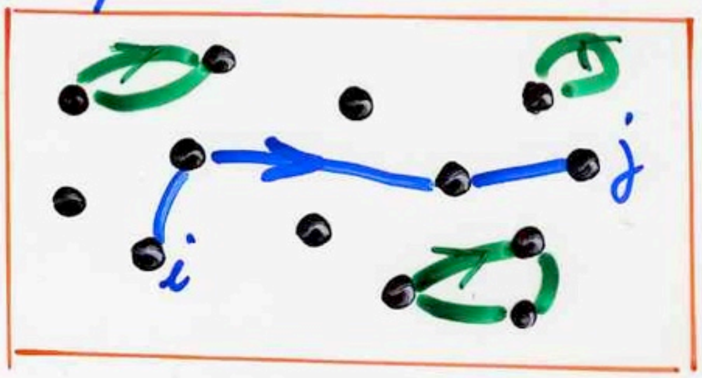
$$v : S \times S \to \mathbb{K}[x]$$

Prop. $$\sum_{\substack{\omega \\ i \rightsquigarrow j}} v(\omega) = \frac{N_{ij}}{D}$$

$$D = \sum_{\substack{\{\gamma_1, \dots, \gamma_r\} \\ \text{2 by 2 disjoint cycles}}} (-1)^r v(\gamma_1) \cdots v(\gamma_r)$$



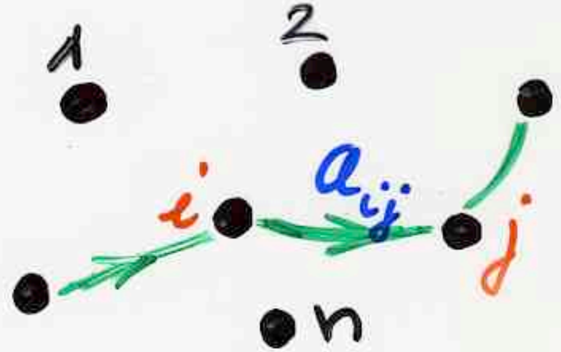
$$N_{ij} = \sum_{\{\eta; \gamma_1, \ \ , \gamma_r\}} (-1)^r v(\eta)\, v(\gamma_1) \cdots v(\gamma_r)$$



$$(I_n - A)^{-1}_{ij} = \frac{\mathrm{cof}_{ji}(I_n - A)}{\det(I_n - A)}$$

$$I_n + A + A^2 + \dots + A^n + \dots$$

$$A = (a_{ij})$$



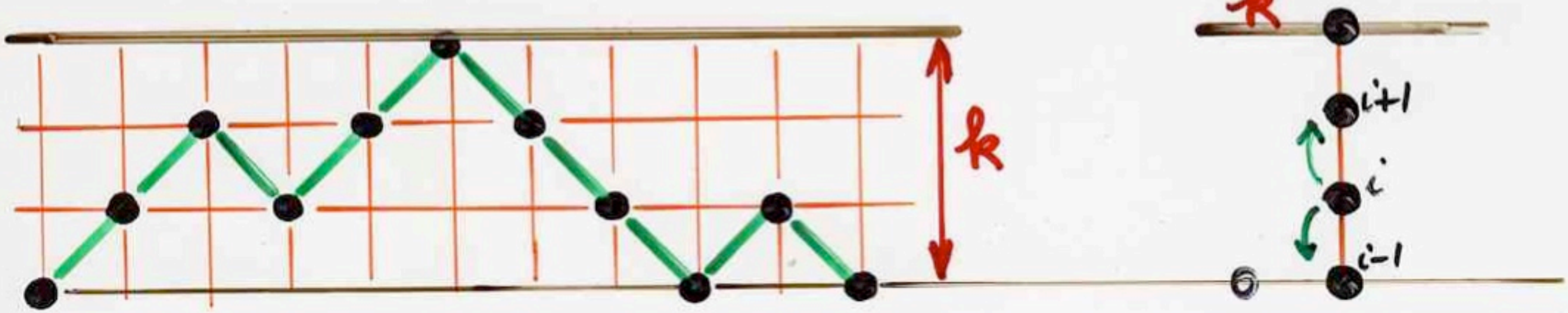
Abdesselam, Brydges (2006)

loop ensembles Cramer's rule

Mayer expansion

ex: Dyck path

bounded at height k



$$\sum_{\omega} t^{|\omega|/2} = \frac{F_k(t)}{F_{k+1}(t)}$$

Dyck paths

bounded k

$$A = (a_{ij}) = \begin{pmatrix} 0 & t & \cdots & 0 \\ t & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t \\ 0 & \cdots & t & 0 \end{pmatrix}$$